KALMAN FILTER AND PREDICTOR FORMULATION FOR PREDICTING THE PRICE SIGNAL AND THE OPTIMAL SPACING INTERVAL

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摘 要

本文目的是想藉由卡門濾淨與預測方法，來預測價格變動訊號，並提出一個一般化之預測公式。此外，我們也將探討預測之訊息增益與取樣區間長短之關係，由此，推導出最適之取樣區間。

Abstract

The purpose of this note is intended to propose a general formula for predicting the price signal by way of Kalman filter and predictor algorithm. Additionally, we will also explore the relationship between the predictor gain and the time interval of spacing samples, and thereby derive the optimal spacing interval.

1. Introduction

The advantages behind the Kalman filter and predictor formulation are achieved by the fact that this approach takes both the process noise and the measurement noise into account and presents the estimator and predictor equations in the form of recursive stochastic dynamics. The details of the approach can be referred to Bozic (1979) or Jazwinski (1970). Here, we shall apply this method to derive a general formula for predicting the future price signal.

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2. Model Formulation

We assume a general (average) price level being surveyed is at range P + ρ_t at time t, and at range P + ρ_{t+1} at time t + 1, T interval later. We use T to represent the spacing between samples made one survey apart. The general price level is denoted by P, and ρ_t, ρ_{t+1} represent deviations from the average. We are interested in estimating these deviations, which are assumed to be statistically random with zero-mean value.

To a first approximation, if the price signal is changing at velocity $\dot{\rho}_t$ and T is not too long

(1) $\rho_{t+1} = \rho_t + T \cdot \dot{\rho}_t$

which is the range equation surveyed for the price signal at time t and t + 1.

Considering the acceleration of the change in the price signal, we have

(2) $T \cdot u_t = \dot{\rho}_{t+1} - \dot{\rho}_t$

which is the acceleration equation. Assume that the acceleration u_t is a zero-mean, stationary white noise process, i.e., E $[u_t]$ = 0, E $[u_{t+1}\ u_t]$ = 0, but it has some known variance E $[u_t^2] = \sigma_u^2$. Such accelerations might be caused by sudden, irregular, or nonsystematic factors of aggregate demand and supply. The variable $w_t = T \cdot u_t$ is also a white noise process, and we have

(3) $\dot{\rho}_{t+1} = \dot{\rho}_t + w_t$

Defining a two component signal vector X_t with one component the range, $x_{1t} = \rho_t$, and the other component the velocity, $x_{2t} = \dot{\rho}_t$, and applying these to eqs. (1) and (3), we obtain

(4) $$\underbrace{\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 \\ w_t \end{bmatrix}}_{W_t}$$

or, in matrix form

(5) $\quad X_{t+1} = A \cdot X_t + W_t$

Assume that in estimating the signal vector X_t, we make noisy measurement at time t. We have the following data equation measured by $\rho_t{}^m$

(6) $\quad \rho_t{}^m = \rho_t + v_t$

The additive noise, v_t, is usually assumed to be Gaussian with zero-mean and variance $\sigma_v{}^2$. The next step is to formulate noise covariance matrices for the system, and for the measurement equation. For the latter, we have

(7) $\quad R_t = E\ (v_t \ \ v_t{}^T) = \sigma_v{}^2$

and the system noise covariance matrix is given by

(8) $$Q_t = E\ (W_t\ W_t{}^T) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\omega{}^2 \end{bmatrix}$$

where $\sigma_\omega{}^2 = E\ (w^2)$.

3. Kalman Filter and Predictor for the Model

Applying Kalman filter and predictor formula to eqs. (5) and (6), we obtain

(9) $$\hat{X}_{t+1\,|\,t} = A \cdot \hat{X}_{t\,|\,t-1} + G_t \cdot \left[\begin{bmatrix} \rho_t{}^m \\ \dot{\rho}_t{}^m \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{\rho}_{t\,|\,t-1} \\ \hat{\dot{\rho}}_{t\,|\,t-1} \end{bmatrix} \right]$$

(10) $$G_t = A \cdot MSE_{t\,|\,t-1} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \cdot \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot MSE_{t\,|\,t-1} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} \sigma_v{}^2 & 0 \\ 0 & 0 \end{bmatrix} \right]^{-1}$$

(11) $MSE_{t|t-1} = A \cdot MSE_{t-1|t-1} \cdot A^T + Q_{t-1}$

where MSE denotes the mean square prediction error covariance matrix.

To start Kalman processing we have to initialize the Kalman gain matrix G_t. For this purpose the mean square error covariance matrix MSE_t has to be specified. A reasonable ad hoc initialization can be established using measurement at times $t = 1$ and $t = 2$.

From measurement data we can make the following estimates:

$$(12) \quad \hat{X}_2 = \begin{bmatrix} \hat{x}_{12} = \hat{\rho}_2 = \rho_2^m \\ \hat{x}_{22} = \hat{\dot{\rho}}_2 = (\rho_2^m - \rho_1^m) / T \end{bmatrix}$$

To calculate $MSE_{2|2}$, we use the general expression

$$(13) \quad MSE_{2|2} = E\{[x_2 - \hat{x}_2] \cdot [x_2 - \hat{x}_2]^T\}$$

$$= E \begin{bmatrix} -v_2 \\ w_1 - (v_2 - v_1)/T \end{bmatrix} \cdot [-v_2 \quad w_1 - (v_2 - v_1) / T]$$

$$= \begin{bmatrix} \sigma_v^2 & \sigma_v^2 / T \\ \sigma_v^2 / T & \sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix}$$

where we assume the independence of noise sources w and v, and also the independence between individual noise samples across time.

Since we have this error matrix at $t = 2$, we could use it to calculate the predictor gain G_3 at $t = 3$, which is given by

$$(14) \quad G_3 = A \cdot MSE_{3|2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \cdot \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot MSE_{3|2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & 0 \end{bmatrix} \right]^{-1}$$

$$(15) \quad MSE_{3|2} = A \cdot MSE_{2|2} \cdot A^T + Q_2$$

$$= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_v^2 & \sigma_v^2 / T \\ \sigma_v^2 / T & \sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} T^2 \cdot \sigma_\omega^2 + 5\sigma_v^2 & T \cdot \sigma_\omega^2 + 3\sigma_v^2 / T \\ T \cdot \sigma_\omega^2 + 3\sigma_v^2 / T & 2\sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix}$$

thus,

$$(14) \quad G_3 = \frac{1}{\triangle} \cdot \begin{bmatrix} T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2 & 0 \\ T^3 \cdot \sigma_\omega^4 + 9T \cdot \sigma_v^2 \cdot \sigma\omega^2 + 6\sigma_v^4 / T & \triangle \end{bmatrix}$$

where $\triangle = T^2 \cdot \sigma_\omega^4 + 8\sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T^2$

Substituting the predictor gain G_3 into eq. (9), we obtain the predictor of the price signal at $t = 4$

$$(16)\quad \hat{\rho}_{4|3} = \hat{\rho}_{3|2} + T\cdot\hat{\dot{\rho}}_{3|2} + \frac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4 / T^2}{T^2\cdot\sigma_\omega^4 + 8\sigma_v^2\cdot\sigma_\omega^2 + 3\sigma_v^4 / T^2} \cdot(\rho_3^m - \hat{\rho}_{3|2})$$

$$(17)\quad \hat{\dot{\rho}}_{4|3} = \hat{\dot{\rho}}_{3|2}$$

The next step is to find the predictor at t = 5

$$(18)\quad \hat{\rho}_{5|4} = \hat{\rho}_{4|3} + T\cdot\hat{\dot{\rho}}_{4|3} + G_4\cdot(\rho_4^m - \hat{\rho}_{4|3})$$

$$(18)\quad \hat{\dot{\rho}}_{5|4} = \hat{\dot{\rho}}_{4|3}$$

where

$$(19)\quad G_4 = A\cdot MSE_{4|3}\cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \cdot \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot MSE_{4|3}\cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T + R_4 \right]^{-1}$$

$$(20)\quad MSE_{4|3} = A\cdot MSE_{3|3}\cdot A^T + Q_3$$

$$(21)\quad MSE_{3|3} = MSE_{3|2} - K_3\cdot MSE_{3|2}$$

$$(22)\quad K_3 = A^{-1}\cdot G_3$$

After some iterated substitutions and calculations, we obtain the predictor gain G_4

$$(23)\quad G_4 = \begin{bmatrix} \dfrac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4 / T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4 / T^2} & T \\ 0 & 1 \end{bmatrix}$$

Substituting G_4 into eq. (18), we have

$$(24)\quad \hat{\rho}_{5\,|\,4} = \hat{\rho}_{4\,|\,3} + T\cdot\hat{\dot{\rho}}_{4\,|\,3} + \frac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4 / T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/T^2}\cdot(\rho_4^m - \hat{\rho}_{4\,|\,3})$$

The process is then repeated recursively by the same way as stated above. Finally, we could derive a general predictor formula for ρ_{t+1} basing on measurement up to time t as follows:

$$(25)\quad \hat{\rho}_{t+1\,|\,t} = \hat{\rho}_{t\,|\,t-1} + T\cdot\hat{\dot{\rho}}_{t\,|\,t-1} + G_t\cdot(\rho_t^m - \hat{\rho}_{t\,|\,t-1})$$

which can be expressed in an alternative form after recursive substitutions

$$\begin{aligned}(26)\quad \hat{\rho}_{t+1\,|\,t} = {} & \Big[(1 - G_t)\cdot(1 - G_{t-1})\cdot(1 - G_{t-2})\cdot(1 - G_{t-3}) \\ & \ldots\Big]_{G_S = 0,\ S < 3}\cdot\hat{\rho}_{3\,|\,2} + \Big[1 + (1 - G_t) + \\ & + (1 - G_t)\cdot(1 - G_{t-1}) + (1 - G_t)\cdot(1 - G_{t-1}) \\ & \cdot(1 - G_{t-2}) + \ldots\Big]_{G_s = 0,\ S < 4}\cdot T\cdot\hat{\dot{\rho}}_{3\,|\,2} + \\ & \Big[G_t + (1 - G_t)\cdot G_{t-1}\cdot L + (1 - G_t)\cdot(1 - G_{t-1}) \\ & \cdot G_{t-2}\cdot L^2 + \ldots\Big]_{G_S = 0,\ S < 3}\cdot\rho_t^m,\ t > 3\end{aligned}$$

where

$$(27)\quad G_t = \frac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4 / T^2}{(t - 2)T^2\cdot\sigma_\omega^4 + (6t - 10)\cdot\sigma_v^2\cdot\sigma_\omega^2 + t\cdot\sigma_v^4 / T^2}$$

Eq. (26) shows that the predictor of the price signal at any time t + 1 basing on observed information at time t can be quite conveniently determined by a particular form of a sequence of previous Kalman gains Gs associated with previous observed price deviations $L^t\cdot\rho_t^m$, the initial predictors of the price signal $\hat{\rho}_{3\,|\,2}$ and its' velocity $T\cdot\hat{\dot{\rho}}_{3\,|\,2}$.

The predictor gain G_t will become larger (smaller) as we take the spacing intervals between samples longer (shorter). This can be verified by

$$(28)\quad \frac{\partial G_t}{\partial T} = \frac{4T\cdot\sigma_v^2\cdot\sigma_\omega^6 + (8\sigma_v^4\cdot\sigma_\omega^4 / T) + 20\sigma_v^6\sigma_\omega^2 / T^3}{[(t-2)T^2\cdot\sigma_\omega^4 + (6t-10)\cdot\sigma_v^2\cdot\sigma_\omega^2 + t\cdot\sigma_v^4 / T^2]^2}$$

$$> 0$$

We can also find the optimal spacing interval T* which could make the Kalman gain maximum due to the fact that $\partial^2 G_t/\partial T^2 < 0$ for some large t. T* can be achieved by letting eq. (28) be equal to zero.

$$(29)\quad T^* = \pm\frac{\sigma_v}{\sigma_\omega}\cdot(-1 \pm 2i)^{1/2}$$

Eq. (29) shows that the magnitude (or modulus) of the optimal spacing interval is determined by

$$(30)\quad | T^* | = \frac{\sigma_v}{\sigma_\omega}\cdot 5^{1/4}$$

It shows that, given σ_v, the optimal spacing interval becomes smaller (larger) as the standard deviation of the process noise gets larger (smaller), and, given σ_ω, the optimal spacing interval would become smaller (larger) as the standard deviation of the measurement noise goes smaller (larger).

4. Conclusions

We have succeeded in deriving a general predictor formula for predicting the future price signal at any time t + 1. Additionally, we also have explored the relationship between the predictor gain and the spacing interval, and thereby have obtained the optimal spacing interval.

The predictor formula we derived could be usefully applied to expect the future prices signal of those goods in which both the acceleration of the change in the price signal and the measurement noise of the price signal obey the i. i. d. stationary white noise processes with zero-mean and constant variance.

References

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Mathematical Appendix

1. The Computation of G_3:

$$G_3 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} T^2\cdot\sigma_\omega^2 + 5\sigma_v^2 & T\cdot\sigma_\omega^2 + 3\sigma_v^2 / T \\ T\cdot\sigma_\omega^2 + 3\sigma_v^2 / T & 2\sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix} \cdot \begin{bmatrix} T^2\cdot\sigma_\omega^2 + 6\sigma_v^2 & T\cdot\sigma_\omega^2 + 3\sigma_v^2 / T \\ T\cdot\sigma_\omega^2 + 3\sigma_v^2 / T & 2\sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix}^{-1}$$

$$= \frac{1}{\triangle} \cdot \begin{bmatrix} 2T^2\cdot\sigma_\omega^2 + 8\sigma_v^2 & 3T\cdot\sigma_\omega^2 + 5\sigma_v^2 / T \\ T\cdot\sigma_\omega^2 + 3\sigma_v^2 / T & 2\sigma_\omega^2 + 2\sigma_v^2 / T^2 \end{bmatrix} \cdot \begin{bmatrix} 2\cdot\sigma_\omega^2 + 2\sigma_v^2 / T^2 & -T\cdot\sigma_\omega^2 - 3\sigma_v^2 / T \\ -T\cdot\sigma_\omega^2 - 3\sigma_v^2 / T & T^2\cdot\sigma_\omega^2 + 6\sigma_v^2 \end{bmatrix}$$

$$= \frac{1}{\triangle} \cdot \begin{bmatrix} \begin{matrix} (2\sigma_\omega^2 + 8\sigma_v^2)\cdot(2\sigma_\omega^2 + 2\sigma_v^2 / T^2) + \\ (3T\cdot\sigma_\omega^2 + 5\sigma_v^2/T)\cdot(-T\cdot\sigma_\omega^2 - 3\sigma_v^2 / T) \end{matrix} & \begin{matrix} (2T^2\cdot\sigma_\omega^2 +.8\sigma_v^2)\cdot(-T\cdot\sigma_\omega^2 - 3\sigma_v^2 / T) \\ + (3T\cdot\sigma_\omega^2 + 5\sigma_v^2 / T)\cdot(T^2\cdot\sigma_\omega^2 + 6\sigma_v^2) \end{matrix} \\ 0 & \triangle \end{bmatrix}$$

$$= \begin{bmatrix} (T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4 / T^2) / \triangle & (T^3\cdot\sigma_\omega^4 + 9T\cdot\sigma_v^2\cdot\sigma_\omega^2 + 6\sigma_v^4 / T) / \triangle \\ 0 & 1 \end{bmatrix}$$

where $\triangle = T^2 \cdot \sigma_\omega^4 + 8\sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T^2$

2. The Computation of $MSE_{3|3}$:

$$MSE_{3|3} = (I - A^{-1} \cdot G_3) \cdot MSE_{3|2}$$

where

$$A^{-1} \cdot G_3 = K_3 = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2) / \triangle & (T^3 \cdot \sigma_\omega^4 + 9T\sigma_v^2 \cdot \sigma_\omega^2 + 6\sigma_v^4 / T) / \triangle \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2) / \triangle & (T \cdot \sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4/T)/\triangle \\ 0 & 1 \end{bmatrix}$$

and

$$I - A^{-1} \cdot G_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} (T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2) / \triangle & (T \cdot \sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T) / \triangle \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2\sigma_v^2 \cdot \sigma_\omega^2 + 2\sigma_v^4 / T^2) / \triangle & -(T \cdot \sigma_v^2 \cdot \sigma\omega^2 + 3\sigma_v^4 / T) / \triangle \\ 0 & 0 \end{bmatrix}$$

Thus,

$$MSE_{3|3} = (I - A^{-1} \cdot G_3) \cdot MSE_{3|2}$$

$$= \begin{bmatrix} (2\sigma_v^2 \cdot \sigma\omega^2 + 2\sigma_v^4 / T^2) / \triangle & -(T \cdot \sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T) / \triangle \\ 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} T^2 \cdot \sigma_\omega^2 + 5\sigma_V^2 & T \cdot \sigma_\omega^2 + 3\sigma_V^2 / T \\ T \cdot \sigma_\omega^2 + 3\sigma_V^2 / T & 2\sigma_\omega^2 + 2\sigma_V^2 / T^2 \end{bmatrix}$$

$$= \begin{bmatrix} (2\sigma^2 \cdot \sigma_\omega^2 + 2\sigma_V^4/T^2) \cdot (T^2 \cdot \sigma_\omega^2 + 5\sigma_V^2) / \triangle & 0 \\ -(T \cdot \sigma_V^2 \cdot \sigma_\omega^2 + 3\sigma_V^4/T) \cdot (T \cdot \sigma_\omega^2 + 3\sigma_V^2 / T) / \triangle & \\ & \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (T^2 \cdot \sigma_V^2 \cdot \sigma_\omega^4 + 6\sigma_V^4 \cdot \sigma_\omega^2 + \sigma_V^6 / T^2) / \triangle & 0 \\ 0 & 0 \end{bmatrix}$$

3. The Computation of $MSE_{4\,|\,3}$:

$$MSE_{4\,|\,3} = A \cdot MSE_{3\,|\,3} \cdot A^T + Q_3$$

$$= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (T^2 \cdot \sigma_V^2 \cdot \sigma_\omega^4 + 6\sigma_V^4 \cdot \sigma_\omega^2 + \sigma_V^6 / T^2) / \triangle & 0 \\ 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} (T^2 \cdot \sigma_V^2 \cdot \sigma_\omega^4 + 6\sigma_V^4 \cdot \sigma_\omega^2 + \sigma_V^6 / T^2) / \triangle & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

4. The Computation of G_4:

$$G_4 = A \cdot MSE_{4\,|\,3} \cdot I^T \cdot (I \cdot MSE_{4\,|\,3} \cdot I^T + R_4)^{-1}$$

where

$$(MSE_{4\,|\,3} + R_4)^{-1}$$

$$= \begin{bmatrix} (T^2\cdot\sigma_v^2\cdot\sigma_\omega^2 + 6\sigma_v^4\cdot\sigma_\omega^2 + \sigma_v^6/\ T^2)\ /\ \triangle & 0 \\ +\ (T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 8\sigma_v^4\cdot\sigma_\omega^2 + 3\sigma_v^6/\ T^2)\ /\ \triangle & \\ 0 & \sigma_\omega^2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{\triangle}{2T^2\cdot\sigma_v^2\cdot\sigma_\omega^2 + 14\sigma_v^4\cdot\sigma_\omega^2 + 4\sigma_v^6/\ T^2} & 0 \\ 0 & 1/\sigma_\omega^2 \end{bmatrix}$$

and

$$A\cdot MSE_{4\mid 3} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 6\sigma_v^4\cdot\sigma_\omega^2 + \sigma_v^6/T^2)/\triangle & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} (T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 6\sigma_v^4\cdot\sigma_\omega^2 + \sigma_v^6/T^2)/\triangle & T\cdot\sigma_\omega^2 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

Thus,

$$G_4 = \begin{bmatrix} \dfrac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4/\ T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/\ T^2} & T \\ 0 & 1 \end{bmatrix}$$

5. Ths Computation of K_4:

$$K_4 = A^{-1}\cdot G_4$$

$$= \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4/\ T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/\ T^2} & T \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2}{2T^2 \cdot \sigma_\omega^4 + 14\sigma_v^2 \cdot \sigma_\omega^2 + 4\sigma_v^4 / T^2} & 0 \\ 0 & 1 \end{bmatrix}$$

$I - A^{-1} \cdot G_4$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \dfrac{T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2}{2T^2 \cdot \sigma_\omega^4 + 14\sigma_v^2 \cdot \sigma_\omega^2 + 4\sigma_v^4 / T^2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{T^2 \cdot \sigma_\omega^4 + 8\sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T^2}{2T^2 \cdot \sigma_\omega^4 + 14\sigma_v^2 \cdot \sigma_\omega^2 + 4\sigma_v^4 / T^2} & 0 \\ 0 & 0 \end{bmatrix}$$

6. The Computation of G_5:

$MSE_{4|4} = (I - A^{-1} \cdot G_4) \cdot MSE_{4|3}$

$$= \begin{bmatrix} \dfrac{T^2 \cdot \sigma_\omega^4 + 8\sigma_v^2 \cdot \sigma_\omega^2 + 3\sigma_v^4 / T^2}{2T^2 \cdot \sigma_\omega^4 + 14\sigma_v^2 \cdot \sigma_\omega^2 + 4\sigma_v^4 / T^2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (T^2 \cdot \sigma_v^2 \cdot \sigma_\omega^4 + 6\sigma_v^4 \cdot \sigma_\omega^2 + \sigma_v^6 / T^2) / \triangle & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{T^2 \cdot \sigma_v^2 \cdot \sigma_\omega^4 + 6\sigma_v^4 \cdot \sigma_\omega^2 + \sigma_v^6 / T^2}{2T^2 \cdot \sigma_\omega^4 + 14\sigma_v^2 \cdot \sigma_\omega^2 + 4\sigma_v^4 / T^2} & 0 \\ 0 & 0 \end{bmatrix}$$

Thus,

$$MSE_{5|4} = \begin{bmatrix} \dfrac{T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 6\sigma_v^4\cdot\sigma_\omega^2 + \sigma_v^6/\ T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/\ T^2} & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$$A\cdot MSE_{5|4} = \begin{bmatrix} \dfrac{T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 6\sigma_v^4\cdot\sigma_\omega^2 + \sigma_v^6/\ T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/\ T^2} & T\cdot\sigma_\omega^2 \\ 0 & \sigma_\omega^2 \end{bmatrix}$$

$(MSE_{5|4} + R_5)^{-1}$

$$= \begin{bmatrix} \dfrac{3T^2\cdot\sigma_v^2\cdot\sigma_\omega^4 + 20\sigma_v^4\cdot\sigma_\omega^2 + 5\sigma_v^6/\ T^2}{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4/\ T^2} & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{2T^2\cdot\sigma_\omega^4 + 14\sigma_v^2\cdot\sigma_\omega^2 + 4\sigma_v^4\ /\ T^2}{3T^2\cdot\ \sigma_v^2\cdot\sigma_\omega^4 + 20\sigma_v^4\cdot\sigma_\omega^2 + 5\sigma_v^6\ /\ T^2} & 0 \\ 0 & 1/\sigma_\omega^2 \end{bmatrix}$$

Thus,

$$G_5 = A\cdot MSE_{5|4}\cdot(MSE_{5|4} + R_5)^{-1}$$

$$= \begin{bmatrix} \dfrac{T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 + \sigma_v^4/\ T^2}{3T^2\cdot\sigma_\omega^4 + 20\sigma_v^2\cdot\sigma_\omega^2 + 5\sigma_v^4/\ T^2} & T \\ 0 & 1 \end{bmatrix}$$

7. Generalization of G_t:

$$G_t = \begin{bmatrix} \dfrac{T^2 \cdot \sigma_\omega^4 + 6\sigma_v^2 \cdot \sigma_\omega^2 + \sigma_v^4 / T^2}{(t - 2) \cdot T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t\sigma_v^4/T^2} & T \\ 0 & 1 \end{bmatrix}$$

8. The computation of future expectations of the price signal

$$\begin{aligned}
\rho_{6|5} &= \rho_{5|4} + T \cdot \dot{\rho}_{5|4} + G_5 \cdot (\rho_5^m - \rho_{5|4}) \\
&= (1 - G_5) \cdot \rho_{5|4} + T \cdot \dot{\rho}_{5|4} + G_5 \cdot \rho_5^m \\
&= (1 - G_5) \cdot [\rho_{4|3} + T \cdot \dot{\rho}_{4|3} + G_4 \cdot (\rho_4^m - \rho_{4|3})] + \\
&\quad T \cdot \dot{\rho}_{5|4} + G_5 \cdot \rho_5^m \\
&= (1 - G_5) \cdot (1 - G_4) \cdot \rho_{4|3} + (1 - G_5) \cdot T \cdot \dot{\rho}_{4|3} + \\
&\quad (1 - G_5) \cdot G_4 \cdot \rho_4^m + T\dot{\rho}_{5|4} + G_5 \cdot \rho_5^m \\
&= (1 - G_5) \cdot (1 - G_4) \cdot [\rho_{3|2} + T \cdot \dot{\rho}_{3|2} + G_3 \cdot (\rho_3^m - \rho_{3|2})] + \\
&\quad (1 - G_5) \cdot T \cdot \dot{\rho}_{4|3} + (1 - G_5) \cdot G_4 \cdot \rho_4^m + T\dot{\rho}_{5|4} + G_5 \cdot \rho_5^m \\
&= (1 - G_5) \cdot (1 - G_4) \cdot (1 - G_3) \cdot \rho_{3|2} + (1 - G_5) \cdot \\
&\quad (1 - G_4) \cdot T \cdot \dot{\rho}_{3|2} + (1 - G_5) \cdot (1 - G_4) \cdot G_3 \cdot \rho_3^m + \\
&\quad (1 - G_5) \cdot T \cdot \dot{\rho}_{4|3} + (1 - G_5) \cdot G_4 \cdot \rho_4^m + T \cdot \dot{\rho}_{5|4} + G_5 \cdot \rho_5^m
\end{aligned}$$

Thus,

$$\rho_{t+1|t} = [(1 - G_t)\cdot(1 - G_{t-1})\cdot(1 - G_{t-2})\cdot(1 - G_{t-3}) \ldots]_{G_S = 0,\ S<3}$$
$$\cdot \rho_{3|2} + [T + (1 - G_t)\cdot T + (1 - G_t)\cdot(1 - G_{t-1})\cdot T +$$
$$(1 - G_t)\cdot(1 - G_{t-1})\cdot(1 - G_{t-2})\cdot T + \ldots]_{G_S = 0,\ S<4}$$
$$\cdot \rho_{3|2} + [G_t + (1 - G_t)\cdot G_{t-1}\cdot L + (1 - G_t)\cdot(1 - G_{t-1})\cdot$$
$$G_{t-2}\cdot L^2 + \ldots]_{G_S = 0,\ S<3} \cdot \rho_t^m$$

9. The derivatives of G_t:

$$\frac{\partial G_t}{\partial T} = \frac{[(t-2)\,T^2\cdot\sigma_\omega^4 + (6t-10)\cdot\sigma_v^2\cdot\sigma_\omega^2 + t\cdot\sigma_v^4 / T^2]\cdot}{[(t-2)\,T^2\cdot\sigma_\omega^4 + (6t-10)\cdot\sigma_v^2\cdot\sigma_\omega^2 + t\cdot\sigma_v^4 / T^2]^2}$$

$$\frac{[2T\cdot\sigma_\omega^4 - 2\cdot\sigma_v^4 / T^3] - [T^2\cdot\sigma_\omega^4 + 6\sigma_v^2\cdot\sigma_\omega^2 +}{}$$

$$\frac{\sigma_v^4/T^2]\cdot[2\,(t-2)\,T\cdot\sigma_\omega^4 - 2t\cdot\sigma_v^4 / T^3]}{}$$

$$= \frac{1}{\triangle^2}\cdot\Bigg[\frac{t}{T^2}\cdot\sigma_v^4\cdot\frac{-2}{T^3}\cdot\sigma_v^4 + (6t-10)\cdot\sigma_v^2\cdot\sigma_\omega^2\cdot\frac{-2}{T^3}\cdot\sigma_v^4$$

$$+ (t-2)\,T^2\cdot\sigma_\omega^4\cdot\frac{-2}{T^3}\cdot\sigma_v^4 + \frac{1}{T^2}\cdot\sigma_v^4\cdot\frac{2t}{T^3}\cdot\sigma_v^4 + 6\sigma_v^2.$$

$$\sigma_\omega^2\cdot\frac{2t}{T^3}\cdot\sigma_v^4 + T^2\cdot\sigma_\omega^4\cdot\frac{2t}{T^3}\cdot\sigma_v^4 + \frac{t}{T^2}\cdot\sigma_\omega^4\cdot 2T\cdot\sigma_v^4 +$$

$$(6t-10)\cdot\sigma_v^2\cdot\sigma_\omega^2\cdot 2T\cdot\sigma_\omega^4 + (t-2)\,T^2\cdot\sigma_\omega^4\cdot 2T\cdot\sigma_\omega^4$$

$$- \frac{1}{T^2}\cdot\sigma_\omega^4\cdot 2\,(t-2)\,T\cdot\sigma_v^4 - 6\sigma_v^2\cdot\sigma_\omega^2\cdot 2(t-2)\,T\cdot\sigma_\omega^4$$

$$- T^2\cdot\sigma_\omega^4\cdot 2\,(t-2)\,T\cdot\sigma_\omega^4\Bigg] > 0$$

$$= \frac{1}{\triangle^2} \cdot \left[\frac{20}{T^3} \cdot \sigma_v^6 \cdot \sigma_\omega^2 + \frac{8}{T} \cdot \sigma_v^4 \cdot \sigma_\omega^4 + 4T \cdot \sigma_v^2 \cdot \sigma_\omega^6 \right] > 0$$

$$= \frac{(20\sigma_v^6 \cdot \sigma_\omega^2/T^3) + (8\sigma_v^4 \cdot \sigma_\omega^4 / T) + 4T \cdot \sigma_v^2 \cdot \sigma_\omega^6}{[(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]^2} > 0$$

$$\frac{\partial^2 G_t}{\partial T^2} = \frac{[(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]^2 \cdot}{[(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]^4}$$

$$\frac{[(-60\sigma_v^6 \cdot \sigma_\omega^2 / T^4) - (8\sigma_v^4 \cdot \sigma_\omega^4 / T^2) + 4T \cdot \sigma_v^2 \cdot \sigma_\omega^6]}{} +$$

$$\frac{[(-20\sigma_v^6 \cdot \sigma_\omega^2 / T^3) - (8\sigma_v^4 \cdot \sigma_\omega^4 / T) - 4T \cdot \sigma_v^2 \cdot \sigma_\omega^6] \cdot 2}{[(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]^4}$$

$$\frac{\cdot [(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]}{}$$

$$\frac{\cdot [2(t - 2) T \cdot \sigma_\omega^4 - 2t \cdot \sigma_v^4 / T^3]}{}$$

$$= \frac{1}{[(t - 2) T^2 \cdot \sigma_\omega^4 + (6t - 10) \cdot \sigma_v^2 \cdot \sigma_\omega^2 + t \cdot \sigma_v^4 / T^2]^3}$$

$$\{[20t \cdot \sigma_v^{10} \cdot \sigma_\omega^2 / T^6] - 12 (t - 2) T^2 \cdot \sigma_v^2 \cdot \sigma_\omega^{10} + [(-336t$$

$$+ 600) \cdot \sigma_v^8 \cdot \sigma_\omega^4 / T^4] + (-16t + 40) \cdot \sigma_v^4 \cdot \sigma_\omega^8 + [(-148t$$

$$+ 360) \cdot \sigma_v^6 \cdot \sigma_\omega^6 / T^2]\} < 0, \text{ for some large } t$$

10. The derivation of optimal T*:

Setting $\partial G_t/\partial T$ equal to zero, we obtain

$$5\sigma_v^4 + 2T^2 \cdot \sigma_v^2 \cdot \sigma_\omega^2 + T^4 \cdot \sigma_\omega^4 = 0$$

Let $T^2 = x$, we get

$$X^2 + 2\ (\sigma_v^2/\sigma_\omega^2) \cdot x + 5\ (\sigma_v^4/\sigma_\omega^4) = 0$$

Thus,

$$x = -(\sigma_v^2/\sigma_\omega^2) \cdot [1 \pm 2 \cdot (-1)^{1/2}]$$

$$|\ x\ | = (\sigma_v^2/\sigma_\omega^2) \cdot 5^{1/2}$$

That is:

$$|\ T\ | = \frac{\sigma_v}{\sigma_\omega} \cdot 5^{1/4}$$