考試科目	數理統計學	系所別	統計	考試時間	月 日 星期 上午 下午 第 節

1. (15%) Suppose there are two multivariate normal populations, say, $\mathbf{\Pi}_1$ that is $N_p(\boldsymbol{\mu}_1, \Sigma)$ and $\mathbf{\Pi}_2$ that is $N_p(\boldsymbol{\mu}_2, \Sigma)$, where p is the number of variables. Suppose a new observation vector $\boldsymbol{x}$ is known to come from either $\mathbf{\Pi}_1$ or $\mathbf{\Pi}_2$. Let

$$U = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \Sigma^{-1} \boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T \Sigma^{-1} (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2).$$

Please verify the distribution of U.

2. (20%) Let $X_1, X_2, \cdots, X_{2n}$ be iid $N(0,1)$ rv's. Define

$$\begin{aligned} U_n &= \left\{ \frac{X_1}{X_2} + \frac{X_3}{X_4} + \cdots + \frac{X_{2n-1}}{X_{2n}} \right\}, \\ V_n &= X_1^2 + X_2^2 + \cdots + X_n^2 , \end{aligned}$$

and

$$Z_n = \frac{U_n}{V_n} .$$

Please find the limiting distribution of Z_n.

3. (25%) From a box containing N identical balls marked 1 through N, M balls are drawn one after another without replacement. Let X_i denote the number on the ith ball drawn, $i = 1, 2, \cdots, M$, $1 \le M \le N$. Suppose that we wish to estimate N on the basis of observations $X_1, X_2, \cdots, X_M$.

(1) Find the UMVUE of N.

(2) Find the MLE of N.

(3) Compare the MSE's of the UMVUE and the MLE.

4. (20%) Let $X_1, X_2, \cdots, X_n$ be a sample of size n from $U(0, \theta)$, $\theta > 0$, Show that

$$\phi_1(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if } \max(x_1, x_2, \cdots, x_n) > \theta_0 \\ \alpha & \text{if } \max(x_1, x_2, \cdots, x_n) \le \theta_0 \end{cases}$$

is a uniformly most powerful (UMP) test of size α for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ and that the test

$$\phi_2(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if } \max(x_1, x_2, \cdots, x_n) > \theta_0 \text{ or} \\ & \quad \max(x_1, x_2, \cdots, x_n) \leq \theta_0 \alpha^{1/n} \\ \alpha & \text{otherwise} \end{cases} .$$

is UMP size α for testing $H_0' : \theta = \theta_0$ against $H_1' : \theta \neq \theta_0$.

5. (20%) Let X takes on the specified values $v_1, \cdots, v_{k+1}$ with probabilities $\theta_1, \cdots, \theta_{k+1}$ respectively. Suppose that $X_1, X_2, \cdots, X_n$ are independently and identically distributed as X. Suppose that $\theta = (\theta_1, \cdots, \theta_{k+1})$ is unknown and may range over the set $\Theta = \{(\theta_1, \cdots, \theta_{k+1}) : \theta_i \geq 0, 1 \leq i \leq k+1, \sum_{i=1}^{k+1} \theta_i = 1\}$. Let N_j be the number of X_i which equal v_j.

(1) Show that $\boldsymbol{N} = (N_1, N_2, \cdots, N_k)$ is sufficient for θ.

(2) What is the distribution of $(N_1, N_2, \cdots, N_{k+1})$?

考試科目	線性模式	系所別	統計學系	考試時間	6 月 2[illegible] 日 星期二 (三) 下午第[illegible]節

1. Consider a linear model

$$Y = X\beta + e,$$

where the X is a known $n \times p$ matrix with rank p, $\beta = (\beta_1, \beta_2, ...\beta_p)'$ is an unknown vector of interest, and $e = (e_1, e_2, ..., e_n)'$ is a normally distributed random vector with mean vectors $(0, ..., 0)'$ and covariance matrix $\sigma^2 I_n$.

(a) Find UMVU estimates for β and σ^2. (15 points)

(b) Find the least squares estimates for β. (5 points)

(c) Derive the sampling distributions of the estimates that you obtained in parts (a) and (b). (15 points)

2. (Continued) Now remove the normality assumption in Problem 1. Assume that the Gauss-Markov conditions hold; that is,

$$E(e_i) = 0, \quad \text{for all } i,$$

$$Var(e_i) = \sigma^2, \quad \text{for all } i,$$

and

$$E(e_i e_j) = 0, \quad \text{for all } i \neq j.$$

(a) State and prove Gauss-Markov theorem. (20 points)

(b) How do the assumptions impact the answers to parts (a) and (b) of Problem 1? Explain. (10 points)

(c) How do the assumptions impact the answer to part (c) of Problem 1? Explain. (10 points)

3. Consider the one-way layout model,

$$Y_{ij} = \beta_i + e_{ij}, \; j - 1, ..., m, \; i = 1, ..., p.$$

Let $\bar{\beta} - \sum_{i=1}^{p} \beta_j / p$ and $\alpha_j = \beta_j - \bar{\beta}$.

(a) Show that if $\phi = \sum_{i=1}^{p} c_i \beta_i$ with the c_i satisfying $\sum_{i=1}^{p} c_i = 0$, then there are constants w_i such that $\phi = \sum_{i=1}^{p} w_i \alpha_i$. Conversely, show that if $\phi = \sum_{i=1}^{p} w_i \alpha_i$, then there are constants c_i satisfying $\sum_{i=1}^{p} c_i = 0$ such that $\phi = \sum_{i=1}^{p} c_i \beta_i$. (10 points)

(b) What is Scheffè idea for simultaneous confidence intervals of $\phi = \sum_{i=1}^{p} w_i \alpha_i$, $(w_1, ..., w_p)' \in R^p$? (10 points)

(c) Give the Scheffè-intervals of $\phi = \sum_{i=1}^{p} w_i \alpha_i$, $(w_1, ..., w_p)' \in R^p$. (5 points)