國立政治大學 104 學年度博士班招生考試試題

考試科目	數理統計	所別	統計學系	考試時間	5 月 9 日(六) 第一節

1. (15pts) Let X be a random variable on the probability space (Ω, A, P). Prove Chebyshev's inequality: for any $a > 0$,

$$P(|X| \geq a) \leq \frac{1}{a^2} E(X^2)$$

2. (30pts) Let $X_1, X_2, \ldots$ be i.i.d. Uniform$[0, \theta]$, for $\theta > 0$. Show that

   (a) (15pts) $X_{(n)}$ is the MLE of θ.

   (b) (15pts) $n(\theta - X_{(n)})$ converges in distribution to the Exponential distribution with mean θ.

3. (55pts) The Poisson distribution $P(\lambda)$ with parameter $\lambda > 0$ is

$$p_\lambda(\kappa) = \frac{e^{-\lambda}\lambda^\kappa}{\kappa!}, \text{for } \kappa = 0, 1, 2, \ldots$$

   (a) (10pts) Show that the gamma family of distributions constitutes a conjugate family of priors for λ.

   (b) (10pts) $X \sim P(\lambda)$. Find the risk function of $\delta(X) = X$ under the loss function

$$\ell(\lambda, d) = \frac{(\lambda - d)^2}{\lambda}$$

   (c) (5pts) Show that $P(\lambda)$ has moment generating function

$$M(t) = exp\{\lambda(e^t - 1)\}$$

   (d) (10pts) Show that for $0 < x < 1$, $x - \frac{x^2}{2} < log(1 + x) < x$.
   (Hint: use Taylor expansion of $log(1 + x)$ around $x = 0$)

   (e) (10pts) Let $X_1, X_2, \ldots$ be i.i.d. $P(\lambda)$, $\lambda > 0$. Let $S_n = \sum_{i=1}^n X_i$. Show that, for $t > 0$,

$$P(S_n > n(\lambda + \epsilon)) \leq exp\{n\lambda(e^t - 1) - nt(\lambda + \epsilon)\}.$$

   (f) (10pts) Show that for $0 < \epsilon < \lambda/2$, $P(S_n > n(\lambda + \epsilon)) \leq exp\{-\frac{n\epsilon^2}{4\lambda}\}$

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。