考試科目	數理統計	所別	統計所	考試時間	5月24日 星期六 第一節

1. (15 分) For uniform distribution U(0,1) random variables $U_1, U_2, \ldots,$ define $N = \min\{n : \sum_{i=1}^{n} U_i > 1\}$. That is, N is the number of random numbers that must be summed to exceed 1. Compute the density function of N, $E(N)$, and $Var(N)$.

2. (20 分) Let X and Y be independently and identically distributed N(0,σ^2) random variables. Show that:

   (a) X^2+Y^2 and $\dfrac{X}{\sqrt{X^2+Y^2}}$ are independent.

   (b) θ is uniformly distributed on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\theta = \sin^{-1}\dfrac{X}{\sqrt{X^2+Y^2}}$.

   (c) X/Y has a Cauchy distribution.

3. (15 分) Sequential Probability Ratio Test (SPRT), developed by Wald, can be used to test simple null hypothesis (H_0) versus simple alternative hypothesis (H_1). The test is based on the ratio of likelihood function under H_0 and H_1, i.e.,

$$\lambda_n = \frac{\prod_{i=1}^{n} f(X_i \mid H_0)}{\prod_{i=1}^{n} f(X_i \mid H_1)} \quad \text{or} \quad \log\lambda_n = \sum_{i=1}^{n} \log\left(\frac{f(X_i \mid H_0)}{f(X_i \mid H_1)}\right),$$

   where $X_1, \ldots, X_n$ are a random sample from p.d.f. $f(x \mid H_J)$ with $J = 0$ or 1. The rule of applying SPRT is to continue sampling until $\log\lambda_n \ge B$ or $\log\lambda_n \le A$, where $B > A$. The values of A and B depend on the type I error α and type II error β. It is proved that the SPRT minimizes the sample size under H_0 and H_1, given the error probabilities α and β. Suppose the sample is drawn from Bernoulli distribution B(1,θ) with H_0: θ = 3/8 and H_1: θ = 1/2. Let $\alpha = \beta = 0.05$. Compute the expected sample size if H_0 is true, and the expected sample if H_1 is true.

備考	試題隨卷繳交

命題委員：　　　　　　（簽章）　　年　　月　　日

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考試科目	數理統計	所別	統計所	考試時間	5月24日 星期六	第一節

4. (15 分) Let $X_1,\ldots,X_n$ be a random ample from Bernoulli(p) distribution and $Y_n=\frac{\sum_{i=1}^{n}X_i}{n}$. (a) Show that $\sqrt{n}(Y_n-p)\to N(0,p(1-p))$ in distribution.

(b) Show that for p = 1/2, $n[Y_n(1-Y_n)-\frac{1}{4}]\to -\frac{1}{4}\chi_1^2$ in distribution.

5. (15 分) Let $X_1,\ldots,X_n$ be a random sample from uniform distribution U($\theta-\frac{1}{2},\theta+\frac{1}{2}$). Show that for any T such that $X_{(n)}-\frac{1}{2}\le T\le X_{(1)}+\frac{1}{2}$ is a maximum likelihood estimate of θ, where $X_{(1)}=\min\{X_1,\ldots,X_n\}$ and $X_{(n)}=\max\{X_1,\ldots,X_n\}$.

6. (20 分) Let $X_1,\ldots,X_n$ be a random sample from uniform distribution U(0, θ) and $R=X_{(n)}-X_{(1)}$, where $X_{(n)}=\max\{X_1,\ldots,X_n\}$ and $X_{(1)}=\max\{X_1,\ldots,X_n\}$.

(a) Find the distribution of R.

(b) Show that a confidence interval for θ, based on R, with confidence coefficient 1−α is of the form [R, R/c], where c is the root of the equation $c^{n-1}[n-(n-1)c]=\alpha$.

備考	試題隨卷繳交
命題委員：	（簽章） 年 月 日

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