考試科目	數學	所別	統計系	考試時間	4月26日 星期日 上下午第 2 節

(共兩頁)

1. (14%) Evaluate the following limits if they exist:

   (a) $\lim_{x \to 0} \dfrac{\sin^4 x}{x^2(1-\cos x)}$

   (b) $\lim_{n \to \infty} a_n$ if $a_1 = 1$ and $a_n = a_{n-1} + \cos a_{n-1}, n = 2, 3, 4, \ldots$

2. (16%) Evaluate the following integrals:

   (a) Calculate the region bounded by $y = x^2$ and $y = x + 2$.

   (b) Calculate $\int_0^{\infty} \int_0^{x} \dfrac{1}{(1+x^2+y^2)^2} dy dx$.

3. (12%) Evaluate the following calculations:

   (a) $\dfrac{d}{dx}\left(\int_1^{x^2} \dfrac{dt}{t}\right)$

   (b) Calculate $f''(x)$, if $f(x) = \int_2^{3x} \sin^2 t dt$.

4. (8%) Show that the following functions do not have a limit at $(0, 0)$:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

5. (10%) Find the minimum and maximum values of $f(x, y) = (x-4)^2 + y^2$, taken on by the function on the set $D = \{(x, y) : 0 \le x \le 2, x^3 \le y \le 4x\}$.

考試科目	數學	所別	統計系	考試時間	7月26日 星期日 上下午第 2 節

(共兩頁)

6. (10%) Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that

$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n.$$

Similarly, find the inverse of $I - M_n$, where M_n is the $n \times n$ matrix each of whose entries is k.

7. (10%) Use Cramer's Rule to solve

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

8. (10%) Show that the following set of vectors is a basis for $M_{22} = \{$the set of all real 2 by 2 matrices$\}$:

$$A_1 = \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Also, find the coordinate vector of A to the basis of $S = \{A_1, A_2, A_3, A_4\}$ if

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}.$$

9. (10%) Find the eigenvalues and eigenvectors of A, A^{-1}, and $A + 3I$, if

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

考試科目	統計學	所別	統計學系	考試時間	4月26日 星期日 下午 第 1 節

1. Both Mary and John each have a fair coin. Suppose Mary tosses her coin until she gets heads and that John does the same with his coin.

(5 分) (a) Find the probability that Mary and John make the same number of tosses.

(5 分) (b) Find the probability that Mary makes more tosses than John.

2. Let the joint density function for X and Y be given by

$$f(x,y)=\begin{cases}2\exp[-(x+y)] & \text{for } 0<y<x<\infty \\ 0 & \text{elsewhere}\end{cases}$$

(5 分) (a) Find the marginal density of Y.

(5 分) (b) Find $E(X\mid Y=y)$.

(5 分) (c) Find the joint moment generating function of X and Y.

(5 分) (d) Find the density function for $U=X+Y$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a folded normal distribution with density function

$$f(x\mid\theta)=\begin{cases}\sqrt{\frac{2}{\pi\theta}}\exp(\frac{-x^2}{2\theta}), & \text{if } x>0 \\ 0, & \text{otherwise}\end{cases}$$

where $\theta\in(0,\infty)=\Theta$ is the unknown parameter.

(10 分) (a) Find the maximum likelihood estimator (MLE) of θ.

(5 分) (b) Show that the MLE is an unbiased estimator of θ.

(5 分) (c) Is the MLE the best unbiased estimator of θ ? Justify your answer.

4. Let $X_i, i=1,\ldots,n$ be independent, identically distributed Exponential (θ) random variables, that is, each X_i has the density

$$f(x\mid\theta)=\theta e^{-x\theta}, \quad 0<x<\infty,$$

Assume a prior distribution for θ

$$\pi(\theta)=\alpha e^{-\alpha\theta}, \quad 0<\theta<\infty,$$

where α is a unknown positive constant.

(10 分) (a) Find the posterior density for θ.

(5 分) (b) Find the posterior mean for θ.

(5 分) (c) Find the predictive density for a new observation X_{n+1}, (the distribution of X_{n+1} given $X_1,\ldots,X_n$, integrating out θ, i.e. $f(X_{n+1}\mid X_1,\ldots,X_n)$).

考試科目	統計學	所別	統計學系	考試時間	4月26日 星期日 (上)下午第	節

5. This year the Taiwan Legislature passed a law requiring insurance for all drivers Prior to this event drivers did not have to covered by insurance. The main issue was the number of Taiwan motorists who would be forced by the law into buying insurance. There were a total of 2,800,000 license plates for passenger vehicles registered in Taiwan at the time. An investigation of each one of these to determine whether they had insurance coverage would be prohibitively expensive and time-consuming. It was decided that the government would draw a random sample of motorists and estimate the number of Taiwan's driving population who were uninsured from the sample data. A random sample of 200 license plates was drawn using statistically sound sampling methods. Each was investigated to determine its insurance status. The license plates sampled were placed into one of three categories. The categories and the data are as follows. (License plates that were drawn for the sample but where investigations were unable to find the car or owner were classified as missing).

Insured	Not insured	Missing
180	10	10

(5 分) (a) Decide for yourself how to deal with the missing vehicles.

(10 分) (b) Construct a 95% confidence interval estimate of the proportion of all Taiwan passenger vehicles that are not insured. ($z_{0.025} = 1.96$)

6. A customer bought a bottle of orange juice that labeled its net weight as 400 ± 10 ml at Seven-Eleven store. He went home and found that this particular bottle of orange juice does not have its net weight as 400 ml as it claimed. He felt that he was cheated so he complained to the Consumer Right Protection (消基會). The 消基會 immediately took an action on this and came to you, intelligent Statistician, asking you to investigate on this matter. Of course, you took the case. Now how will you go about, as an intelligent and experienced Statistician, to solve the case and report back to the 消基會.

(5 分) (a) For samples collecting, write down how many and how to collect them.

(10 分) (b) For statistical inference, you need to write down

(1) appropriate hypotheses (H_0 & H_A),

(2) test statistics,