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中 華 民 國 105 年 01 月 07 日

中文摘要：令j為服從伽馬分配的獨立隨機變數，形狀參數為已知，而尺度參數為未知，=1,…,， =1,…,。考慮虛無假設0: +1 對某些 =1,…,，對立假設1: 1<· · · <。為任何0<α<0.4，我們構建了一個新的檢定，其檢定規模與概似比檢定（LRT）相同，並且對所有的尺度參數其檢力是大於LRT。據我們所知，這是所考慮的假設問題，第一個提出齊一較強檢力檢定。所提出的檢定屬於交集聯集檢定。我們應用此檢訂於雙母數指數分佈的檢定。

中文關鍵詞：聯集交集檢定 概似比檢定 簡單順序 雙母數指數分配 齊一較強檢力檢定

英文摘要：Let X_ij be an independent gamma random variable with known shape parameter τ_i and unknown scale parameter θ_i for i=1,…,p and j=1,…,n_i. Consider the simple?order testing problem of testing H_0: θ_i ?θ_(i+1) for “some“ i=1,…,p versus ?H_1: θ?_1<??? ?<θ?_p. For any 0<α<0.4, we construct a new test that has the same size as the likelihood ratio test (LRT) and is uniformly more powerful than the LRT. To our knowledge, this is the first a uniformly more powerful test described for these problems. The proposed test is an intersection?union test (IUT). We apply the results to test the scale parameters of two?parameter exponential distributions.

英文關鍵詞：Intersection?union test; likelihood ratio test; simple order; two-parameter exponential distribution; uniformly more powerful test.

MORE POWERFUL TESTS FOR SIMPLE-ORDER TESTING PROBLEM WITH SCALE PAREMETERS IN GAMMA DISTRIBUTIONS

Abstract: Let X_{ij} be an independent gamma random variable with known shape parameter τ_i and unknown scale parameter θ_i for $i = 1, \ldots, p$ and $j = 1, \ldots, n_i$. Consider the simple-order testing problem of testing $H_0: \theta_i \geq \theta_{i+1}$ for some $i = 1, \ldots, p$ versus $H_1: \theta_1 < \cdots < \theta_p$. For any $0 < \alpha < 0.4$, we construct a new test that has the same size as the likelihood ratio test (LRT) and is uniformly more powerful than the LRT. To our knowledge, this is the first a uniformly more powerful test described for these problems. The proposed test is an intersection-union test (IUT). We apply the results to test the scale parameters of two-parameter exponential distributions.

Key words: Intersection-union test; likelihood ratio test; simple order; two-parameter exponential distribution; uniformly more powerful test.

Mathematics Subject Classification (2000): 62F03; 62F30; 62H15

1. Introduction

Let $X_{i1}, \ldots, X_{in_i}$ denote as independent random samples from gamma distributions with probability density function (pdf) given by

$$f(x_{ij};\ \tau_i, \theta_i) = \frac{x_{ij}^{\tau_i - 1}}{\Gamma(\tau_i)\theta_i^{\tau_i}} e^{-\frac{x_{ij}}{\theta_i}},$$

where τ_i is a known shape parameter and θ_i is an unknown scale parameter for $i = 1, \cdots, p\ (p \geq 3)$. We consider the simple-order testing problem of testing

$$H_0: \theta_i \geq \theta_{i+1} \text{ for some } i = 1, \ldots, p$$
$$\text{versus} \qquad (1.1)$$
$$H_1: \theta_1 < \cdots < \theta_p$$

We use the symbol H_1 to denote the set of $\theta_i,\ i = 1, \cdots, p$, specified by the hypothesis, as well as the statement of the hypothesis. Li and Sinha (1995) discussed the hypotheses and gave the motivations why we want to test the hypotheses. Consequently, they derived the likelihood ratio test (LRT) and failed to construct a test that is uniformly more powerful than the LRT.

The related literatures for testing problem (1.1) are much less to find. Regarding the similar testing problem, there are a fewer literatures. Tripathi et al. (1993) proposed a test based on a generalized minimum chi-squared procedure for testing the homogeneity of the scale parameters versus general unrestricted alternatives when $p \geq 2$ and shape parameters are unknown. Their test is applicable in versatile testing problem with general unrestricted alternatives and it is asymptotic in nature. For the null hypothesis is the homogeneity for scale parameters versus alternative hypothesis is order constraint with at least one strict equality, Bhattacharya (2001) proposed a simple procedure based on Fisher's method of combing probability values to test the hypotheses with a common shape parameter but unknown when, $p \geq 3$. Bhattacharya (2002) proposed two tests where one is to use quadratic forms involving ratios of cumulants as test statistic and the other is a stepwise procedure which uses Fisher's method combining probability values when shape parameters are equal but unknown. The similar testing problem is to test the order of normal variances. Several researchers have studied the homogeneity for normal variances against the restricted nondecreasing order alternatives such as Bartlett (1937), Cochran (1941), Chacho (1963), Hartley (1940,1950), Vincent (1961), Fujino (1979), Mudholkar et al. (1993), Mudholkar et al. (1995)..

For the testing problem (1.1), Li and Sinha (1995) showed that the size-α LRT that rejects H_0 if

$$\frac{\sum_{j=1}^{n_{i+1}} X_{i+1j}/(\tau_{i+1}n_{i+1})}{\sum_{j=1}^{n_i} X_{ij}/\tau_i n_i} > f_{\alpha,2\tau_{i+1}n_{i+1},2\tau_i n_i}$$

for all $i = 1, \ldots, p-1$, where $f_{\alpha,2\tau_{i+1}n_{i+1},2\tau_i n_i}$ is the $100(1-\alpha)$ percentile of central F distribution with $2\tau_{i+1}n_{i+1}$ and $2\tau_i n_i$ degrees of freedom. As mentioned in Berger (1989) and Liu and Berger (1995), the LRT has drawbacks in some cases. Hence, researchers have tried to improve the LRT by enlarging the rejection region of the LRT in order to increase its power. Until now, the study about uniformly more powerful test only focus on one-sided or two-sided testing problem for normal means under various conditions are because it can be to show that the size of the proposed test is α. These studies are including Gutmann (1987), Nomakuchi and Sakata (1987), Berger (1989), Iwasa (1991), Shirley (1992), Liu and Berger (1995), Liu (1999, 2000), McDermott and Wang (2002), and Saikali and Berger (2002). Sasabuchi (2007) proposed some tests that are more powerful than the LRT derived by Sasabuchi et al. (2003) for testing problem about homogeneity of multivariate normal mean vectors when the covariance matrices are common but unknown.

For the case of the sign testing about gamma scale parameters or normal variances, Li and Sinha (1995) and Liu and Chan (2012) constructed more powerful tests than the LRT, respectively; the tests proposed by Liu and Chan (2010) are more powerful than the test proposed by Li and Sinha (1995) under some conditions. To our knowledge, uniformly more powerful test for testing problem (1.1) has not been obtained yet. From the past studies or techniques, they considered that there is no way to construct a uniformly more powerful test for testing problem (1.1). Even if constructing a uniformly more powerful test, it is most difficult to prove the size of the uniformly more powerful test is α under null hypothesis. Different from the past, we employ two statistics, sufficient statistic and ancillary statistic, and use the independence of two statistics to construct a uniformly more powerful test. Importantly, we can show that the size of our proposed test is α.

In this paper, for testing problem of the form (1.1), we proposed a new test that has the same size as the LRT and is uniformly more power than the LRT. First, we consider the testing problem (1.1) when $H_{1i}: \theta_i < \theta_{i+1} < \theta_{i+2}$, $p = 3$. The new test, $\phi_{i+2,i+1,i}$, $i = 1,2,\cdots,p-2,$ is constructed. The rejection region of the new test contains the rejection of the LRT and an additional set, but the size of the new test is still α. So the new test is uniformly more powerful than the LRT. Then, by recognizing that $p > 3$, H_1 can be written as the intersection of sets each defined by two inequalities, we use the intersection-union method to combine tests of the form $\phi_{i+2,i+1,i}$ to obtain a test ϕ_g that is uniformly more powerful for general problem (1.1).

The rest of the paper is organized as follows. In Section 2, we address some definitions and a lemma that will later be used to construct more powerful tests and show that various tests are size-α tests. In Section 3, we describe the testing problem for $p = 3$ and construct test $\phi_{i+2,i+1,i}$. Also, we demonstrate the power of the new test. In Section 4, we construct a uniformly more powerful test based on $\phi_{i+2,i+1,i}$ for testing problem (1.1) for $p > 3$. Section 5 gives applications with two distributions and an illustrative data example. We give a comment and conclusions in Section 6. Proof of the lemma is outlined in the Appendix. Finally, we draw our concluding remarks in Section 5 and collect the proofs in the appendices.

Our notation will be simplified by considering these transformed data. Let $Y_i = \frac{2\sum_{j=1}^{n_i} X_{ij}}{\theta_{i0}}$, $i = 1, \dots, p$, then Y_i~Gamma$(n_i\eta_i, 2\theta_i/\theta_{i0})$. Y_is are central Chi-square random variables with $k_i(= 2n_i\eta_i)$ degree of freedom as $\frac{\theta_i}{\theta_{i0}}$=1. Let $S_i = \frac{Y_i}{k_i}, V_i = k_iS_i + k_{i+1}S_{i+1}$ and $W_i = \frac{S_{i+1}}{S_i}$. Throughout the remainder of this paper, we will express our results in terms of the S_is, V_is. and W_is The complete data vector will be denoted by $\boldsymbol{S}, \boldsymbol{V}$ and $\boldsymbol{W}$; $\boldsymbol{s}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ will denote observed. Throughout the remainder of this paper, $f_{k,m}(\cdot)$ and $F_{k,m}(\cdot)$ denote the pdf and cdf of a central F random variable with k and m degrees of freedom, respectively. Also, $F_{k,m}^{-1}(\cdot)$ denotes the inverse of $F_{k,m}(\cdot)$value and $f_{\alpha,m,k}$ denotes (1-α) ×100% percentile of the $f_{k,m}(\cdot)$.

2. **Preliminary definitions and lemmas**

Before describing our proposed test, we define the functions and sets which are used to construct the rejection region of the test and give two lemmas where one is to ensure that the size of the test is α.The following definitions and lemma will be in subsequent section to define a couple of tests and prove those tests are size-α tests.

Definition 2.1 *For any* $s_i > 0$ *and* $k_i > 0$, *let* L_i *be the three-dimensional set defined by*

$$L_i = \left\{(s_i, s_{i+1}, s_{i+2}) : \frac{s_{i+1}}{s_i} \geq f_{\alpha,2k_{i+1},2k_i}, \frac{s_{i+2}}{s_{i+1}} \geq f_{\alpha,2k_{i+2},2k_{i+1}}\right\}.$$

The set L_i is a triangular pyramid. We will eventually express the LRT in terms of L_i $i = 1,2,\cdots,p-2$. for testing problem (1.1) as subhypotheses any $p \geq 3$.

Let L_i^C denote a cross sectional plane of L_i as $S_i = s$ and L_i^v denote a vertical plane of $L_i$as S_{i+2}=s.

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All three sets are equivalent. $\{(S_i, S_{i+1}, S_{i+2}) \in L_i\} = \{(V_i, W_i, S_{i+2}) \in L_i\} = \{(V_{i+1}, W_{i+1}, S_i) \in L_i\}$. That can be ckecked from definition 2.2.

Definition 2.2 *For any* $s_{i+2} > 0, v_i, w_i$ *and* $k_i > 0$*, let* $L_i^C(s_{i+2})$*and* $L_i^L(s_i)$ *be the two-dimensional sets defined by*

$$L_i^C(s_{i+2}) = \left\{(v_i, w_i, s_{i+2}): f_{\alpha,2k_{i+1},2k_i} \le w_i, 0 < v_i < k_{i+1}c_{i+2,\alpha}^{s_{i+2}} \right\} \cup$$

$$\left\{(v_i, w_i, s_{i+2}): f_{\alpha,2k_{i+1},2k_i} \le w_i < \frac{k_i}{\frac{v_i}{c_{i+2,\alpha}^{s_{i+2}}} - k_{i+1}}, k_{i+1}c_{i+2,\alpha}^{s_{i+2}} \le v_i < \right.$$

$$\left. c_{i+1,\alpha}^{k_i} c_{i+2,\alpha}^{s_{i+2}} \right\}, \qquad (2.1)$$

$$L_i^L(s_i) = \left\{(v_{i+1}, w_{i+1}, s_i): f_{\alpha,2k_{i+2},2k_{i+1}} \le w_{i+1} < \right.$$

$$\left. \frac{v_{i+1}/c_{i+1}^{s_i,\alpha} - k_{i+1}}{k_{i+2}}, c_{i+2}^{k_{i+2},\alpha} c_{i+1}^{s_i,\alpha} \le v_{i+1} \right\}, \ (2.2)$$

Where $c_{i+2,\alpha}^{s_{i+2}} = s_{i+2}/f_{\alpha,2k_{i+2},2k_{i+1}}$, $c_{i+1,\alpha}^{k_i} = (k_{i+1} + k_i/f_{\alpha,2k_{i+1},2k_i})$,

$c_{i+1}^{s_i,\alpha} = s_i f_{\alpha,2k_{i+1},2k_i}$ and $c_{i+2}^{k_{i+2},\alpha} = (k_{i+2} f_{\alpha,2k_{i+2},2k_{i+1}} + k_{i+1})$

Also, the set L_i can be expressed as $L_i = \int_0^\infty L_i^C(s_{i+2}) ds_{i+2} = \int_0^\infty L_i^L(s_i)\, ds_i$.

For $i = 1$, $k_2 s_2 + k_1 s_1 = v_1$ is a plane perpendicular to the $S_2 - S_1$ coordinate plane with any $v_1 > 0$. For $i = 2$, $k_3 s_3 + k_2 s_2 = v_2$ is a plane perpendicular to the $S_3 - S_2$ coordinate plane with any $v_2 > 0$. Examples of $L_1^C(3)$ and $L_1^L(2)$ $k_2 s_2 + k_1 s_1 = v_1$, for $s_3 = 3$ and $s_1 = 2$, when $k_3 = k_2 = k_1 = 5$, are shown in Figure 1. In Figure 1(a), the dotted line is $k_2 s_2 + k_1 s_1 = v_1$ with $v_1 = \left(k_2 + k_1/f_{\alpha,2k_2,2k_1}\right) s_3 / f_{\alpha,2k_3,2k_2} c_{k_2,k_1}^1 c_{s_3}$. In Figure 1(b), the dotted line is $k_3 s_3 + k_2 s_2 = v_2$ with $v_2 = c_{k_3,k_2}^1 c_{s_1}$. $v_2 = (k_3 f_{\alpha,2k_3,2k_2} + k_2) s_1 f_{\alpha,2k_2,2k_1}$.

Definition 2.3 *For any given* s_{i+2} *and* v_i*,that are defined in Definition 2.2 such that* $s_{i+2} > 0$ *and* $v_i >$*, we define*

$$P_{v_i s_{i+2}}^C = \alpha - \int_{L_i^C(v_i, s_{i+2})} f_{k_2,k_1}(w_i\ ) dw_i \ ,$$

where$L_i^C(v_i, s_{i+2}) = \{w_i: (v_i, w_i, s_{i+2}) \in L_i^C(s_{i+2})\ \}$. Specifically,

$$P^C_{v_i,s_{i+2}} = \begin{cases} 0, & 0 < v_i \le k_{i+1} c^{s_{i+2}}_{i+2,\alpha}, \\ 1 - F_{k_{i+1},k_i}\left(\dfrac{k_i}{v_i / c^{s_{i+2}}_{i+2,\alpha} - k_{i+1}}\right), & k_{i+1} c^{s_{i+2}}_{i+2,\alpha} \le v_i < c^{k_i}_{i+1,\alpha} c^{s_{i+2}}_{i+2,\alpha}, \\ \alpha, & c^{k_i}_{i+1,\alpha} c^{s_{i+2}}_{i+2,\alpha} \le v_i. \end{cases}$$

Definition 2.4 *For any given* s_i *and* v_{i+1}*, such that* $s_i > 0$ *and* $(k_{i+2} + k_{i+1})s_i \le v_{i+1} < \infty$*, we define*

$$P^L_{v_{i+1}s_i} = \alpha - \int_{L_i^L(v_{i+1},s_i)} f_{k_{i+2},k_{i+1}}(w_{i+1}) dw_{i+1},$$

where $L_i^L\ (v_{i+1}, s_i) = \{w_{i+1}: (v_{i+1}, w_{i+1}, s_i) \in L_i^L(s_i)\}$. Specifically,

$$P^L_{v_{i+1}s_i} = \begin{cases} \alpha, & (k_{i+2} + k_{i+1}) s_i \le v_{i+1} < c^{k_{i+1},\alpha}_{i+2} c^{s_i,\alpha}_i, \\ 1 - F_{k_{i+2},k_{i+1}}\left(\dfrac{v_{i+1}/c^{s_i,\alpha}_i - k_{i+1}}{k_{i+2}}\right), & c^{k_{i+1},\alpha}_{i+2} c^{s_i,\alpha}_i \le v_{i+1}. \end{cases}$$

The specific formulas for $P^C_{v_i s_{i+2}}$and $P^L_{v_{i+1}s_i}$are verified by using the definition of $L_i^C(s_{i+2})$ and $L_i^L(s_i)$, respectedly. Note that $0 \le P^C_{v_i,s_{i+2}} \le \alpha$ for all $v_i > 0$ and $s_{i+2} > 0$, and $0 \le P^L_{v_{i+1}s_i} \le \alpha$ for all $v_{i+1} \ge (k_{i+2} + k_{i+1})s_i$ and $s_i > 0$. The line between $(s_{i+2}/(f_{\alpha,2k_{i+1},2k_i} f_{\alpha,2k_{i+2},2k_{i+1}}), s_3/f_{\alpha,2k_{i+2},2k_{i+1}}, s_{i+2})$ and $(s_{i+2}, s_{i+2}, s_{i+2})$ with a fixed s_{i+2} or that between $(s_i, s_i f_{\alpha,2k_{i+1},2k_i}, s_1 f_{\alpha,2k_{i+1},2k_i} f_{\alpha,2k_{i+2},2k_{i+1}})$ and (s_i, s_i, s_i) with a fixed s_i satisfies the equation $s_{i+2} = b s_{i+1} - a s_i$, where $a = f_{\alpha,2k_{i+1},2k_i}(f_{\alpha,2k_{i+2},2k_{i+1}} - 1)/(f_{\alpha,2k_{i+1},2k_i} - 1)$ and $b = (f_{\alpha,2k_{i+2},2k_{i+1}} f_{\alpha,2k_{i+1},2k_i} - 1)/(f_{\alpha,2k_{i+1},2k_i} - 1)$. Two cases of the line are given in Figure 1 for $k_3 = k_2 = k_1 = 5$. When s_3 or s_1 moves from 0 to ∞, it becomes a plane between the line $s_3/f_{\alpha,2k_3,2k_2} = s_2 = s_1 f_{\alpha,2k_2,2k_1}$, the edge of L_1, and the line $s_3 = s_2 = s_1$. We now define the two sets that contain this plane and are used to construct the first proposed test.

Definition 2.5 *For* $0 < \alpha < 0.4$, s_{i+2}*>0 and* $0 < d < 1$*, let* $A_i^C(s_{i+2})$ *be the set defined by*

$$A_i^C(s_{i+2}) = \{(v_i, w_i, s_{i+2}): l_{v_i,s_{i+2}}^{C,D} \le w_i \le l_{v_i,s_{i+2}}^{C,U}, \quad 0 < v_i \le (k_{i+1} + k_i)s_{i+2},\}$$

where

$$l_{v_i,s_{i+2}}^{C,U} = \begin{cases} \infty, & 0 < v_i < k_{i+1} c_{i+2,\alpha}^{s_{i+2}}, \\ F_{2k_2,2k_1}^{-1}\left(F_{2k_2,2k_1}\left(\dfrac{k_1}{v_1/c_{s_3} - k_2}\right) + dP_{k_2,k_1}(v_1,s_3)\right), & k_{i+1} c_{i+2,\alpha}^{s_{i+2}} \le v_i < c_{i+1,\alpha}^{k_i} c_{i+2,\alpha}^{s_{i+2}} \\ \min\left\{F_{2k_2,2k_1}^{-1}\left(F_{2k_2,2k_1}\left(\dfrac{av_1 + k_1 s_3}{bv_1 - k_1 s_3}\right) + dP_{k_2,k_1}(v_1,s_3)\right), \dfrac{k_i s_{i+2}}{v_1 - k_2 s_{i+2}}\right\}, & c_{i+1,\alpha}^{k_i} c_{i+2,\alpha}^{s_{i+2}} \le v_i \le (k_i \end{cases}$$

$$l_{v_i,s_{i+2}}^{C,D} = \max\{F_{2k_2,2k_1}^{-1}(F_{2k_2,2k_1}(l_{v_i,s_{i+2}}^{C,U}) - \alpha), 1\},$$

$$a_{k_2,k_1}^1(v_1,s_3) = F_{2k_2,2k_1}^{-1}\left(F_{2k_2,2k_1}\left(\frac{k_1}{v_1/c_{s_3} - k_2}\right) + dP_{k_2,k_1}(v_1,s_3)\right)$$

And

$$a_{k_2,k_1}^2(v_1,s_3) = F_{2k_2,2k_1}^{-1}\left(F_{2k_2,2k_1}\left(\frac{av_1 + k_1 s_3}{bv_1 - k_1 s_3}\right) + dP_{k_2,k_1}(v_1,s_3)\right),$$

for i$=1,2,\cdots,p-2$.

Definition 2.6 *For* $0 < \alpha < 0.4$ *and* $0 < d < 1$, *let* $A_i^L(s_i)$ *be the set defined by*

$$A_i^L(s_i) = \{(v_{i+1}, w_{i+1}): l_{v_{i+1},s_i}^{L,L} \le w_{i+1} \le l_{v_{i+1},s_i}^{L,U}, \quad 0 < v_{i+1} \le (k_{i+2} + k_{i+1})s_i\}$$

where

$$l_{v_{i+1},s_i}^{L,U} = \begin{cases} \min\left\{F_{2k_3,2k_2}^{-1}\left(F_{2k_3,2k_2}\left(\dfrac{bv_2 - ak_2 s_1}{v_2 + bk_3 s_1}\right) + dP_{k_3,k_2}(v_2,s_1)\right), \dfrac{v_{i+1}}{s_i} - k_{i+1}\right\}, & (k_{i+2} + k_{i+1})s_i \le v_{i+1} < c \\ F_{2k_3,2k_2}^{-1}\left(F_{2k_3,2k_2}\left(\dfrac{v_2/c_{s_1} - k_2}{k_3}\right) + dP_{k_3,k_2}(v_2,s_1)\right), & c_{k_{i+2},k_{i+1}}^1 c_i^{s_i,\alpha} \le v_{i+1}, \end{cases}$$

$$l_{v_{i+1},s_i}^{L,L} = \max\left\{F_{2k_3,2k_2}^{-1}\left(F_{2k_3,2k_2}\left(l_{k_3,k_2}^1(v_2,s_1)\right) - \alpha\right), 1\right\},$$

$$a_{k_3,k_2}^1(v_1,s_3) = F_{2k_3,2k_2}^{-1}\left(F_{2k_3,2k_2}\left(\frac{bv_2 - ak_2 s_1}{v_2 + bk_3 s_1}\right) + dP_{k_3,k_2}(v_2,s_1)\right)$$

and

$a^2_{k_3,k_2}(v_1,s_3) = F^{-1}_{2k_3,2k_2}\left(F_{2k_3,2k_2}\left(\frac{v_2/c_{s_1}-k_2}{k_3}\right) + dP_{k_3,k_2}(v_2,s_1)\right)$Let

$A_i^{CS} = \int_0^\infty A_i^C(s_{i+2})ds_{i+2}$ and $A_i^L = \int_0^\infty A_i^L(s_i)\,ds_i$

Examples of the two sets with a given $s_3 = 3$ and $s_1 = 2$ are shown in Figures 1(a) and 1(b), respectively.

Subsequently, we define sets that are used to construct the second test; the idea is from Berger (1989). Before defining sets, we need an integer J to determine how many additional sets of a constructed more powerful test. For $0 < \alpha < 0.4$ and $i = 1,2$, let J_i be the integer that satisfies $f_{\alpha,2k_{i+1},2k_i} > f_{2\alpha,2k_{i+1},2k_i} > \cdots > f_{J_i\alpha,2k_{i+1},2k_i} > 1$. Define $J = \min\{J_i\}$, $i = 1,2$..

Definition 2.7 *For* $0 < \alpha < 0.4$ *and* $j = 1,\ldots,J$, *let* $B^j_{k_3,k_2,k_1}$ *be the three-dimensional set defined by*

$$B_i^j = \left\{(s_i,s_{i+1},s_{i+2}): \max\{f_{(j+1)\alpha,2k_{i+1},2k_i},1\} \le \frac{s_{i+1}}{s_i} \le f_{j\alpha,2k_{i+1},2k_i}, \max\{f_{(j+1)\alpha,2k_{i+2},2k_{i+1}},1\} \le \frac{s_{i+2}}{s_{i+1}} \le f_{j\alpha,2k_{i+2},2k_{i+1}}\right\}$$

.

Also, B_i^j, $j = 1,\ldots,J$, *can be expressed in terms of* (v_1,w_1,s_3) *given by*

$$B_i^j = \left\{(v_1,w_1,s_3): \frac{k_1}{\frac{v_1}{c^j_{s_3}}-k_2} \le w_1 \le f_{j\alpha,2k_2,2k_1}, c^j_{k_2,k_1}c^j_{s_3} < v_1 < c^{j+1}_{k_2,k_1}c^j_{s_3}, 0 < s_3\right\} \cup \{(v_1,w_1,s_3): f_{(j+1)\alpha,2k_2,2k_1} \le w_1 < f_{j\alpha,2k_2,2k_1}, c^{j+1}_{k_2,k_1}c^j_{s_3} \le v_1 < c^j_{k_2,k_1}c^{j+1}_{s_3}, 0 < s_3\} \cup \left\{(v_1,w_1,s_3): \max\{f_{(j+1)\alpha,2k_2,2k_1},1\} \le w_1 < \frac{k_1}{v_1/c^{j+1}_{s_3}-k_2}, c^j_{k_2,k_1}c^{j+1}_{s_3} \le v_1 < c^{j+1}_{k_2,k_1}c^{j+1}_{s_3}, 0 < s_3\right\},$$

where $c_{s_3}^{j} = \frac{s_3}{f_{j\alpha,2k_3,2k_2}}$, $c_{s_3}^{J+1} = s_3$, $c_{k_2,k_1}^{j} = k_2 + \frac{k_1}{f_{j\alpha,2k_2,2k_1}}$ and $c_{k_2,k_1}^{J+1} = k_2 + k_1$, or B_{k_3,k_2,k_1}^{j}, $j = 1, \ldots, J$, can be expressed in terms of (v_2, w_2, s_1) given by

$$B_i^j = \left\{(v_2, w_2, s_1): \max\{f_{(j+1)\alpha,2k_3,2k_2}, 1\} \le w_2 \le \frac{v_2/c_{s_1}^{j+1} - k_2}{k_3}, c_{k_3,k_2}^{j+1} c_{s_1}^{j+1} < v_1 < c_{k_3,k_2}^{j} c_{s_1}^{j+1}, 0 < s_1\right\} \cup$$

$$\{(v_2, w_2, s_1): f_{(j+1)\alpha,2k_3,2k_2} \le w_2 < f_{j\alpha,2k_3,2k_2}, c_{k_3,k_2}^{j} c_{s_1}^{j+1} \le v_2 < c_{k_3,k_2}^{j+1} c_{s_1}^{j}, 0 < s_1\} \cup \left\{(v_2, w_2, s_1): \frac{v_2/c_{s_1}^{j} - k_2}{k_3} \le w_2 < f_{j\alpha,2k_3,2k_2}, c_{k_3,k_2}^{j+1} c_{s_1}^{j} \le v_2 < c_{k_3,k_2}^{j} c_{s_1}^{j}, 0 < s_1\right\},$$

where $c_{s_1}^{j} = s_1 f_{j\alpha,2k_2,2k_1}$, $c_{s_1}^{J+1} = s_1$, $c_{k_3,k_2}^{j} = k_3 f_{j\alpha,2k_3,2k_2} + k_2$ and $c_{k_3,k_2}^{J+1} = k_3 + k_2$.

An example of B_1^j is presented in Figure 2 for $k_3 = k_2 = k_1 = 10$, $\alpha = 0.1$, $s_3 = 2$ and $s_1 = 2$. In this example , $J = 4$, each additional region B^j is like diamond for $j = 1, \cdots, 4$.

The following lemmas are the keys to ensure that the size of proposed tests are α; its proof is given in Appendix A

Lemma *2.1 Let $W_i = S_{i+1}/S_i$ and $V_i = k_{i+1}S_{i+1} + k_i S_i$ for $i = 1,2,\cdots,p-1$, where $S_i \sim \Gamma(k_i, \theta_i/k_i)$. $h_{w_i|v_i}(w_i; v_i, \theta_{i+1}, \theta_i)$ denotes as the conditional pdf of W_i given V_i. Then (i) For $w_i \ge 1$ and $\theta_i \ge \theta_{i+1}$, $h_{w_i|v_i}(w_i; v_i, \theta_{i+1}, \theta_i) \le h_{w_i|v_i}(w_i; v_i, \theta_i, \theta_i)$.*

(ii) For $\theta_i = \theta_{i+1} = \theta_{i*}$, V_i and W_i are complete sufficient statistic and ancillary statistic for θ_{i*}, respectively.

(iii) For $\theta_i = \theta_{i+1}$, V_i and W_i are statistical independent.

It can be proved easily by Basu's theorem, and we thus omit its proof in this paper.

If $\theta_{i0} = \theta_{i+1} = \theta_i$, $i = 1,2$, then, W_i has a respective central F distribution with $2k_{i+1}$ and $2k_i$ degrees of freedom, and $h_{W_i|V_i}(w_i; \theta_{i0}, \theta_{i0}) = f_{2k_{i+1},2k_i}(w_i)$ based on Lemma 2.2. Hence, Inequality (2.3) is rewritten as

$$h_{W_i|V_i}(w_i; \theta_{i+1}, \theta_i) \le f_{2k_{i+1},2k_i}(w_i)$$

(2.4)

for $\theta_i \ge \theta_{i+1}$ when $w_i \ge 1$.

$\{(S_i, S_{i+1}, S_{i+2}) \in A_i^{SC}\}$is equivalent to $\{(V_i, W_i, S_{i+2}) \in A_i^{SC}\}$

$\{(S_i, S_{i+1}, S_{i+2}) \in A_i^{\mathrm{LG}}\}$is equivalent to $\{(V_{i+1}, W_{i+1}, S_i) \in A_i^{LG}\}$

(S_i, S_{i+1}, S_{i+2}) is one-to-one (V_i, W_i, S_{i+2})

Lemma 2.2 *Let* $0 < \alpha < 0.4$, $\theta_2 \le \theta_1$, *and* $\theta_3 > 0$.
Then, $(i) P_{(\theta_\mathrm{i},\theta_{\mathrm{i+1}},\theta_{\mathrm{i+2}})}\left((S_\mathrm{i}, S_{\mathrm{i+1}}, S_{\mathrm{i+2}}) \in A_i^{\mathrm{CS}}\right) \le \alpha,$
$(ii) P_{(\theta_\mathrm{i},\theta_{\mathrm{i+1}},\theta_{\mathrm{i+2}})}\left((S_\mathrm{i}, S_{\mathrm{i+1}}, S_{\mathrm{i+2}}) \in A_i^{LG}\right) \le \alpha$

Lemma 2.3 *Let* $0 < \alpha < 0.4$, $\theta_3 \le \theta_2$, *and* $\theta_1 > 0$. *Then,*
$P_{(\theta_i,\theta_{i+1},\theta_{i+2})}\left((S_i, S_{i+1}, S_{i+2}) \in A_{i+2,i+1}\right) \le \alpha.$

Proof. For $V_2 = k_3 S_3 + k_2 S_2$ and $W_2 = S_3/S_2$, $P\big((S_1, S_2, S_3) \in A_{k_3,k_2}\big) = P\big(l_{k_3,k_2}^2(V_1, S_3) \le W_2 \le l_{k_3,k_2}^1(V_1, S_3), (k_3 + k_2)S_1 < V_2, 0 < S_1\big)$. Let $f_{V_2}(v_2)$ and $f_{S_1}(s_1)$ be the pdfs of V_2 and S_1, respectively, and V_2 and S_1 are independent. For $\theta_3 = \theta_2 = \theta_{20}$, the first inequality of the following equations is based on Lemma 2.1 and Inequality (2.4), and W_2 follows a central F with $2k_3$ and $2k_2$ degrees of freedom.

$$P_{(\theta_1,\theta_2,\theta_3)}\left((S_1, S_2, S_3) \in A_{k_3,k_2}\right)$$

$$= \int_0^\infty \int_{(k_3+k_2)s_1}^\infty \int_{l_{k_3,k_2}^2(v_2,s_1)}^{l_{k_3,k_2}^1(v_2,s_1)} h_{W_2|V_2}(w_2; \theta_3, \theta_2)\, dw_2 f_{V_2}(v_2) f_{S_1}(s_1) dv_2 ds_1$$

$$\le \int_0^\infty \int_{(k_3+k_2)s_1}^\infty \int_{l_{k_3,k_2}^2(v_2,s_1)}^{l_{k_3,k_2}^1(v_2,s_1)} f_{2k_3,2k_2}(w_2)\, dw_2 f_{V_2}(v_2) f_{S_1}(s_1) dv_2 ds_1$$

$$\le \int_0^\infty \int_0^\infty \int_{l_{k_3,k_2}^2(v_2,s_1)}^{l_{k_3,k_2}^1(v_2,s_1)} f_{2k_3,2k_2}(w_2)\, dw_2 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3$$

$$= \int_0^\infty \int_0^\infty \left[F_{2k_3,2k_2}\left(l_{k_3,k_2}^1(v_2, s_1)\right)\right.$$

$$\left. - F_{2k_3,2k_2}\left(l_{k_3,k_2}^2(v_2, s_1)\right)\right] dw_1 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3$$

(2.4)

The expression in brackets in (2.4) is clearly bounded above by α for $(k_3 + k_2)s_1 < v_2 < \infty$ because

$$F_{2k_3,2k_2}\left(l^1_{k_3,k_2}(v_2, s_1)\right) - F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2, s_1)\right) \leq$$

$$F_{2k_3,2k_2}\left(l^1_{k_3,k_2}(v_2, s_1)\right) - F_{2k_3,2k_2}\left(F^{-1}_{2k_3,2k_2}\left(F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2, s_1)\right) - \alpha\right)\right) = \alpha$$

3. Uniformly more powerful test for $p = 3$

In this section, we consider the testing problem (1.1) when $p = 3$; that is given as follows

$$\begin{array}{c} H_0: \theta_{i+1} \leq \theta_i \text{ or } \theta_{i+2} \leq \theta_{i+1} \\ \text{versus} \\ H_1: \theta_i < \theta_{i+1} < \theta_{i+2} \end{array} \tag{3.1}$$

For testing problem (3.1), the null hypothesis can be expressed as the union of two sets and the alternative hypothesis can be expressed as the intersection of two sets. The hypotheses (3.1) is expressed as

$$\begin{array}{c} H_0: \{\theta_{i+1} \leq \theta_i, \theta_{i+2} > 0\} \text{ or } \{\theta_{i+2} \leq \theta_{i+1}, \theta_i > 0\} \\ \text{versus} \\ H_1: \{\theta_{i+1} > \theta_i, \theta_{i+2} > 0\} \text{ and } \{\theta_{i+2} > \theta_{i+1}, \theta_i > 0\} \end{array} \tag{3.2}$$

Subsequently, we construct a new test $\phi_{i,i+1,i+2}$ that is a size-α test and uniformly more powerful than the LRT. To simply the notation, here we only state the case $i = 1$.
Before defining the test ϕ_{321}, we give the following definition.

Definition 3.1 *Let* x_{ij}, $i = 1,2,3$, $j = 1, \ldots, n_i$ *be observed values. Define* $s_i = \sum_{j=1}^{n_i} x_{ij}/k_i$ *with* $k_i = \tau_i n_i$ *for* $i = 1,2,3$, *and* $\mathbf{s} = (s_1, s_2, s_3)$. *Also, define the corresponding random variable* $S_i = \sum_{j=1}^{n_i} X_{ij}/k_i$ *and a three-dimension random vector* $\mathbf{S} = (S_1, S_2, S_3)$.

Since X_{ij}, $i = 1,2,3$ and $j = 1, \ldots, n_i$, are independent gamma random variables, S_i, $i = 1,2,3$, are independent random variables and $2k_iS_i/\theta_i$, $i = 1,2,3$, follow a respective central Chi - squared distribution with $2k_i$ degrees of freedom. From preceding section, $W_i = S_{i+1}/S_i$ and $V_i = k_{i+1}S_{i+1} + k_iS_i$, $i = 1,2$. Note that V_1 and S_3 are independent and V_2 and S_1 are independent.

Based on Lemma 2.2, for $\theta_{i+1} = \theta_i = \theta_{i0}$, $i = 1,2$, the random variable W_i have a central F with k_{i+1} and k_i degrees of freedom. The following two lemmas show that the probability of belonging $\mathbf{S}$ to A_{21} and A_{32}, respectively, is less than α for $\theta_i \geq \theta_{i+1}$, $i = 1,2$.

Lemma 3.1 *Let* $0 < \alpha < 0.4$, $\theta_2 \le \theta_1$ *and* $\theta_3 > 0$. *Then* $P_{(\theta_1,\theta_2,\theta_3)}\left((S_1,S_2,S_3) \in A_{k_2,k_1}\right) \le \alpha$.

Proof. For $V_1 = k_2 S_2 + k_1 S_1$ and $W_1 = S_2/S_1$, $P\big((S_1,S_2,S_3) \in A_{k_2,k_1}\big) = P\big(l^1_{k_2,k_1}(V_1,S_3) \le W_1 \le l^2_{k_2,k_1}(V_1,S_3), 0 < V_1 \le (k_2 + k_1)S_3,\ 0 < S_3\big)$. Let $f_{V_1}(v_1)$ and $f_{S_3}(s_3)$ be the pdfs of V_1 and S_3, respectively. V_1 and S_3 are independent. For $\theta_1 = \theta_2 = \theta_{10}$, the first inequality of the following equations is based on Lemma 2.1 and Inequality (2.4), and W_1 follows a central F with $2k_2$ and $2k_1$ degrees of freedom.

$$
\begin{aligned}
&P_{(\theta_1,\theta_2,\theta_3)}\left((S_1,S_2,S_3) \in A_{k_2,k_1}\right)\\
&= P_{(\theta_1,\theta_2,\theta_3)}\left(l^1_{k_2,k_1}(V_1,S_3) \le W_1 \le l^2_{k_2,k_1}(V_1,S_3), 0 < V_1 \le (k_2+k_1)S_3, 0 < S_3\right)\\
&= \int_0^\infty \int_0^{(k_2+k_1)s_3} \int_{l^1_{k_2,k_1}(v_1,s_3)}^{l^2_{k_2,k_1}(v_1,s_3)} h_{W_1|V_1}(w_1;\theta_2,\theta_1)\, dw_1 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3\\
&\le \int_0^\infty \int_0^{(k_2+k_1)s_3} \int_{l^1_{k_2,k_1}(v_1,s_3)}^{l^2_{k_2,k_1}(v_1,s_3)} f_{2k_2,2k_1}(w_1)\, dw_1 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3\\
&\le \int_0^\infty \int_0^\infty \int_{l^1_{k_2,k_1}(v_1,s_3)}^{l^2_{k_2,k_1}(v_1,s_3)} f_{2k_2,2k_1}(w_1)\, dw_1 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3\\
&= \int_0^\infty \int_0^\infty \Big[F_{2k_2,2k_1}\left(l^2_{k_2,k_1}(v_1,s_3)\right)\\
&- F_{2k_2,2k_1}\left(l^1_{k_2,k_1}(v_1,s_3)\right)\Big]\, dw_1 f_{V_1}(v_1) f_{S_3}(s_3) dv_1 ds_3
\end{aligned}
$$

(3.3)

The expression in brackets in (3.3) is clearly bounded above by α for $0 < v_1 \le (k_2 + k_1)s_3 < \infty$ because

$$
F_{2k_2,2k_1}\left(l^2_{k_2,k_1}(v_1,s_3)\right) - F_{2k_2,2k_1}\left(l^1_{k_2,k_1}(v_1,s_3)\right) \le F_{2k_2,2k_1}\left(l^2_{k_2,k_1}(v_1,s_3)\right) - F_{2k_2,2k_1}\left(F^{-1}_{2k_2,2k_1}\left(F_{2k_2,2k_1}\left(l^2_{k_2,k_1}(v_1,s_3)\right) - \alpha\right)\right) = \alpha. \qquad \square
$$

Lemma 3.2 *Let* $0 < \alpha < 0.4$, $\theta_3 \le \theta_2$ *and* $\theta_1 > 0$. *Then*$P_{(\theta_1,\theta_2,\theta_3)}\left((S_1,S_2,S_3) \in A_{k_3,k_2}\right) \le \alpha$.

Proof. For $V_2 = k_3 S_3 + k_2 S_2$ and $W_2 = S_3/S_2$, $P((S_1,S_2,S_3) \in A_{32\cdot 1}) = P\big(l^1_{k_3,k_2}(V_1,S_3) \le W_2 \le l^2_{k_3,k_2}(V_1,S_3), (k_3 + k_2)S_1 < V_2, 0 < S_1\big)$. Let $f_{V_2}(v_2)$ and $f_{S_1}(s_1)$ be the pdfs of V_2 and S_1, respectively. V_2 and S_1 are independent. For $\theta_2 = \theta_3 = \theta_{20}$, the first inequality of the following equations is based on Lemma 2.1 and Inequality (2.4), and W_2 follows a central F with $2k_3$ and $2k_2$ degrees of freedom.

$$P_{(\theta_1,\theta_2,\theta_3)}\left((S_1,S_2,S_3)\in A_{k_3,k_2}\right)$$
$$= P_{(\theta_1,\theta_2,\theta_3)}\left(l^1_{k_3,k_2}(V_2,S_1)\le W_2\le l^2_{k_3,k_2}(V_2,S_1),(k_3+k_2)S_1<V_2,0<S_1\right)$$
$$= \int_0^\infty\int_{(k_3+k_2)s_1}^\infty\int_{l^1_{k_3,k_2}(v_2,s_1)}^{l^2_{k_3,k_2}(v_2,s_1)} h_{W_2|V_2}(w_2;\theta_3,\theta_2)\,dw_2 f_{V_2}(v_2)f_{S_1}(s_1)dv_2ds_1$$
$$\le \int_0^\infty\int_{(k_3+k_2)s_1}^\infty\int_{k_3,k_2(v_2,s_1)}^{l^2_{k_3,k_2}(v_2,s_1)} f_{2k_3,2k_2}(w_2)\,dw_2 f_{V_2}(v_2)f_{S_1}(s_1)dv_2ds_1$$
$$\le \int_0^\infty\int_0^\infty\int_{l^1_{k_3,k_2}(v_2,s_1)}^{l^2_{k_3,k_2}(v_2,s_1)} f_{2k_3,2k_2}(w_2)\,dw_2 f_{V_1}(v_1)f_{S_3}(s_3)dv_1ds_3$$
$$= \int_0^\infty\int_0^\infty\Big[F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2,s_1)\right)$$
$$- F_{2k_3,2k_2}\left(l^1_{k_3,k_2}(v_2,s_1)\right)\Big]dw_1 f_{V_1}(v_1)f_{S_3}(s_3)dv_1ds_3$$
(3.4)

The expression in brackets in (3.4) is clearly bounded above by α for $(k_3+k_2)s_1<v_2<\infty$ because
$F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2,s_1)\right)-F_{2k_3,2k_2}\left(l^1_{k_3,k_2}(v_2,s_1)\right)\le$
$F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2,s_1)\right)-F_{2k_3,2k_2}\left(F^{-1}_{2k_3,2k_2}\left(F_{2k_3,2k_2}\left(l^2_{k_3,k_2}(v_2,s_1)\right)-\alpha\right)\right)=\alpha$ □

We now define test ϕ_{k_3,k_2,k_1}. In fact, we can define the whole family of tests indexed by constant d, where $0<d<1$; this appears in Definitions 2.4 and 2.5.

Definition 3.2 *Consider the testing problem (3.2). For any* α *that satisfies* $0<\alpha<0.4$ *and* $0<d<1$, *we define* ϕ_{k_3,k_2,k_1} *as the test that rejects* H_0 *if* $\mathbf{S}=R_1\cap R_2$, *where* $R_1=\{\mathbf{s}:(s_1,s_2,s_3)\in A_{k_2,k_1}\}$ *and* $R_2=\{\mathbf{s}:(s_1,s_2,s_3)\in A_{k_3,k_2}\}$.

The following lemma shows that the rejection region of the LRT is a subset of that of ϕ_{k_3,k_2,k_1}.

Lemma 3.3 *Consider the testing problem (3.1) and* $0<\alpha<0.4$. *Let* $R_L=\{\boldsymbol{s}:(s_1,s_2,s_3)\in L_{321}\}$ *be the rejection region of the size-*α *LRT. Then* $R_L\subset R_i$, $i=1,2$. *Hence, the rejection region of* ϕ_{k_3,k_2,k_1}, $R_1\cap R_2$, *contains* R_L.

Proof. For v_1, w_1, and s_3, the set L_{k_3,k_2,k_1} is expressed as (2.1). From Definition 2.4, for $0<v_1<k_2c_{s_3}$, $l^2_{k_2,k_1}(v_1,s_3)=\infty$ and $l^1_{k_2,k_1}(v_1,s_3)=f_{\alpha,2k_2,2k_1}$. For $k_2c_{s_3}\le v_1<(k_2+k_1/f_{\alpha,2k_2,2k_1})c_{s_3}$, $l^2_{k_2,k_1}(v_1,s_3)>\frac{k_1}{v_1/c_{s_3}-k_2}$ and $f_{\alpha,2k_2,2k_1}>l^1_{k_2,k_1}(v_1,s_3)$. Hence $L_{k_3,k_2,k_1}\subset R_1$. For v_2, w_2, and s_1, the set L_{k_3,k_2,k_1} is expressed as (2.2). From Definition 2.5, for $(k_3f_{\alpha,2k_3,2k_2}+k_2)c_{s_1}\le v_2$,

$l^1_{k_3,k_2}(v_2, s_1) > \frac{v_2/c_{s_1} - k_2}{k_3}$ and $f_{\alpha,2k_3,2k_2} > l^2_{k_3,k_2}(v_2, s_1)$. and Hence, $L_{k_3,k_2,k_1} \subset R_2$. Since $L_{k_3,k_2,k_1} \subset R_i$, $i = 1,2$, the rejection region of ϕ_{k_3,k_2,k_1}, $R_1 \cap R_2$, contains R_L. □

The following theorem shows that ϕ_{k_3,k_2,k_1} is a size-α test and uniformly more powerful than the LRT.

Theorem 3.1 *Consider the testing problem (3.2). When* $0 < \alpha < 0.4$, ϕ_{k_3,k_2,k_1} *has the size of exactly* α, *and* ϕ_{k_3,k_2,k_1} *is uniformly more powerful than the size-*α *LRT.*

Proof. From Lemma 3.3, we know that the rejection region of the *size-*α LRT, R_L, is a subset of the rejection region of ϕ_{k_3,k_2,k_1}. Hence, ϕ_{k_3,k_2,k_1} is uniformly more powerful than the *size-*α LRT. Also,

$$\text{the size of } \phi_{k_3,k_2,k_1} \geq \text{the size of LRT} = \alpha. \quad (3.5)$$

For $\theta_2 \leq \theta_1$ and $\theta_3 > 0$,

$$P_{(\theta_1,\theta_2,\theta_3)}(\mathbf{S} \in R_1 \cap R_2) \leq P_{(\theta_1,\theta_2,\theta_3)}(\mathbf{S} \in R_1) = P_{(\theta_1,\theta_2,\theta_3)}\left(\mathbf{S} \in A_{k_2,k_1}\right) \leq \alpha. \quad (3.6)$$

The last inequality in (3.6) is based on Lemma 3.1. In addition, for $\theta_3 \leq \theta_2$ *and* $\theta_1 > 0$,

$$P_{(\theta_1,\theta_2,\theta_3)}(\mathbf{S} \in R_1 \cap R_2) \leq P_{(\theta_1,\theta_2,\theta_3)}(\mathbf{S} \in R_2) = P_{(\theta_1,\theta_2,\theta_3)}\left(\mathbf{S} \in A_{k_3,k_2}\right) \leq \alpha. \quad (3.7)$$

The last inequality in (3.7) is based on Lemma 3.1. Because (3.6) and (3.7) are true for any $(\theta_1, \theta_2, \theta_3) \in H_0$, the size of ϕ_{k_3,k_2,k_1} is less than or equal to α. Along with (3.5), this implies that the size of ϕ_{k_3,k_2,k_1} is exactly equal to α. □

Figure 3 shows an example of the rejection region of ϕ_{321} for $s_3 = 3$. The example shown is for $\alpha = 0.1$, $k_1 = k_2 = k_3 = 5$, and $d = 1/2$ when $s_3 = 3$. In Figure 3, the solid lines represent $l^1_{k_2,k_1}(v_1, s_3)$ and $l^2_{k_2,k_1}(v_1, s_3)$, and the area surrounded by the two lines is R_1 with given s_3. The dotted lines represent $l^1_{k_3,k_2}(v_2, s_1)$ and $l^2_{k_3,k_2}(v_2, s_1)$, and the area surrounded by the two lines is R_2 with given s_3. The intersection of the two areas is the rejection region of ϕ_{k_3,k_2,k_1} with given s_3, and it will become a cube as s_3 moves from 0 to ∞. For testing problem (3.2), numerical results regarding the power of the two tests, the LRT and ϕ_{k_3,k_2,k_1}, are provided in Table 1.

In this case, we select $\alpha = 0.1$, $d = 1/2$, and $k_1 = k_2 = k_3 = 10$, and let $\delta_1 = \theta_2/\theta_1$ and $\delta_2 = \theta_3/\theta_2$. Further, $\beta_L(\delta_1, \delta_2)$ and $\beta_N(\delta_1, \delta_2)$ denote the power functions of the LRT and ϕ_{k_3,k_2,k_1}, respectively. The first two rows of Table 1 lists the values of $(\delta_1, \delta_2) = (\delta_1, 1)$. These values represent the boundary points for $\theta_3 = \theta_2$ and $\theta_1 > 0$, and the computed powers are lesser than 0.1. These values imply that both the two tests are biased, and the difference between the powers of ϕ_{k_3,k_2,k_1} and α is considerably smaller than that between the powers of the LRT and α. Test ϕ_{k_3,k_2,k_1} improves greatly on the LRT; $\beta_N(0.9,1) = 0.0352$ and $\beta_L(0.9,1) = 0.0004$. The rest of the table provides powers for different values of (δ_1,δ_2). $\delta_1 = 1$ represents the boundary points for $\theta_2 = \theta_1$ and $\theta_3 > 0$; the values of the boundary points are shown in boldface. A larger improvement in the power of ϕ_{k_3,k_2,k_1} occurs for lower values of (δ_1,δ_2) such as $(0.9,1)$ or $(0.9,0.9)$. It is noteworthy that when $(\delta_1,\delta_2) = (1.5,1.5)$, the power of the LRT does not exceed 0.1, whereas that of ϕ_{k_3,k_2,k_1} does. A comparison of the numerical results on the power of the two tests reveals that ϕ_{k_3,k_2,k_1} is uniformly more powerful than the LRT.

4. More powerful test in general problem

Considering testing problem (1.1) with $0 < \alpha < 0.4$, we describe a size-α test that is uniformly more powerful than the size-α LRT. We denote this test as ϕ_g. The intersection-union method is used to construct ϕ_g. A summary of this method may be found in Berger (1982), Liu and Berger (1995), Berger (1997), and Saikali and Berger (2002).

To use the intersection-union method, the testing problem (1.1) is expressed as

$$H_0: \cup_{i=1}^{p^*} H_{i0} \text{ versus } H_1: \cap_{i=1}^{p^*} H_{i1}$$

where $p^* = (p-1)/2$ if p is odd and $p^* = p/2$ if p is even. If p is odd, $H_{i0}: \theta_{2i} \le \theta_{2i-1}$ or $\theta_{2i+1} \le \theta_{2i}$ versus $H_{i1}: \theta_{2i} > \theta_{2i-1}$ and $\theta_{2i+1} > \theta_{2i}$, $i = 1, \ldots, p^*$; there exists only one expression for representing (1.1). However, when p is even, there exists several ways to represent (1.1); one of them is $H_{i0}: \theta_{2i} \le \theta_{2i-1}$ or $\theta_{2i+1} \le \theta_{2i}$ versus $H_{i1}: \theta_{2i} > \theta_{2i-1}$ and $\theta_{2i+1} > \theta_{2i}$, $i = 1, \ldots, p^* - 1$, and $H_{p^*0}: \theta_{p-1} \le \theta_{p-2}$ or $\theta_p \le \theta_{p-1}$, versus $H_{p^*1}: \theta_{p-1} > \theta_{p-2}$ and $\theta_p > \theta_{p-1}$. Note that p^* is the total number of any divisions of the indices $\{1, \ldots, p\}$ into the minimal number of subset of size three such that each value $1, \ldots, p$ appears at least once. To construct a uniformly more powerful test, any such division of $\{1, \ldots, p\}$ will work, but different divisions lead to different tests especially if p is even.

Consider testing H_{i0} versus H_{i1} for $i = 1, \ldots, p^*$. Let C_i denote the size-α rejection region of $\phi_{k_{2i+1},k_{2i},k_{2i-1}}$ (for some d) from Section 3. Because $H_0: \cup_{i=1}^{p^*} H_{i0}$, we can define an IUT based on the C_i.

Here $\mathbf{S} = (S_1, \ldots, S_p)$ is denoted as a p-dimensional random vector where $S_i = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / k_i$, $i = 1, \ldots, p$, and $k_i = \tau_i n_i$.

Definition 4.1 *For the testing problem (1.1) and* $0 < \alpha < 0.4$, *let* ϕ_g *be the test that rejects* H_0 *if* $\mathbf{S} \in \cap_{i=1}^{p} C_i$.

Theorem 4.1 *For* $0 < \alpha < 0.4$, *the test* ϕ_g *is a size-*α *test of* H_0

versus H_1, *and is uniformly more powerful than the size-*α *LRT.*

Proof. Because each C_i is a *size-*α rejection region of for testing H_{i0}, by Theorem 1 in Berger (1982), the size of ϕ_g is $\leq \alpha$. The rejection region of the size-α LRT is

$R_L = \left\{\mathbf{s}: \frac{s_{i+1}}{s_i} \geq f_{\alpha,2k_{i+1},2k_i}, i = 1, \dots, p-1\right\} \subset \left\{\mathbf{s}: \frac{s_2}{s_1} \geq f_{\alpha,2k_2,2k_1}, \frac{s_3}{s_2} \geq f_{\alpha,2k_3,2k_2}\right\} \subset C_i,$

for $i = 1, \dots, p^*$. Hence, R_L is contained in the rejection region of ϕ_g, the size of $\phi_g \geq$ the size of the LRT α, and ϕ_g is uniformly more powerful than the LRT. □

5. Application to two distributions and an illustrative example

For a simple-order testing problem of normal variances stated as (1.1), we construct a test that is uniformly more powerful than the LRT. In this section, by using the same technique, we can easily construct a uniformly more powerful test for a simple-order testing problem of the scale parameters in two-parameter exponential distributions and normal distributions, respectively.

5.1 Two-parameter exponential distributions

Consider $X_{i1}, \dots, X_{in_i}$ follow a two-parameter exponential distribution with pdf

$$f(x_{ij}) = \frac{1}{\theta_i} e^{(x_{ij}-\mu_i)/\theta_i}, x_{ij} \geq \mu_i,$$

where μ_i and θ_i are location and scale parameter, respectively, for $i = 1, \dots, p$ and $j = 1, \dots, n_i$. The distribution is the case of gamma distribution with $\tau_i = 1$ and a location parameter μ_i is considered in the distribution for $i = 1, \dots, p$.

Let $S_i = \sum_{j=1}^{n_i}(X_{ij} - Y_i)/(n_i - 1)$, where $Y_i = \min\{X_{i1}, \dots, X_{in_i}\}$ for $i = 1, \dots, p$. Then $2(n_i - 1)S_i/\theta_i$, $i = 1, \dots, p$, follow a central Chi-square distribution with $2(n_i - 1)$ degrees of freedom because $(n_i - 1)S_i/\theta_i$, $i = 1, \dots, p$ follow a gamma distribution with shape parameter $n_i - 1$ and scale parameter unity from Sinha (1986). From Li and Sinha (1995), the LRT of the testing problem (1.1) can be modified and the test that rejects H_0 if

$$\frac{S_{i+1}}{S_{1i}} \geq f_{\alpha,2(n_{i+1}-1),2(n_i-1),}$$

for $i = 1, \dots, p-1$. Therefore, a more power test ϕ_g can be constructed from Section 2 and 3 for the testing problem (3.2). Subsequently, we illustrate the test ϕ_g on a data which following a two-parameter exponential distribution. An illustrative example is given in subsection 5.3.

5.2 An illustrative example

The following data, taken from Proschan (1963), represents time of successive failures of the air-conditioning system of each member of a fleet of Boeing 720 jet airplanes. For each airplane, the interval between their successive failures was shown to follow a two-parameter exponential distribution, where the location parameter μ_i is guarantee time of the successive failures and the scale parameter θ_i is the expected mean time in addition to guarantee time, $i = 1,2,3$. The interval data are listed below in the order of occurrence *w.r.t.* planes 7908, 7914, and 8044. For plane 7908, it is taken by last ten observations.

Plane 7908 (μ_1 and θ_1) : 34, 31, 18, 18, 67, 57, 62, 7, 22, 34;

Plane 7914 (μ_2 and θ_2) : 50, 44, 102, 72, 22, 39, 3, 15, 197, 188, 79, 88, 46, 5, 5, 36, 22, 139, 210, 91, 30, 23, 13, 14;

Plane 8044 (μ_3 and θ_3) : 487, 18, 100, 7, 98, 5, 85, 91, 43, 230, 3, 130;

Considering the testing problem (5.1) when $p = 3$, the test $\phi_{18,46,22}$ can be constructed. It shows that the null hypothesis is not rejected by the LRT, but is rejected by test $\phi_{18,46,22}$. The respective computed values are $s_2/s_1 = 2.1741$, $s_3/s_2 = 1.7262$, $v_1 = 3304.25$, $v_2 = 5112.083$, $l^1_{21\cdot3}(v_1, s_3) = 1.8212$, $l^2_{21\cdot3}(v_1, s_3) = 2.2282$, $l^1_{32\cdot1}(v_2, s_1) = 1.7061$, $l^2_{32\cdot1}(v_2, s_1) = 2.1725$, $f_{0.05,46,22} = 2.0450$, and $f_{0.05,22,46} = 1.778$. Thus, we reject H_0 because of $l^1_{21\cdot3}(v_1, s_3) < 2.1741 < l^{12}_{21\cdot3}(v_1, s_3)$ and $l^1_{32\cdot1}(v_2, s_1) < 1.7262 < l^2_{32\cdot1}(v_2, s_1)$.

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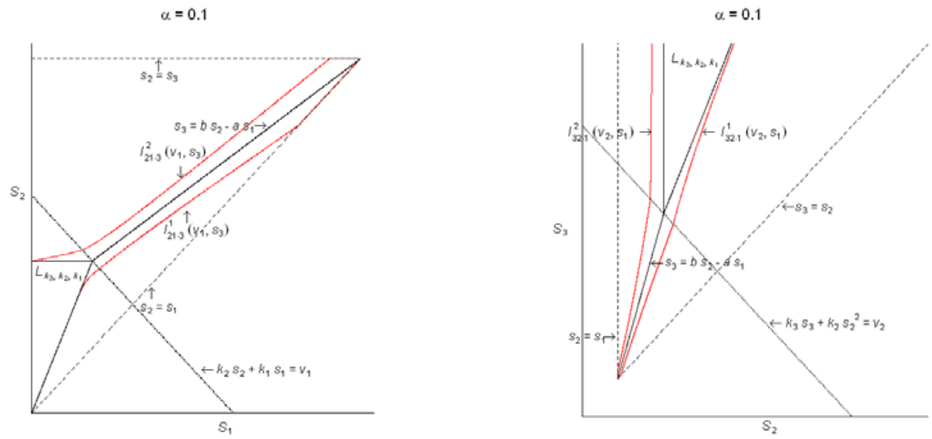
Table 1. Power of the LRT, and ϕ_{321} for $p = 3$, $\alpha = 0.1$, $d = 1/2$and $k_1 = k_2 = k_3 = 10$ with $\delta_1 = \theta_2/\theta_1$ and $\delta_2 = \theta_3/\theta_2$.

	0.9	0.95	1	1.5
$\beta_L(\delta_1, 1)$	**0.0004**	**0.0006**	**0.0008**	**0.0065**
$\beta_N(\delta_1, 1)$	**0.0352**	**0.0397**	**0.0440**	**0.0692**
$\beta_L(\delta_1, \delta_1)$	0.0002	0.0004	**0.0008**	0.0531
$\beta_N(\delta_1, \delta_1)$	0.0266	0.0350	**0.0440**	0.1390
$\beta_L(\delta_1, 1.4142\delta_1)$	0.0021	0.0039	**0.0068**	0.1539
$\beta_N(\delta_1, 1.4142\delta_1)$	0.0549	0.0652	**0.0753**	0.2172
$\beta_L(\delta_1, 2.4142\delta_1)$	0.0216	0.0323	**0.0459**	0.3038
$\beta_N(\delta_1, 2.4142)$	0.0715	0.0820	**0.0942**	0.3170
$\beta_L(\delta_1, 10\delta_1)$	0.0656	0.0818	**0.0999**	0.3465
$\beta_N(\delta_1, 10\delta_1)$	0.0657	0.0819	**0.1000**	0.3465



(a) $s_3 = 2$ (b) $s_1 = 1$

Figure 1. (a) The sets L_{k_3,k_2,k_1} and $A_{21\cdot3}$ and functions $s_3 = bs_2 - abs_1$, $k_2s_2 + k_1s_1 = v_1$, $l^1_{21\cdot3}(v_1, s_3)$ and $l^2_{21\cdot3}(v_1, s_3)$ for $\alpha = 0.1$, $k_1 = k_2 = k_3 = 5$, and $d = 1/2$ when $s_3 = 2$; (b) the sets L_{k_3,k_2,k_1} and $A_{32\cdot1}$ and functions $s_3 = bs_2 - abs_1$, $k_3s_3 + k_2s_2 = v_2$, $l^1_{32\cdot1}(v_2, s_1)$, and $l^2_{32\cdot1}(v_2, s_1)$ for $\alpha = 0.1$, $k_1 = k_2 = k_3 = 5$, and $d = 1/2$ when $s_1 = 1$.

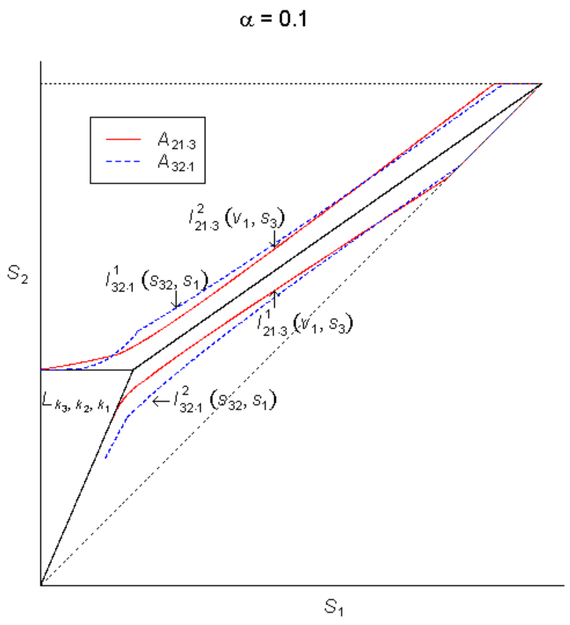


Figure 2. The rejection region of ϕ_{k_3,k_2,k_1} for $\alpha = 0.1$, $k_1 = k_2 = k_3 = 5$, and $d = 1/2$ when $s_3 = 3$.

科技部補助計畫衍生研發成果推廣資料表

日期:2016/01/01

科技部補助計畫	計畫名稱: 檢定	
	計畫主持人: 劉惠美	
	計畫編號: 103-2633-M-004-001-	學門領域: 數理統計與機率

無研發成果推廣資料

103年度專題研究計畫研究成果彙整表

計畫主持人：劉惠美			計畫編號：103-2633-M-004-001-				
計畫名稱：檢定							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數（含實際已達成數）	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	1	1	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	2	2	100%	人次	
		博士生	1	0	0%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
其他成果（無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）		無					

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

科技部補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以100字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：□已發表 □未發表之文稿 ■撰寫中 □無

   專利：□已獲得 □申請中 ■無

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   其他：（以100字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以500字為限）

   提出創新的檢定，比最大概似比檢定的檢力大。正在撰寫修飾投稿至Annals of Stat.