Pricing Arithmetic Average Reset Options With Control Variates

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Based on the closed-form solutions of partial barrier options, we derive the prices of general reset options with $m$ reset levels and continuous reset dates. Furthermore, we provide some special characteristics of reset call and put options. We explore the phenomena of delta jump existing for reset call and put options during the entire reset period whenever the stock price touches the barriers. For practical application, we use the reset call options with continuous reset dates as control variates to evaluate the prices of six arithmetic average reset options listed on Taiwan Stock Exchange from 1998 to 1999.

Path-dependent options, whose payoffs are influenced by the path of the prices of underlying assets, have become increasingly popular in recent years. One of path-dependent options is reset option. The strike prices of these options would be adjusted only on the specified reset dates if the price of underlying asset is below one of the reset levels. In practice, reset options have been traded for many years. The Chicago Board Options Exchange (CBOE) and the New York Stock Exchange (NYSE) both introduced S&P 500 index put warrants with three-month reset period in late 1996. Morgan Stanley issued a reset warrant with an initial strike price of $44.73 in July 1997. The strike price would be adjusted to $39.76 on August 5, 1997 if the price of its underlying asset fell below $39.76. In 1998, Grand Cathay Securities in Taiwan issued reset option that the strike price would be adjusted if the six-day average closing price of United Microelectronics Corporation (whose security code in the Taiwan Stock Exchange (TSE) is 2323) fell below 90% of the initial strike price $58.5 during the first three months after the warrant was issued.

Since reset options are new derivatives in the financial market, the related literature is

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rare. Gray and Whaley [1997] examined the pricing of S&P 500 bear market warrants with periodic reset and compared them vis-à-vis a standard S&P500 put as the index level and volatility change. Gray and Whaley [1999] also provided a closed-from solution for reset option with single reset date. Cheng and Zhang [2000] studied the reset options that the strike price will be reset to the prevailing stock price if the option is out of money. A closed-form pricing formula in terms of the multivariate normal distribution is derived under risk-neutral framework. The terminal payoff of reset options with \( n \) reset dates and initial strike price \( K_0 \), studied by Cheng and Zhang [2000], is as follows:

\[
C_T = \max\{S(T) - \min\{K_0, S(t_1), \ldots, S(t_n)\}, 0\}
\]

In practice, the terminal payoff of reset option is more often set as

\[
C_{T}^{\text{Reset}} = \max\left[ S(T) - K^*, 0 \right] \equiv S(T) - K^*
\]

\[
K^* = K_i \quad \text{if} \quad D_i \geq \min\{S(t_1), \ldots, S(t_n)\} > D_{i+1}, \quad \text{for} \quad i = 0, \ldots, m,
\]

where \( K_i, \ i = 1, \ldots, m, \) are the reset strike prices and \( D_i, \ i = 1, \ldots, m, \) are the reset levels; \( D_0 (\equiv \infty) > D_1 > D_2 \cdots > D_m > D_{m+1} (\equiv 0) \).

The reset options described in (1) and (2) are actually adopted in the real world. Liao and Wang [2002] provided exact closed-form solutions for the general reset options with \( m \) reset levels and \( n \) pre-decided reset dates under the risk-neutral framework. The phenomena of Delta and Gamma jumps across reset dates as well as the properties of Delta and Gamma waviness near reset dates are also investigated in that paper.

Nevertheless, taking a reset option with the first three months for reset dates as an example, the dimension of the multivariate normal distribution is 91. It is difficult to calculate the accurate cumulative probability of multivariate normal distribution with high dimension, and hence, the accurate prices of reset options with many reset dates are also difficult to obtain. Furthermore, the closed-form solution of arithmetic average reset option does not
exist due to the fact that the sum of lognormal variables is not lognormal. In order to overcome this problem, the motivation for this paper is to use a similar contingent claim to approximate the value of a reset option or to take this similar contingent claim as control variate in Monte Carlo simulation to obtain the price of arithmetic average reset option. The similar contingent claim that we use in this paper is a multiple partial reset option with \( m \) reset levels and its continuous reset period is less than time to maturity.

Accordingly, using martingale method, we derive closed-form solutions for single-barrier options with monitoring period less than time to maturity (i.e. partial barrier options). The formulas are the same as the results of Heynen and Kat [1994] and Hui [1997]\(^1\). Then, we use partial barrier options as building blocks to derive the closed-form valuation of a general reset option with \( m \) reset levels and continuous reset period which is less than time to maturity. Furthermore, we conduct a Monte Carlo simulation using the closed-form solution of reset option with \( m \) reset levels and continuous reset period as the control variate to value the price of an arithmetic average reset option with \( m \) reset levels and \( n \) reset dates.

An outline of the paper is as follows. We establish the model and derive the closed-form valuations of single-barrier options and general reset options with continuous reset dates in Section I. In Section II, we present some properties of reset options with continuous reset dates and compare the actual issuing prices of reset options with simulated prices. Section III provides the conclusion of this paper.

### I. PRICING PARTIAL RESET OPTIONS

We assume that the dynamics of underlying asset price \( S(t) \) are described by the following stochastic differential equation:
\[ \frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma dW^Q_t \]

where \( \delta \geq 0 \) and \( \sigma > 0 \) are constants, and \( r \) is the constant spot interest rate over the trading period \([0, T]\). \( W^Q_t \) stands for a one-dimensional standard Brownian motion defined on a filtered probability space \( (\Omega, F, Q) \), where \( Q \) is the spot martingale measure or risk neutral probability measure. The savings account, \( B_t \), is defined as

\[ B(T) = B(t)e^{r(T-t)} \]

Let \( T \) be the expiry date and the reset period be the time interval \([0, T_0]\), where \( T_0 \) is the last day of reset period, \( T_0 < T \). The payoffs of single-barrier down-and-out or down-and-in call options and down-and-out or down-and-in put options at expiry are as follows:

\[ \begin{align*}
DOC_T &= \left[ S(T) - K \right]^+ I_{\{\text{Min } S(\nu) > D, \nu \leq T \}} \\
DIC_T &= \left[ S(T) - K \right]^+ I_{\{\text{Min } S(\nu) \leq D, \nu \leq T \}}
\end{align*} \]

\[ \begin{align*}
DOP_T &= \left[ K - S(T) \right]^+ I_{\{\text{Min } S(\nu) > D, \nu \leq T \}} \\
DIP_T &= \left[ K - S(T) \right]^+ I_{\{\text{Min } S(\nu) \leq D, \nu \leq T \}}
\end{align*} \]

where \( I(\cdot) \) is the indicator function; \( S(t) > D \) and \( K \geq D \).

Similarly, the payoffs at expiry of single-barrier up-and-out or up-and-in call options and up-and-out or up-and-in put options are given as follows:

\[ \begin{align*}
UOC_T &= \left[ S(T) - H \right]^+ I_{\{\text{Max } S(\nu) < G, \nu \leq T \}} \\
UIC_T &= \left[ S(T) - H \right]^+ I_{\{\text{Max } S(\nu) \geq G, \nu \leq T \}}
\end{align*} \]

\[ \begin{align*}
UOP_T &= \left[ H - S(T) \right]^+ I_{\{\text{Max } S(\nu) < G, \nu \leq T \}} \\
UIP_T &= \left[ H - S(T) \right]^+ I_{\{\text{Max } S(\nu) \geq G, \nu \leq T \}}
\end{align*} \]

where \( S(t) < G \) and \( G \geq H \).

In order to derive the values of all types of single-barrier call and put options with reset period less than time to maturity, we first provide the following theorem:
Theorem 1. Assume that \( X(t) = u t + \sigma W_t \), \( x \geq y \), \( 0 \geq a \), \( b \geq 0 \) and \( \tau > \lambda \), then we have

\[
P[X(\tau) \geq x, \text{Min} X(t) \geq y] = N_2 \left[ \frac{u - x - u \lambda - y \lambda}{\sigma \sqrt{\tau}}, \frac{u \lambda - y \lambda}{\sigma \sqrt{\tau}} \right] - e^{\frac{2 u y}{\sigma^2}} N_2 \left[ \frac{u + 2 y - x - y + u \lambda - y \lambda}{\sigma \sqrt{\tau}}, \frac{y + u \lambda - y \lambda}{\sigma \sqrt{\tau}} \right]
\]

(3)

\[
P[X(\tau) \leq x, \text{Min} X(t) \leq y] = N_2 \left[ \frac{x - u t - y}{\sigma \sqrt{\tau}}, \frac{y - u \lambda}{\sigma \sqrt{\tau}} \right] + e^{\frac{2 u y}{\sigma^2}} N_2 \left[ \frac{x - u t - 2 y - y + u \lambda}{\sigma \sqrt{\tau}}, \frac{y + u \lambda}{\sigma \sqrt{\tau}} \right]
\]

(4)

\[
P[X(\tau) \geq a, \text{Max} X(t) \geq b] = N_2 \left[ \frac{-a + u t - b + u \lambda}{\sigma \sqrt{\tau}}, \frac{y - u \lambda}{\sigma \sqrt{\tau}} \right] + e^{\frac{2 u b}{\sigma^2}} N_2 \left[ \frac{-a + 2 b + u t - b - u \lambda}{\sigma \sqrt{\tau}}, \frac{-b - u \lambda}{\sigma \sqrt{\tau}} \right]
\]

(5)

\[
P[X(\tau) \leq a, \text{Max} X(t) \geq b] = N_2 \left[ \frac{-a + u t - b + u \lambda}{\sigma \sqrt{\tau}}, \frac{y - u \lambda}{\sigma \sqrt{\tau}} \right] + e^{\frac{2 u b}{\sigma^2}} N_2 \left[ \frac{-a + 2 b - u t - b - u \lambda}{\sigma \sqrt{\tau}}, \frac{-b - u \lambda}{\sigma \sqrt{\tau}} \right]
\]

(6)

where \( N_2 \) is the cumulative probability of bivariate normal distribution with zero means and instantaneous correlation coefficient \( \rho \). We prove Theorem 1 in Appendix A.

Using Theorem 1, we can straightforwardly derive the prices of all single-barrier options as shown in Theorem 2.

Theorem 2. Let \( DOC_t \), \( DIP_t \), \( UIC_t \) and \( UIP_t \) be the prices at time \( t \) of down-and-out call, down-and-in put, up-and-in call and up-and-in put with reset period \( \lambda = T_0 - t \), which is less than time-to-maturity \( \tau = T - t \), respectively. Then, we have

\[
DOC_t = S(t)e^{-\delta \tau} N_2 \left[ d_1(K, \tau), d_1(D, \lambda), \frac{\lambda}{\tau} \right] - Ke^{-\tau r} N_2 \left[ d_2(K, \tau), d_2(D, \lambda), \frac{\lambda}{\tau} \right]
\]

\(- S(t)e^{-\delta \tau} \left( \frac{D}{S(t)} \right)^{-\frac{1}{2} \sqrt{\rho \sigma}} d_1(D, K, h_1(D), \sqrt{\frac{\lambda}{\tau}}) - Ke^{-\tau r} \left( \frac{D}{S(t)} \right)^{-\frac{1}{2} \sqrt{\rho \sigma}} N_2 \left[ g_2(D, K, h_2(D), \sqrt{\frac{\lambda}{\ tau}} \right]
\]

(7)
\[
DIP_t = \left[-S(t)e^{-\delta \tau} N_{2}\left[- d_1(K, \tau), -d_1(D, \lambda), \frac{\lambda}{\tau}\right] + Ke^{-r \tau} N_{2}\left[- d_2(K, \tau), -d_2(D, \lambda), \frac{\lambda}{\tau}\right]\right] - \left\{S(t)e^{-\delta \tau} \left(\frac{D}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[- g_1(D, K), -h_1(D), -\frac{\lambda}{\tau}\right] - Ke^{-r \tau} \left(\frac{D}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[- g_2(D, K), -h_2(D), -\frac{\lambda}{\tau}\right]\right\}
\]

\[
UIC_t = \left[S(t)e^{-\delta \tau} N_{2}\left[d_1(H, \tau), d_1(G, \lambda), \frac{\lambda}{\tau}\right] - He^{-r \tau} N_{2}\left[d_2(H, \tau), d_2(G, \lambda), \frac{\lambda}{\tau}\right]\right] + \left\{S(t)e^{-\delta \tau} \left(\frac{G}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[g_1(G, H), -h_1(G), -\frac{\lambda}{\tau}\right] - He^{-r \tau} \left(\frac{G}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[g_2(G, H), -h_2(G), -\frac{\lambda}{\tau}\right]\right\}
\]

\[
UIP_t = \left[He^{-r \tau} N_{2}\left[d_2(H, \tau), d_2(G, \lambda), -\frac{\lambda}{\tau}\right] - S(t)e^{-\delta \tau} N_{2}\left[d_1(H, \tau), d_1(G, \lambda), -\frac{\lambda}{\tau}\right]\right] + \left\{He^{-r \tau} \left(\frac{G}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[-g_2(G, H), h_2(G), \frac{\lambda}{\tau}\right] - S(t)e^{-\delta \tau} \left(\frac{G}{S(t)}\right)^{1-2(\tau-\delta)\sigma^2} N_{2}\left[-g_1(G, H), h_1(G), \frac{\lambda}{\tau}\right]\right\}
\]

where \( \lambda = T_0 - t \), \( \tau = T - t \), and

\[
d_1(x, y) = \frac{\ln \left(\frac{S(t)}{x}\right) + (r - \delta + \frac{1}{2} \sigma^2) y}{\sigma \sqrt{y}}, \quad d_2(x, y) = d_1(x, y) - \sigma \sqrt{y}
\]

\[
g_1(x, y) = \frac{\ln \left(\frac{x^2}{yS(t)}\right) + (r - \delta + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}}, \quad g_2(x, y) = g_1(x, y) - \sigma \sqrt{\tau}
\]

\[
h_1(x) = \frac{\ln \left(\frac{x}{S(t)}\right) + (r - \delta + \frac{1}{2} \sigma^2) \lambda}{\sigma \sqrt{\lambda}}, \quad h_2(x) = h_1(x) - \sigma \sqrt{\lambda}
\]

By put-call parity, we have the following relations:

\[
DIC_t = C_t - DOC_t, \quad UOP_t = P_t - UIP_t, \quad UOC_t = C_t - UIC_t, \quad DOP_t = P_t - DIP_t
\]

where

\[
C_t = S(t)e^{-r \tau} N[d_1(K, \tau)] - Ke^{-r \tau} N[d_2(K, \tau)], \quad P_t = Ke^{-r \tau} N[-d_2(K, \tau)] - S(t)e^{-r \tau} N[-d_1(K, \tau)]
\]
We prove Theorem 2 in Appendix B.

Using the fact that
\[
\lim_{T_0 \to T} N_2 \left[ x, y, \sqrt{\frac{T_0 - t}{T - t}} \right] = \lim_{T_0 \to T} P \left[ \varepsilon_T \leq x, \varepsilon_{T_0} \leq y \right] = N \left[ \min(x, y) \right]
\]
where \( \varepsilon \) are standard normal random variables, we know that when \( T_0 \) approaches \( T \), the prices of the eight types of barrier options will become the barrier options with reset period being equal to time to maturity. Taking down-and-out call option as an example, when \( T_0 \) approaches \( T \), we have

\[
\lim_{T_0 \to T} DOC_i = S(t) e^{-\delta t} N[d_1(K, \tau)] - K e^{-\tau t} N[d_2(K, \tau)]
\]

\[
- S(t) e^{-\delta t} \left( \frac{D}{S(t)} \right)^{\frac{2(t-\delta)}{\sigma^2}} N[g_1(D, K)] + K e^{-\tau t} \left( \frac{D}{S(t)} \right)^{-\frac{2(t-\delta)}{\sigma^2}} N[g_2(D, K)]
\]

which is the same as the result of Rubinstein and Reiner [1991].

Now, consider the reset call option with \( m \) reset levels and continuous reset dates with reset period less than time to maturity. The payoff at expiry date \( T \) is as follows:

\[
C_T^{\text{reset}} = \max \left[ S(T) - K^*, 0 \right] = \left[ S(T) - K^* \right]^+
\]

\[
K^* = K_i \quad \text{if} \quad D_i \geq \min \left\{ S(s) : s \leq T \right\}, \quad \text{for} \quad i = 0, \ldots, m,
\]

or equivalently,

\[
C_T^{\text{reset}} = \sum_{i=0}^{m} \left[ S(T) - K_i \right]^+ \left\{ I \left( \min_{s \leq T_0} S(s) > D_{i+1} \right) - I \left( \min_{s \leq T_0} S(s) > D_i \right) \right\}
\]

where \( K_{i-1} \geq K_i \geq D_i, \quad i = 1, \ldots, m, \) and \( D_0 (\equiv \infty) > D_1 > D_2 \cdots > D_m > D_{m+1} (\equiv 0) \).

Correspondingly, The payoff at expiry date \( T \) of the reset put option with \( m \) reset levels and continuous reset dates with reset period less than time to maturity is as follows:

\[
P_T^{\text{reset}} = \max \left[ H^* - S(T), 0 \right] = \left[ H^* - S(T) \right]^+
\]

\[
H^* = H_i \quad \text{if} \quad G_{i+1} > \max \left\{ S(s) : s \leq T \right\}, \quad \text{for} \quad i = 0, \ldots, m.
\]
Similarly, we have
\[
P_{T}^{\text{Reset}} = \sum_{i=0}^{m} \left[ H_i - S(T) \right] \left\{ I \left( \max S(s) \geq G_i \right) - I \left( \max S(s) \geq G_{i+1} \right) \right\}
\]
where \( G_i \geq H_i \geq H_{i+1}, \ i = 1, \ldots, m \), and \( G_{m+1} (\equiv \infty) > G_m > \cdots > G_1 > G_0 (\equiv 0) \).

Therefore, we can replicate the reset call option with the following trading strategies:

1. Purchase one unit of European call option with strike price \( K_m \).
2. Purchase one unit of European down-and-out call option with strike price \( K_{i-1} \), barrier \( D_i \), \( i = 1, \ldots, m \), for each \( i \).
3. Short sell one unit of European down-and-out call option with strike price \( K_i \), barrier \( D_i \), \( i = 1, \ldots, m \), for each \( i \).

In the same way, we can replicate the reset put options with the following trading strategies:

1. Purchase one unit of European put option with strike price \( H_0 \).
2. Purchase one unit of European up-and-in put option with strike price \( H_i \), barrier \( G_i \), \( i = 1, \ldots, m \), for each \( i \).
3. Short sell one unit of European up-and-in put option with strike price \( H_{i-1} \), barrier \( G_i \), \( i = 1, \ldots, m \), for each \( i \).

Consequently, we have the following results:

**Theorem 3.** Let \( C_{T}^{\text{Reset}} \) and \( P_{T}^{\text{Reset}} \) denote the prices at time \( t \) of reset options with \( m \) reset levels and continuous reset dates with reset period less than time to maturity, respectively. \( G_i > S(t) > D_i \), \( K_{i-1} \geq K_i \geq D_i \) and \( G_i \geq H_i \geq H_{i-1} \) for \( i = 1, \ldots, m \). \( D_{i-1} > D_i \) and \( G_i > G_{i-1} \) for \( i = 2, \ldots, m \). The arbitrage-free prices of general reset options are

\[
C_{T}^{\text{Reset}} = S(t) e^{-\delta \tau} N[d_1(K_m, \tau)] - K_m e^{-\delta \tau} N[d_2(K_m, \tau)] + \sum_{i=1}^{m} \left( DOC_{i}^{i-1,i} - DOC_{i}^{i,i} \right)
\]

\[
P_{T}^{\text{Reset}} = H_0 e^{-\delta \tau} N[-d_2(H_0, \tau)] - S(t) e^{-\delta \tau} N[-d_1(H_0, \tau)] + \sum_{i=1}^{m} \left( UIP_{i}^{i,i} - UIP_{i}^{i-1,i} \right)
\]


where $DOC_{ij}$ refers to the down-and-out call with strike price $K_j$ and barrier level $D_i$. $UIP_{ij}$ refers to the up-and-in put with strike price $H_j$ and barrier level $G_i$. $DOC_{ij}$ and $UIP_{ij}$ are given by (7) and (8), respectively.

The reset call or put options with reset period less than time to maturity can serve as the upper bounds or control variates of the reset call or put options with discrete reset dates, especially when there are many discrete reset dates and the distance between any two adjacent reset dates is small.

II. NUMERICAL ANALYSES OF RESET OPTIONS

Characteristics of Reset Options with Continuous Reset Dates

First, we discuss the properties of reset call options with continuous reset dates. Consider a one-year-maturity reset call option with initial strike price 100. The strike price will be adjusted if the closing price of underlying stock falls below 80%, 70% and 60% of the initial strike price 100. We compare the prices of the reset options with 3 reset levels and reset period of one month and three months to the plain vanilla call option. The results are shown in Table I.

From Table I, some characteristics of reset call options are similar to the standard European call option. The values of reset call options are increasing functions of stock price, risk-free interest rate, and the volatility of stock returns. In addition, there are five properties that uniquely exist in reset call options: (1) The values of reset call options are increasing with the duration of reset period. (2) Under the same reset levels $D_j$, lower reset strike prices $K_j$ will have higher values for the reset call options. (3) Due to the more protection toward the holders of reset call options, the values of reset call options with continuous reset dates are always greater than the value of plain vanilla call option. (4) Because the reset call
options are increasing functions of the reset period, the reset options with continuous reset dates can serve as the upper bound or control variates for the reset call options with discrete reset dates. (5) In cases of higher values of stock price than reset levels and the lower volatility of stock returns, the price bounds for the reset call options with discrete reset dates will be smaller. For example, when the stock price, risk-free rate, volatility of stock returns and reset strike prices are 115, 0.05, 0.3, and (90, 80, 70) respectively, the price bound of reset call option with discrete reset dates at each day of the one-month monitoring period is [24.8642, 24.8643].

We now investigate the properties of reset put options with continuous reset dates. Consider a one-year-maturity reset put option with initial strike price 100. The strike price will be adjusted if the closing price of underlying stock rises above 115%, 120% and 125% of the initial strike price 100. We compare the prices of the reset options with 3 reset levels and reset period of one month and three months to the plain vanilla put option. The results are presented in Table II.

Table II shows the characteristics of reset put options are also similar to the plain vanilla European put option; namely, the values of reset put options are decreasing functions of stock price and risk-free rate and are increasing with the volatility of stock returns. Additionally, the reset put options have the following properties: (1) Similar to reset call options, the values of reset put options are increasing with the duration of reset period and are greater than the value of plain vanilla put option due to the more protection toward the holders. These values can serve as the upper bounds or control variates for the reset put options with discrete reset dates. (2) Under the same reset levels $G_j$, higher reset strike prices $H_j$ will induce higher values of the reset put options. (3) In cases of lower values of stock price than reset levels and the lower volatility of stock returns, the price bounds for the reset put options with discrete reset dates will be smaller. For example, when the stock price,
risk-free rate, the volatility of stock returns and reset strike prices are 85, 0.05, 0.3, and (105,110,115), respectively, the price bound for reset put option with discrete reset dates at each day of one-month monitoring period is [16.5400,16.5409].

**Delta Jump of Reset Options**

Since we can replicate the reset options with barrier options, similar to barrier options, the reset options also have the phenomenon of delta jump whenever the stock price touches the barriers. Using chain rule of differentiation we can derive the delta of reset options. The results are presented in the following theorem.

**Theorem 4.** The delta of reset call option and reset put option are as follows:

$$\frac{\partial C_{\text{Reset}}}{\partial S(t)} = e^{-\delta \tau} N[d_1(K_m, \tau)] + \sum_{i=1}^{m} \left( \frac{\partial \text{DOC}_{i-1,i}}{\partial S(t)} - \frac{\partial \text{DOC}_{i,i}}{\partial S(t)} \right)$$

$$\frac{\partial P_{\text{Reset}}}{\partial S(t)} = -e^{-\delta \tau} N[-d_1(H_0, \tau)] + \sum_{i=1}^{m} \left( \frac{\partial \text{UIP}_{i,i}}{\partial S(t)} - \frac{\partial \text{UIP}_{i-1,i}}{\partial S(t)} \right)$$

where

$$\frac{\partial \text{DOC}_{i,i}}{\partial S(t)} = e^{-\delta \tau} \left[ \sum_{j=1}^{2} \left( \frac{D_j}{S(t)} \right)^{2} A_j^{i,i} \left( \frac{\lambda}{\tau} \right) + B_j^{i,i} \left( \frac{\lambda}{\tau} \right) \right]$$

$$\frac{\partial \text{UIP}_{i,i}}{\partial S(t)} = -e^{-\delta \tau} \left[ \sum_{j=1}^{2} \left( \frac{G_j}{S(t)} \right)^{2} X_j^{i,i} \left( \frac{\lambda}{\tau} \right) + Y_j^{i,i} \left( \frac{\lambda}{\tau} \right) \right]$$

$$A_{1,2}^i(\rho) = -N_2 \left[ g_{1,2}(D_i, K_j), h_{1,2}(D_i), \rho \right] + \frac{\exp[-(g_{1,2}(D_i, K_j))^2]}{\sigma \sqrt{2 \pi \tau}} \left( \frac{h_{1,2}(D_i) - \rho g_{1,2}(D_i, K_j)}{\sqrt{1 - \rho^2}} \right) \right]$$
Delta jump can occur at any time within the reset period. If the stock price has touched the \( y^{th} \) barrier before time \( t \), the delta would become the following expressions:

\[
\frac{\partial C_{\text{Reset}}^t}{\partial S(t)} = e^{-\delta \tau} N[d_1(K_m, \tau)] + \sum_{j=y+1}^{m} \left( \frac{\partial DOC_{i,j}^{t-1,i}}{\partial S(t)} - \frac{\partial DOC_{i,j}^{t,i}}{\partial S(t)} \right)
\]

\[
\frac{\partial P_{\text{Reset}}^t}{\partial S(t)} = e^{-\delta \tau} N[-d_1(H_j, \tau)] + \sum_{j=y+1}^{m} \left( \frac{\partial UIP_{i,j}^{t-1,i}}{\partial S(t)} - \frac{\partial UIP_{i,j}^{t,i}}{\partial S(t)} \right)
\]

Because the strike prices of \( DOC_{i,j}^{t-1,i} \) and \( DOC_{i,j}^{t,i} \) or \( UIP_{i,j}^{t-1,i} \) and \( UIP_{i,j}^{t,i} \) are different, \( \frac{\partial DOC_{i,j}^{t-1,i}}{\partial S(t)} - \frac{\partial DOC_{i,j}^{t,i}}{\partial S(t)} \) and \( \frac{\partial UIP_{i,j}^{t-1,i}}{\partial S(t)} - \frac{\partial UIP_{i,j}^{t,i}}{\partial S(t)} \) will not be equal to zero. As a result, at time \( u \in (t, T] \), the delta will jump if the stock price touches the \( j^{th} \) barrier, \( j = y+1, \ldots, m \).

For example, consider two cases as follows: (1) One-year-maturity reset call option that strike price will be adjusted if the closing price of underlying stock falls below 87.5%, 75%, 62.5%, 50% and 37.5% of the initial strike price 80. The volatility of stock returns is 0.5. (2) One-year-maturity reset put option that strike price will be adjusted if the closing price of
underlying stock rises above 8/7, 10/7, 12/7, 14/7, and 16/7 of the initial strike price 35. The volatility of stock returns is 0.3. Both options have reset period three months. Their delta are plotted in Figure 1 and 2, respectively.

The phenomena of delta jumps exist in both cases. The delta jumps happen whenever the stock price touches the barriers. For reset call option, when the stock price is above the highest barrier, the behavior of delta is similar to the delta of plain vanilla call option. However, when the stock price falls and touches the barrier, due to the strike price being adjusted to a new lower level, the delta of reset call option will jump to a higher level. For a reset put option, when the stock price is below the lowest barrier, the behavior of delta is similar to the delta of plain vanilla put option. Nevertheless, when the stock price rises more close to the barrier, due to the fact that the strike price could be adjusted to a new higher level if the stock price actually touches the barrier, the reset put option is more valuable. The associated delta of reset put option will increase and even become positive. After the stock price actually touches the barrier, the strike price is adjusted to a new higher level. Consequently, the reset put option again behaves as the same as plain vanilla put option, and the delta jumps to a lower level.

For time $t \in [0, T_0)$, taking reset call option as an example, if the minimum stock price has touched the $j^{th}$ barrier, as time $t \rightarrow T_0$, the delta of call option at time $T_0$ is as follows:

$$\lim_{t \rightarrow T_0} \frac{\partial C_{\text{Reset}}}{\partial S(t)} = \lim_{t \rightarrow T_0} \left[ e^{-\delta \tau} N[d_1(K_m, \tau)] + \sum_{i=1}^{m} \left( \frac{\partial DOC_{i^{-1},j}}{\partial S(t)} - \frac{\partial DOC_{i,j}}{\partial S(t)} \right) \right] = e^{-\delta \tau} N[d_1(K_j, \tau)]$$

However, the delta at time $t \geq T_0$ is

$$e^{-\delta \tau} N[d_1(K_j, \tau)] I(S(T_0) \geq D_{j+1}) + \sum_{i=1}^{m} e^{-\delta \tau} N[d_1(K_i, \tau)] I[D_i > S(T_0) \geq D_{j+1}]$$

Comparing (11) and (12), we see that except the case that stock price at time $T_0$ is higher
than the \((y+1)^{th}\) barrier, the delta at time \(t \geq T_0\) is different from the delta at time before \(T_0\). As a result, it is obvious that the delta will jump in the reset period \([0, T_0]\) whenever the stock price touches the lower barriers of reset call option\(^2\).

**Numerical Results of The Prices of Arithmetic Average Reset Options**

In practice, the reset options use arithmetic average of stock prices. However, due to the problem that the sum of lognormal distributions is not lognormal, exact closed-form solutions of arithmetic average reset options do not exist. For this reason, we use the reset options with continuous reset period as control variates for Monte Carlo simulations to obtain the fair prices of arithmetic average reset options.

For the Monte Carlo simulation, we take reset call option as an example. We partition the interval \([t, T]\) into \(n\) subintervals of equal length \(h\) and let \(T_j = t + jh\) be the partition points, \(j = 1, \ldots, n\). We then have

\[
    \ln S(t_j) = \ln S_t + (r - \delta - \frac{1}{2}\sigma^2)(t_j - t) + \sigma \sqrt{h} \left( \sum_{j=1}^{j} e_i \right), \quad j = 1, \ldots, n
\]

where \(e_j, \; j = 1, \ldots, n\), are identical and independent standard normal random variables. For the \(g^{th}\) simulated path, \(g = 1, \ldots, M\), we record the minimum value of stock prices during the reset period. If the minimum value is below the \(l^{th}\) reset level for \(l = 1, \ldots, m\), then the value of control variate, partial reset call option, at time \(t \in [0, T]\) is

\[
    C_t^{Control}(g) = \exp[-r(T - t)] \text{Max}[S(T) - K_t, 0]
\]

For the value of arithmetic average reset call option, suppose that there are \(x\) days in one year, and for each day, there are \(y\) partition points. We have \(n = xy(T - t)\).\(^3\) Then, the arithmetic average value of stock prices is computed based on the partition point \(T_j = t + jh\), \(j = y, 2y, \ldots, n\). Similarly, for the \(g^{th}\) simulated path, we record the minimum arithmetic
average value of stock prices during the reset period. If the minimum arithmetic average value is below the \( f^{th} \) reset level for \( f = 1, \ldots, m \), then the value of arithmetic average reset call option at time \( t \in [0, T] \) is

\[
C_t^{\text{Reset}}(g) = \exp[-r(T-t)] \max[S(T) - K_f, 0]
\]

As a result, the value of arithmetic average reset call option with control variate is

\[
\hat{C}_t^{\text{Reset}} = \frac{1}{M} \sum_{g=1}^{M} \left[ C_t^{\text{Reset}}(g) - C_t^{\text{Control}}(g) \right] + C_t^{\text{Reset}}
\]

where \( C_t^{\text{Reset}} \) is computed based on (10). We can also use antithetic variates that is first proposed by Boyle [1977] in option pricing by creating a perfectly negatively correlated asset. Then, we have \( 2M \) paths to compute the option value.

Based on the above method, our empirical products are the six arithmetic average reset options listed on TSE from 1998 to 1999. The reset conditions of these six arithmetic average reset options are shown in Table III. From the Table, we see that the number of discrete reset dates is large (from thirty days to ninety days) and the time interval between any two adjacent reset dates is only one day. Consequently, these kinds of reset options with many discrete reset dates can use the reset options with continuous reset period as control variates in Monte Carlo simulations to value their fair prices with smaller standard deviations.

During 1998 and 1999, we use the RP (Repurchase Agreement) rates as the proxies of risk-free interest rates, which are between 4% and 5%. The numerical results from 10,000 simulations are summarized in Table IV. From the Table, we can see that the standard deviations of 10,000 simulations are small by using the control variates. Meanwhile, it is clear that the markups of reset options in Taiwan are between 10% and 20%. The high markup may result from transaction costs and regulation restrictions (e.g., it is not allowed to issue put options in Taiwan.)
Ⅲ. CONCLUSION

Using martingale method, we provide closed-form solutions for the eight types of partial barrier options and reset options with continuous reset dates. We also show some special characteristics of reset call and put options. Both of reset call and put options are increasing with the duration of reset period, greater than the values of plain vanilla options due to the more protection toward the holders, and can serve as the upper bounds for the reset call and put options with discrete reset dates.

We also provide the delta of reset call and put options and investigate the phenomena of delta jumps existing in reset call and put options during all the reset period whenever the stock price touches the barriers.

Finally, the closed-form solutions of reset options with continuous reset period can be used as control variates in Monte Carlo simulations to compute the fair prices of arithmetic average reset options. For numerical analyses, we use the six arithmetic average reset options listed on TSE from 1998 to 1999. The simulation results show that the markups of reset options issued in practice are between 10% and 20%. The high markup may be caused by transaction costs and regulation restrictions.
TABLE I

Prices of Plain Vanilla Call Option and Reset Calls with 3 Reset levels and Reset Periods of One Month and Three Months

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$S(t)$</th>
<th>$r = 0.05$</th>
<th>$r = 0.07$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(K_1, K_2, K_3)$</td>
<td>One Month</td>
<td>Three Months</td>
</tr>
<tr>
<td>30%</td>
<td>0.85</td>
<td>6.4171</td>
<td>8.6997</td>
</tr>
<tr>
<td></td>
<td>$(80,70,60)$</td>
<td>9.7863</td>
<td>11.8476</td>
</tr>
<tr>
<td></td>
<td>$(85,75,65)$</td>
<td>7.7947</td>
<td>8.7552</td>
</tr>
<tr>
<td></td>
<td>$(90,80,70)$</td>
<td>14.2978</td>
<td>15.1891</td>
</tr>
<tr>
<td></td>
<td>$(80,70,60)$</td>
<td>14.2761</td>
<td>14.8774</td>
</tr>
<tr>
<td></td>
<td>$(85,75,65)$</td>
<td>24.8643</td>
<td>24.9671</td>
</tr>
<tr>
<td></td>
<td>$(90,80,70)$</td>
<td>24.8643</td>
<td>24.9051</td>
</tr>
<tr>
<td>50%</td>
<td>0.85</td>
<td>13.1563</td>
<td>17.0589</td>
</tr>
<tr>
<td></td>
<td>$(80,70,60)$</td>
<td>18.5189</td>
<td>21.0248</td>
</tr>
<tr>
<td></td>
<td>$(85,75,65)$</td>
<td>15.7593</td>
<td>19.0633</td>
</tr>
<tr>
<td></td>
<td>$(90,80,70)$</td>
<td>22.7227</td>
<td>25.2422</td>
</tr>
<tr>
<td></td>
<td>$(80,70,60)$</td>
<td>22.4564</td>
<td>24.3382</td>
</tr>
<tr>
<td></td>
<td>$(85,75,65)$</td>
<td>22.2197</td>
<td>23.5390</td>
</tr>
<tr>
<td></td>
<td>$(90,80,70)$</td>
<td>32.2106</td>
<td>33.4457</td>
</tr>
<tr>
<td>115%</td>
<td>0.85</td>
<td>32.1191</td>
<td>32.1840</td>
</tr>
<tr>
<td></td>
<td>$(80,70,60)$</td>
<td>32.1602</td>
<td>32.7688</td>
</tr>
<tr>
<td></td>
<td>$(85,75,65)$</td>
<td>32.2137</td>
<td>33.8162</td>
</tr>
</tbody>
</table>

Here, $\delta = 0, K_0 = 100, D_1 = 80, D_2 = 70, D_3 = 60, r = 0, T = 1$. Some properties uniquely exist in reset call options:

1. The values of reset call options are increasing with the duration of reset period.
2. Under the same reset levels $D_j$, lower reset strike prices $K_j$ have higher values for the reset call options.
3. Due to the more protection toward the holders of reset call options, the values of reset call options with continuous reset dates are always greater than the value of plain vanilla call option.
4. Because the reset call options are increasing functions of the reset period, the reset options with continuous reset dates and plain vanilla call option can serve as the upper and lower bounds for the reset call option with discrete reset dates, respectively.
5. In cases of higher values of stock price than reset levels and the lower volatility of stock returns, the price bounds of the reset call options with discrete reset dates will be smaller.
### Table II

Prices of Plain Vanilla Put Option and Reset Puts with 3 Reset levels and Reset Periods of One Month and Three Months

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( S(t) )</th>
<th>((H_1,H_2,H_3))</th>
<th>( T_0 )</th>
<th>( \sigma )</th>
<th>( S(t) )</th>
<th>((K_1,K_2,K_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>One Month</td>
<td>Three Months</td>
<td>( 0 )</td>
<td>One Month</td>
<td>Three Months</td>
<td></td>
</tr>
<tr>
<td>85</td>
<td>(105,110,115)</td>
<td>16.5400</td>
<td>16.6618</td>
<td>85</td>
<td>(105,110,115)</td>
<td>15.0754</td>
</tr>
<tr>
<td></td>
<td>(110,115,120)</td>
<td>16.5419</td>
<td>16.7554</td>
<td></td>
<td>(110,115,120)</td>
<td>15.2431</td>
</tr>
<tr>
<td></td>
<td>(115,120,125)</td>
<td>16.5430</td>
<td>16.8630</td>
<td></td>
<td>(115,120,125)</td>
<td>15.2460</td>
</tr>
</tbody>
</table>

| 30% | 100 | 9.6038 | 10.5116 |
| | (105,110,115) | 9.8220 | 11.2897 |
| | (110,115,120) | 10.0714 | 12.1746 |
| | (115,120,125) | 10.4177 | 11.8191 |

| 50% | 100 | 18.3119 | 19.8668 |
| | (105,110,115) | 19.1299 | 21.3226 |
| | (110,115,120) | 20.0331 | 20.6667 |

Here, \( \delta = 0, H_0 = 100, G_1 = 115, G_2 = 120, G_3 = 125, t = 0, T = 1 \). The properties of reset put options are: (1) The values of reset put options are increasing with the duration of reset period, greater than the value of plain vanilla put options due to the more protection toward the holders and can serve as the upper bounds for the reset put options with discrete reset dates. (2) Under the same reset levels \( G_j \), higher reset strike prices \( H_j \) will induce higher values for the reset put options. (3) In cases of lower values of stock price than reset levels and the lower volatility of stock returns, the price bounds for the reset put options with discrete reset dates will be smaller.
TABLE III

Reset Conditions of Six Reset Options on the Taiwan Stock Exchange

<table>
<thead>
<tr>
<th>Code in TSE</th>
<th>The Reset Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0517</td>
<td>The strike price would be adjusted if the six-day average closing price of 2323 on the TSE fell below 90% of initial strike price $58.5 during the first three months after the warrant was issued.</td>
</tr>
<tr>
<td>0522</td>
<td>The strike price would be adjusted if the six-day average closing price of 2323 on the TSE fell below 98%, 96%, 94%, 92%, 90% of initial strike price $81 during the first three months after the warrant was issued.</td>
</tr>
<tr>
<td>0523</td>
<td>The strike price would be adjusted if the six-day average closing price of 2303 on the TSE fell below 80% of initial strike price $57 during the first three months after the warrant was issued.</td>
</tr>
<tr>
<td>0527</td>
<td>The strike price would be adjusted if the six-day average closing price of 2311 on the TSE fell below 80% of initial strike price $95.5 during the first three months after the warrant was issued.</td>
</tr>
<tr>
<td>0528</td>
<td>The strike price would be adjusted if the six-day average closing price of 2373 on the TSE fell below 95%, 90%, 85 of initial strike price $58.5 during the first three months after the warrant was issued.</td>
</tr>
<tr>
<td>0538</td>
<td>The strike price would be adjusted if the six-day average closing price of 2303 on the TSE fell below 85% of initial strike price $81 during the first one month after the warrant was issued.</td>
</tr>
</tbody>
</table>

This table shows the terms of six reset options listed on TSE. These reset options use arithmetic average of stock prices as trigger and have multiple reset levels.
TABLE IV

Numerical Results of Six Arithmetic Average Reset Options

<table>
<thead>
<tr>
<th>Code in TSE</th>
<th>Issued Date</th>
<th>Issued Price</th>
<th>Historical Volatility</th>
<th>$r = 0.04$</th>
<th>$r = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Fair Price (Std)</td>
<td>Markup</td>
</tr>
<tr>
<td>0517</td>
<td>1998/10/22</td>
<td>16.43</td>
<td>50.36%</td>
<td>13.8718(0.0079)</td>
<td>17.17 %</td>
</tr>
<tr>
<td>0522</td>
<td>1999/04/28</td>
<td>20.25</td>
<td>43.51%</td>
<td>17.6036(0.0055)</td>
<td>15.03 %</td>
</tr>
<tr>
<td>0523</td>
<td>1999/05/27</td>
<td>13.68</td>
<td>45.93%</td>
<td>12.3613(0.0116)</td>
<td>10.48 %</td>
</tr>
<tr>
<td>0527</td>
<td>1999/06/09</td>
<td>24.83</td>
<td>49.07%</td>
<td>21.9984(0.0209)</td>
<td>12.97 %</td>
</tr>
<tr>
<td>0528</td>
<td>1999/06/14</td>
<td>15.21</td>
<td>45.13%</td>
<td>13.3833(0.0065)</td>
<td>13.64 %</td>
</tr>
<tr>
<td>0538</td>
<td>1999/10/20</td>
<td>21.87</td>
<td>49.46%</td>
<td>17.9045(0.0144)</td>
<td>21.96 %</td>
</tr>
</tbody>
</table>

The numerical results are based on the data of the six reset options exhibited in Table II. Std is the standard deviation of 10,000 Monte Carlo simulations. The time to maturity is partitioned into 720 subintervals of equal length. The Markup is counted by $(\text{Issued Price} - \text{Fair Price}) / \text{Fair Price} \times 100\%$. From this table, it is clear that the markups of reset options in Taiwan are between 10% to 20%. The high markup may result from the transaction costs and regulation restrictions (in Taiwan, it is not allowed to issue put options to date.)
Here, $\delta = 0, K_0 = 80, [K_1, \ldots, K_5] = [70,60,50,40,30], [D_1, \ldots, D_5] = [70,60,50,40,30], \ r = 0.05, \ T = 1, \ \nu = 0.5$. The reset period is three months.
FIGURE 2
Delta of Reset Put with Five Reset Levels and Three-Months Reset Period

Here, $\delta = 0$, $H_0 = 35$, $[H_1, \ldots, H_5] = [40, 50, 60, 70, 80]$, $[D_1, \ldots, D_3] = [40, 50, 60, 70, 80]$, $r = 0.05$, $T = 1$, $\nu = 0.5$.

The reset period is three months.
APPENDIX A

Proof of Theorem 1

Assume \( X(t) = ut + \sigma W_t \), \( b \geq a \), \( b \geq c \), \( b \geq 0 \) and \( T > t \), where \( W_t \) stands for the one-dimensional standard Brownian motion on a filtered probability space \((\Omega, F, P, (F_t)_{t=0}^T)\), where the filtration \((F_t)_{t=0}^T\) is generated by \( W = (W_t)_{t=0}^T \). Define \( Z_t = \frac{X_t}{\sigma} = vt + W_t \) and \( v = \frac{u}{\sigma} \). Accordingly, we have

\[
P(Z_T \leq a, Z_i \leq c, M_i^Z \geq b) = E_p(I_D)
\]

where \( M_i^Z = \max_{0\leq s \leq i} Z_s \), and \( D = \{ Z_T \leq a, Z_i \leq c, M_i^Z \geq b \} \). Let \( P^* \) be the probability measure equivalent to \( P \) such that its Radon-Nikodym derivative is

\[
\frac{dP^*}{dP} = \exp(-vW_T - \frac{1}{2}v^2T) = \exp(-vZ_T + \frac{1}{2}v^2T)
\]

By Girsanov’s theorem, the process \( W_i^* \), defined by

\[
W_i^* = W_t + vt = Z_t
\]

follows a standard Brownian motion under probability measure \( P^* \). From the reflection principle of Brownian motion, we have

\[
P(Z_T \leq a, Z_i \leq c, M_i^Z \geq b) = E_{p^*}\left[\exp(vZ_t - \frac{1}{2}v^2t)I_D\right]
\]

\[
= \exp(2bv)E_{p^*}\left[\exp(-vZ_t - \frac{1}{2}v^2t)I(Z_T \geq 2b - a, Z_i \geq 2b - c, M_i^Z \geq b)\right]
\]

Since \( b \geq c \), then

\[
\{ M_i^Z \geq b \} \supset \{ Z_i \geq b \} \supset \{ Z_i \geq 2b - c \}
\]

Consequently,

\[
P(Z_T \leq a, Z_i \leq c, M_i^Z \geq b)
\]
\[
= \exp(2bv)E_{P_R}\left[ \exp\left(-vZ_t - \frac{1}{2}v^2t\right)I(Z_t \geq 2b - a, Z_t \geq 2b - c) \right]
\]

Let the probability measure \( P_R \) on \((\Omega, F)\) be defined by the Radon-Nikodym derivative

\[
\frac{dP_R}{dP} = \exp(-vZ_t - \frac{1}{2}v^2T). \text{ Then, the process } W_t^R, \text{ defined by } dW_t^R = dW_t^* + vdP_t, \text{ is a standard Brownian motion under probability measure } P_R. \text{ Hence, } Z_t = W_t^R - vt, \text{ and we obtain}
\]

\[
P(Z_t \leq a, Z_t \leq c, M^Z_t \geq b) = \exp(2bv)P_R(Z_t \geq 2b - a, Z_t \geq 2b - c)
= \exp(2bv)N_2\left[ \frac{a - 2b - vt}{\sqrt{T}}, \frac{c - 2b - vt}{\sqrt{t}}, \sqrt{\frac{T}{t}} \right]
\]

In the same way,

\[
P(Z_t \geq a, Z_t \leq c, M^Z_t \geq b) = \exp(2bv)N_2\left[ \frac{-a + 2b + vt}{\sqrt{T}}, \frac{c - 2b - vt}{\sqrt{t}}, \sqrt{\frac{T}{t}} \right]
\]

where \( N_2(\cdot, \cdot, \rho) \) is the cumulative bivariate normal distribution with mean vector 0 and instantaneous correlation coefficient \( \rho \). Let \( \hat{Z}_t = -Z_t \) and \( m^Z_t = \text{Min} Z_s, \text{ then}
\]

\[
P(X_t \geq x, X_t \geq c', m^X_t \leq y)
= P(\hat{Z}_t \leq -\frac{x}{\sigma}, \hat{Z}_t \leq -\frac{c'}{\sigma}, M^Z_t \geq -\frac{y}{\sigma}) = \exp\left(\frac{2u\gamma}{\sigma^2}N_2\left[ \frac{2\gamma}{\sigma}, \frac{2\gamma + ut - c'}{\sigma}, \sqrt{\frac{T}{t}} \right] \right)
\]

As a result, we have

\[
P(X_t \geq x, X_t \geq c', m^X_t \geq y) = P(X_t \geq x, X_t \geq c') - P(X_t \geq x, X_t \geq c', m^X_t \leq y)
= N_2\left[ \frac{uT - x}{\sigma\sqrt{T}}, \frac{uT - c' - x}{\sigma\sqrt{t}}, \sqrt{\frac{T}{t}} \right] - \exp\left(\frac{2u\gamma}{\sigma^2}N_2\left[ \frac{2\gamma + ut - x}{\sigma}, \frac{2\gamma + ut - c'}{\sigma}, \sqrt{\frac{T}{t}} \right] \right)
\]

Because

\[
P(X_t \geq x, X_t \leq y, m^X_t \geq y) = P(X_t \geq x, X_t \leq y) - P(X_t \geq x, X_t \leq y, m^X_t \leq y)
= P(X_t \geq x, X_t \leq y) - P(X_t \geq x, X_t \leq y) = 0,
\]

it follows that
\[
P(X_T \geq x, m_i^X \geq y) = P(X_T \geq x, X_i \geq y, m_i^X \geq y)
\]
\[
= N_2 \left[ \frac{uT - x}{\sigma \sqrt{T}}, \frac{ut - y}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right] - \exp \left( \frac{2uv}{\sigma^2} \right) N_2 \left[ \frac{2y + uT - x}{\sigma \sqrt{T}}, \frac{y + ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right]
\]

This completes the proof of (3).

To prove (6), first, note that
\[
P(Z_T \leq a, Z_i \geq b, M_i^Z \geq b) = P(Z_T \leq a, Z_i \geq b) = P(vT + W_T \leq a, vt + W_i \geq b)
\]
\[
= N_2 \left[ \frac{a - vT}{\sqrt{T}}, \frac{-b + vt}{\sqrt{t}}, \sqrt{\frac{t}{T}} \right]
\]

Therefore, we have the following expression:
\[
P(X_T \leq a, M_i^X \geq b) = P(Z_T \leq \frac{a}{\sigma}, M_i^Z \geq \frac{b}{\sigma})
\]
\[
= P(Z_T \leq \frac{a}{\sigma}, Z_i \geq \frac{b}{\sigma}, M_i^Z \geq \frac{b}{\sigma}) + P(Z_T \leq \frac{a}{\sigma}, Z_i \leq \frac{b}{\sigma}, M_i^Z \geq \frac{b}{\sigma})
\]
\[
= N_2 \left[ \frac{a - uT}{\sigma \sqrt{T}}, \frac{-b + ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right] + \exp \left( \frac{2bu}{\sigma^2} \right) N_2 \left[ \frac{a - 2b - uT}{\sigma \sqrt{T}}, \frac{-b - ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right]
\]

Similarly,
\[
P(X_T \geq a, M_i^X \geq b) = P(Z_T \geq \frac{a}{\sigma}, M_i^Z \geq \frac{b}{\sigma})
\]
\[
= P(Z_T \geq \frac{a}{\sigma}, Z_i \geq \frac{b}{\sigma}, M_i^Z \geq \frac{b}{\sigma}) + P(Z_T \geq \frac{a}{\sigma}, Z_i \leq \frac{b}{\sigma}, M_i^Z \geq \frac{b}{\sigma})
\]
\[
= N_2 \left[ \frac{-a + uT}{\sigma \sqrt{T}}, \frac{-b + ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right] + \exp \left( \frac{2bu}{\sigma^2} \right) N_2 \left[ \frac{-a + 2b + uT}{\sigma \sqrt{T}}, \frac{-b - ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right] \quad (A.1)
\]

Then, using (A.1), we obtain
\[
P(X_T \leq x, m_i^X \leq y) = P(Z_T \leq \frac{x}{\sigma}, M_i^Z \geq \frac{y}{\sigma}) = P(-Z_T \geq -\frac{x}{\sigma}, -m_i^Z \geq \frac{y}{\sigma})
\]
\[
= N_2 \left[ \frac{x - uT}{\sigma \sqrt{T}}, \frac{y - ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right] + \exp \left( \frac{2uy}{\sigma^2} \right) N_2 \left[ \frac{x - 2y - uT}{\sigma \sqrt{T}}, \frac{y + ut}{\sigma \sqrt{t}}, \sqrt{\frac{t}{T}} \right]
\]

This completes the proof of Theorem 1.
APPENDIX B

Proof of Theorem 2

Under the risk-neutral probability measure $Q$, the arbitrage-free price of down-and-out call option is as follows:

$$
DOC_t = e^{-rt} E^Q \left\{ \left[ S(T) - K \right]^+ I\left( \min_{t \leq s \leq T} S(s) > D \right) \right\} F_t \\
= e^{-rt} E^Q \left\{ S(T) I_D \right\} F_t - K e^{-rt} E^Q \left\{ I_D \right\} F_t
$$

where $D = \left\{ S(T) \geq K, \min_{t \leq s \leq T} S(s) > D \right\}$ and $F_t$ is specified as in Appendix A. Consequently, we have

$$
e^{-rt} E^Q \left\{ S(T) I_D \right\} F_t = S(t) e^{-\delta t} E^Q \left\{ \exp \left( \sigma (W^*_t - W^*_i) - \frac{1}{2} \sigma^2 (T-t) \right) I_D \right\} F_t
$$

Let the probability measure $Q^G$ be defined by Radon-Nikodym derivative

$$
\frac{dP^G}{dQ^G} = \exp(v W^*_t - \frac{1}{2} v^2 T). \text{ Then, } W^*_G, \text{ defined by}
$$

$$
d W^*_G = d W^*_t - v d t
$$

is a standard Brownian motion under probability measure $P^G$. Therefore,

$$
e^{-rt} E^Q \left\{ S(T) I_D \right\} F_t = S(t) e^{-\delta t} E^P \left\{ I_D \right\} F_t
$$

$$
= S(t) e^{-\delta t} P^G \left( S(T) \geq K, \min_{t \leq s \leq T} S(s) > D \right| F_t \right)
$$

$$
= S(t) e^{-\delta t} P^G \left( S(t) e^{X_T - X_t} \geq K, \min_{t \leq s \leq T} S(s) e^{X_s - X_t} > D \right| F_t \right)
$$

$$
= S(t) e^{-\delta t} P^G \left( (X_T - X_t) \geq \ln \left( \frac{K}{S(t)} \right), \min_{t \leq s \leq T} (X_s - X_t) > \ln \left( \frac{D}{S(t)} \right) \right| F_t \right)
$$

where $X_t = (r - \delta + \frac{1}{2} \sigma^2) t + \sigma W^*_G$ and both $X_T - X_t$ and $\min_{t \leq s \leq T} (X_s - X_t)$ are independent of
the $\sigma$–field $F_t$. Consequently,

$$e^{-rt} E_Q \{S(T)I_D \mid F_t\} = S(t)e^{-\delta r} P_g \left[ X_r \geq \ln \left( \frac{K}{S(t)} \right), m^X > \ln \left( \frac{D}{S(t)} \right) \right]$$

Using (3), we can obtain

$$I_1 = e^{-rt} E_Q \{S(T)I_D \mid F_t\} = S(t)e^{-\delta r} \left\{ N_2 \left[ d_1(K, \tau), d_1(D, \lambda), \sqrt{\frac{\lambda}{\tau}} \right] - \left( \frac{D}{S(t)} \right)^{\frac{2(\epsilon-\delta)}{\sigma^2}} N_2 \left[ g_1(D, K), h_1(D), \sqrt{\frac{\lambda}{\tau}} \right] \right\}$$

Correspondingly, we also have

$$I_2 = Ke^{-rT} E_Q \{I_D \mid F_t\} = Ke^{-rT} \left\{ N_2 \left[ d_2(K, \tau), d_2(D, \lambda), \sqrt{\frac{\lambda}{\tau}} \right] - \left( \frac{D}{S(t)} \right)^{\frac{2(\epsilon-\delta)}{\sigma^2}} N_2 \left[ g_1(D, K), h_1(D), \sqrt{\frac{\lambda}{\tau}} \right] \right\}$$

As a result, $DOC_t = I_1 - I_2$. Then, following the same procedure by using (4) to (6), we can derive the prices of $DIP_t, UIC_t$ and $UIP_t$.

The put-call parity of $DOC_t$ and $DIC_t$ is as follows:

$$DOC_t = e^{-rt} E_Q \left\{ [S(T) - K]^+ I(\text{Min}_{t \leq r} S(s) > D) \mid F_t\right\}$$

$$= e^{-rt} E_Q \left\{ [S(T) - K]^+ [1 - I(\text{Min}_{t \leq r} S(s) \leq D)] \mid F_t\right\}$$

$$= e^{-rt} E_Q \left\{ [S(T) - K]^+ \mid F_t\right\} - e^{-rt} E_Q \left\{ [S(T) - K]^+ I(\text{Min}_{t \leq r} S(s) \leq D) \mid F_t\right\}$$

$$= C_t - DIC_t$$

Similarly, we have

$$UOP_t = P_t - UIP_t, \quad UOC_t = C_t - UIC_t, \quad DOP_t = P_t - DIP_t$$

where $C_t$ and $P_t$ are defined in (9). This completes the proof of Theorem 2.
ENDNOTES

1. Heynen and Kat [1994] used integral expression and Hui [1997] used P.D.E. approach to deriving the closed-form solutions of partial barrier options; however, they did not derive the prices of reset options. This paper provides a systematic way by using a simpler evaluation approach (martingale method) and offers insights into the stochastic properties of the price process of underlying asset.

2. Similar phenomenon also happens for the reset put option at time $T_\nu$.

3. Here, we can adjust those parameters to make $\nu$ integers.

4. Alternatively, we could use implied volatilities to compute the six arithmetic average reset option on TSE. However, there were no ordinary options traded for these securities. Thus, we only use historical volatilities to compute the prices of these reset options.
REFERENCES


