Looking for the magic number: the optimal district magnitude for political parties in d'Hondt PR and SNTV

J.-W. Lin *

Sun Yat-sen Institute for Social Sciences and Philosophy, Academia Sinica, 128 Academia Sinica Road, Section 2, Nankang, Taipei 115, Taiwan

Accepted 26 February 2001

Abstract

District magnitude is regarded by many as the principal dimension that spans the classification of electoral systems. It is believed that larger parties prefer smaller district magnitudes and vice versa. Problems arise when one tries to be exact: how large must a party be for the single-member district system to be its most favorable choice? Will any party find a particular magnitude most preferable? This article extends existing theories of effective thresholds and proposes a seat–vote equation different from the cube law. With reasonable assumptions, I demonstrate that a certain district magnitude maximizes the expected seat share of a particular median-sized party in elections using the d'Hondt PR or SNTV formulae. The validity of this threshold model is verified by an empirical study on recent elections in Finland and Taiwan.

Keywords: D'Hondt PR; District magnitude; Effective threshold; Seat–vote equation; SNTV

"Well, I should like to be a *little* larger, Sir, if you wouldn't mind," said Alice. (Lewis Carroll, *Alice's Adventures in Wonderland*)

1. Introduction

The seat–vote relationship and the impact of district magnitude have been two themes central to the study of electoral systems. The seat–vote relationship indicates how many votes it takes to win a seat, or how many seats can be obtained given a particular vote share won by a political party. District magnitude, the number of seats to be elected in a district, is in fact a major variable in the seat–vote function. It is well known that larger district magnitude increases the proportionality of the seat-vote relationship (Taagepera and Shugart, 1989, p. 113), and that the seat bonus of large parties decreases as district magnitude increases (Sartori, 1968).

District magnitude is such a decisive factor that some scholars see it as the principal dimension that spans other methods of classifying electoral systems. Sartori (1968), for example, puts the single-member district system and large district proportional representation (PR) systems on the same continuum by considering their district magnitudes. Cox (1991) has suggested that plurality systems can be further distinguished by their district magnitudes, and has demonstrated the 'equivalence' between the single non-transferable vote (SNTV) system and the d'Hondt PR system.[1]

Since the proportionality of electoral systems goes up with district magnitude, it is generally believed that large parties prefer smaller district magnitudes and vice versa. The problem comes when one tries to be exact. How large must a party be to make the single-member district system its first preference? Parties of what size will find the ideal PR system most favorable?[2] Fundamentally, can we predict a party's ideal choice of electoral systems by knowing its size? Is a party either going to prefer the single-member district system or the ideal PR system? Are 'median-sized' systems always the result of a compromise?[3]

A key to these questions lies in the optimal district magnitude for political parties of various sizes, even though many other variables also matter. Formally put, district magnitude m is optimal for a party with vote share v if these two variables render an expected seat share s that cannot be further increased by changing m. Our major task is therefore to establish a function $s(m, v)$ and examine whether a maximum exists for s.

Without doubt, efforts have been made to build a seat–vote equation containing these variables. In the next section, I first review the famous cube law that specifies a particular relationship between s, v and m. After pointing out the limits of the cube

[1] Not all scholars accept this continuum. Nohlen (1984), for example, argues that plurality and PR are two incompatible principles of representation.

[2] Under the ideal PR system, the seat share equals the vote share. Such a system rarely exists in reality, but can nevertheless be the ideal system for a party.

[3] In reality, district magnitude is only one of the factors that determine seat allocation. The same district magnitude can produce different seat allocations when the division of votes or the number of candidates vary. A party calculating its optimal district magnitude can only make reasonable *assumptions* about these factors, because the election is yet to take place. This article assumes rational nomination and captures other factors by a variable λ. Even this variable can be given an exact estimate under some assumptions.

law, I propose a new model in the subsequent section. A test of the model using the electoral data in Finland and Taiwan will also be provided. I then use this seat–vote equation to find the conditions under which a maximum for s exists. On the basis of these analyses, the concluding section will address some important issues in electoral reform.

2. The cube law revisited

Formulated originally to predict the seat–vote relationship in two-party elections, the cube law is perhaps the most well-known seat–vote equation. The name cube law originates from the equation $s_K/s_L=(v_K/v_L)^3$, where K and L stand for two different parties. The model has been modified by many scholars, among which Taagepera (1986) gives the most general formulation.[4] In the Taagepera model, the seat share of party K in multi-seat PR elections will be:

$$s_K = v_K^n/\Sigma v_i^n, \tag{1}$$

where $n=(\log V/\log DM)^{1/M}$, V, D and M designate respectively the total number of votes, the total number of districts, and the district magnitude.

Reed (1996) has demonstrated that this model predicts the Japanese election admirably well. Apart from its empirical achievement, however, the model itself raises several problems to be discussed. First, the model is rather complicated. With x parties running, x+3 variables are needed to predict a party's seat share.[5] Second, a link is missing between the cube law and some other important aspects of electoral studies. On the one hand, the cube law does not specify any behavioral assumptions because it adopts the "physics-style" approach. A theory without actors as such would invite some criticism, especially from the rational choice approach (see Reed, 1996 for example). On the other hand, the cube law does not incorporate the concept of threshold, on which numerous theoretical and statistical studies on the seat–vote relationship are based. These studies have defined the threshold of inclusion (representation) as the minimum vote share that can earn a party a seat, and the threshold of exclusion as the maximum vote share that may be insufficient to win a seat under the most unfavorable conditions (Lijphart, 1994, p. 25). With m seats to be elected in a district, the seat share of a party is 0 if its vote share is below the threshold of inclusion, and is at least $1/m$ if its vote share is above the threshold of exclusion. Eq. (1), however, implies that $s_K=0$ if and only if $v_K=0$, manifesting no threshold effect.[6]

Discussions above suggest what an improved seat–vote equation should look like.

---

[4] See also Taagepera and Shugart, 1989, pp. 156–172, for more detailed discussions.

[5] Or x+2 variables when the formula is applied to a single district (D=1). Taagepera further reduces the equation into $s_K=v_K^n/[(v_K^n+(N-1)^{1-n}(1-v_K)^n]$, where N is the effective number of parties, by assuming that party K faces $N-1$ other parties of equal size.

[6] This is a mathematical statement. In reality, a party still wins no seat if $v_K>0$ but $ms_K<0.5$ when we round off the number of seats to the nearest integer.

First, the calculation of expected seats should be based on behavior assumptions about the parties or candidates. Second, the model should reveal the threshold effect of electoral systems. Third, the model should employ the minimum number of variables and remain applicable to the maximum number of cases. If possible, it should use variables at the district level to help achieve this goal and prevent the ecological fallacy from happening.[7] The next section attempts to build such a model, and will compare its predictive power with that of the cube law. Whether the model implies an optimal district magnitude can then be studied.

3. The threshold model

The following seat–vote equation embodies the preceding ideas in the most straightforward way:

$$s_i(m,v_i) = \begin{cases} 0 \quad \text{if } v_i \leq t_1 \\ 0 < s_i < 1 \quad \text{if } t_1 < v_i < t_2, \\ 1 \quad \text{if } v_i \geq t_2 \end{cases} \tag{2}$$

where s_i and v_i indicate the seat share and vote share of party i in a district of magnitude m, and t_1 and t_2 denote the threshold of inclusion and the exclusion threshold for winning *all* seats.[8] Accordingly, the seat share below t_1 and above t_2 is $0/m=0$ and $m/m=1$. With a vote share in between, a party has *some* chance to win $1 \leq k < m$ seat(s), creating a seat share of $0 < k/m < 1$. The particular seat–vote equation thus derived will be named the *threshold model* to characterize its foundation.

The reader should be reminded that t_2 is a generalization of what is commonly called the threshold of exclusion, above which a party is guaranteed to win *one* seat. We can specify the functional form of the threshold model as soon as the thresholds of inclusion and exclusion are identified. The present article will focus on electoral systems in which the exclusion threshold is $1/(m+1)$ (Grofman, 1975; Lijphart et al., 1986; Taagepera and Shugart, 1989). A notable system that has this threshold is the d'Hondt PR (Rae et al., 1971; Rae, 1971).[9] The single nontransferable vote (SNTV) can have the same threshold if political parties seek to maximize their expected seat share, have precise expectations of the distribution of vote support, and can allocate their total vote equally among their nominees (Cox, 1991, p. 121). The parties must also nominate the optimal number of candidates to keep their votes from being wasted (Cox and Rosenbluth, 1994). For SNTV, these assumptions furnish the behavioral foundation of the threshold model and yield outcomes of seat allocation

[7] Sankoff and Mellos (1972), for instance, criticize a nation-based model for tending to overlook the concentration of votes.

[8] Henceforward, small letters are used to designate variables at the district level.

[9] Grofman (1999, p. 319) suggests the same exclusion threshold for the single transferable vote.

that are in equilibrium: because the seat–vote function is based on the optimal vote division, no party can move from that state and improve its electoral performance. The model to be characterized below is thus not mechanical, but founded on the premises of rational choice.

4. The exclusion threshold for winning all seats

For a variety of electoral systems, and most notably for the d'Hondt PR, the exclusion threshold for winning k seats is $k/(m+1)$ (Rae, 1971, p. 193; Lijphart, 1994, p. 26). It can be further extended to define the exclusion threshold for winning *all* seats:

$$t_2 = \frac{m}{m+1}. \tag{3}$$

In a single-member district election (m=1), winning half of the total votes (t_2=0.5) assures victory. The threshold approaches 1 when m becomes very large, confirming the intuition that it is more difficult for a party to monopolize all seats when district magnitude increases.

It should be emphasized that t_2 is simply the threshold across which a party takes all seats *for sure*. In reality, it may require fewer votes for a party to win all seats. However, we do not expect such a thing to happen *all* the time. The average seat share to be expected when $v<t_2$ is therefore less than 1.

5. The inclusion threshold

In contrast to t_2, which depends solely on district magnitude, t_1 is harder to define. Also termed the threshold of representation, t_1 is the minimum vote share necessary to earn a party its first seat, based on the most favorable condition in terms of how the other parties divide their votes (Lijphart, 1994, p. 25). In theory, the inclusion threshold cannot be higher than the exclusion threshold, but can be as low as 0 when all losing parties (candidates) receive no vote at all. Still, many scholars have worked on various formulae to give an estimate of the most likely threshold above which a party can win its first seat. For example, Rokkan (1968, p. 13) has proposed $1/(m+n-1)$ (n being the number of parties running in a district) as the threshold of inclusion for the d'Hondt PR.[10] Taagepera and Shugart (1989, p. 117) have picked 50%/m and call it the *average threshold*. Lijphart (1994) has calculated $\frac{50\%}{m+1} + \frac{50\%}{2m}$ as the *effective threshold* and later (1997, p.74) modified it to 75%/(m+1).

[10] The same inclusion threshold is suggested by Taagepera (1998, p. 406) for the district level.

All these thresholds are functions of district magnitude, the key variable that this article studies. They differ from each other because of the intervening variables such as the number of competitors, the distribution of vote, and nomination strategy (for SNTV). To be parsimonious but general, I define the inclusion threshold as:

$$t_1 = \lambda/m, \tag{4}$$

where λ captures all the intervening factors. Thus defined, t_2 is lowered by the decrease of λ or the increase of m. In accordance with other characterizations of the inclusion threshold, the second order derivative of m to t_1 is positive in Eq. (4). Since the inclusion threshold cannot be higher than the exclusion threshold, $0 \leq \lambda/m \leq 1/(m+1)$ and thus $0 \leq \lambda \leq m//(m+1) < 1$. It also follows that $t_1 \longrightarrow 0$ as $m \longrightarrow +\infty$.

While the negative impact of district magnitude on the inclusion threshold is well known, λ embodies factors that are exogenous to the electoral system, and can be estimated through empirical investigations. The value of λ can also be theoretically interesting. For instance, $\lambda=0.5$ if we apply Rokkan's inclusion threshold and make the Duvergerian assumption that $n=m+1$ (Reed, 1990; Cox, 1994). Still, the variance of λ does not change the prediction of s very much as long as the focus is on multi-seat elections.[11] It is m that stands out as the most determining variable in the seat–vote equation.

6. The seat–vote equation

As indicated by Eq. (2), the threshold model requires that $s=0$ when $v=t_1$, $s=1$ when $v=t_2$, and $0<s<1$ when $t_1<v<t_2$. These conditions make the commonly used logistic function inapplicable: with s as the dependent variable, the range of the logistic function is (0, 1) when the domain is [0, 1]. The problem to be solved is therefore how to let $s(m, v)$ pass two points $(t_1, 0)$ and $(t_2, 1)$, with the two thresholds characterized as in Eqs. (3) and (4).

I take the straightforward assumption that the function connecting $(t_1, 0)$ and $(t_2, 1)$ is linear, and give two justifications. First, no study so far has demonstrated that a particular non-linear function describes the reality better. Even if it is to be used, we have no clue to determine the shape of the non-linear function. Second, a linear function is a safe approximation of whatever non-linear relationships there might be: we do know that s increases with v, even though the speed may not be constant.

On the basis of the linear assumption and the thresholds specified above, I plug in two points $(\lambda/m, 0)$ and $\left(\frac{m}{m+1}, 1\right)$ into a linear function $s=\alpha+\beta v$ and rewrite s as a function of m and v. The result is that, for $v \in \left(\lambda/\text{m}, \frac{m}{m+1}\right)$,

[11] In a five seat district, for example, $t_1=0.08$ if $\lambda=0.4$ and $t_1=0.12$ if $\lambda=0.6$. Such a shift is big enough for λ, in contrast to the small increase of t_1. In general, the impact of λ shrinks as m increases.

$$s(m,v) = \frac{\lambda m - vm^2 + \lambda - mv}{\lambda m + \lambda - m^2}, \tag{5}$$

Two properties of Eq. (5) are worth exploring. First, the negative relationship between λ and s indicates that the decrease of λ makes it easier for all parties to win their first seat. Nonetheless, since the total number of seats remains unchanged, the lowering of the inclusion threshold brings more benefit to the small parties than to the larger ones. Second, it can be shown that, for all vote shares between the two thresholds, the threshold model predicts a seat share that is never less than what a party is guaranteed to gain. To see this, suppose $v=k/(m+1)$ and demonstrate that $s(m,\ v) \geq k/m$. Since $s\left(m, \frac{k}{m+1}\right) = \frac{\lambda m + \lambda - mk}{\lambda m + \lambda - m^2}$ and $k \leq m$, it is easy to see that $s\left(m, \frac{k}{m+1}\right) - \frac{k}{m} \geq 0$. Therefore, although the model is built upon two thresholds, the predictions in between do not violate the basic assumptions.

With Eq. (5), we are ready to examine the relationship between m and s. As this is a complicated issue that deserves special attention, I will leave it for a separate section. Before exploring its theoretical implications, it will be helpful to inspect how well the threshold model explains the seat–vote relationship in the real world.

7. Empirical test for the threshold model

Eq. (5) cannot only be manipulated to derive the optimal district magnitude, but also used to predict a party's seat share given its vote share and the district magnitude. The discrepancy between prediction and reality then tells the fitness of the threshold model. This section selects two recent elections to fulfill this task. The first is the Finnish parliamentary election of 1999, where d'Hondt PR is used to allocate 200 seats in the 15 constituencies (average magnitude=13.3). The second case is Taiwan's Legislative Yuan election of 1998, which employs SNTV in 29 districts for 168 seats (average magnitude=5.8). By their variant district magnitudes and the absence of legal threshold, these two cases illuminate well the impact of district magnitude on seat allocation.[12]

To validate the threshold model, it is not enough to find an insignificant probability of the specified variables being unrelated to the dependent variable. Instead, we must demonstrate how the model, as defined by Eqs. (2) to (5), predicts the actual seat share won by major political parties in the electoral districts.[13] The method that serves this purpose is not the standard significance test, but a straightforward comparison between the actual and predicted seat shares.

[12] District magnitudes in Japan, for example, are usually less than five and thus much less variant.

[13] The purpose of the following test is to show how to operationalize the model, and that it actually works in two cases embedded in very different political environments. More cases have to be included to reduce the selection bias of the test.

In addition to the percentages of correct predictions, a scatterplot of the actual and predicted seat shares can indicate the sources of missed guesses. For instance, the threshold model underestimates the parties' seat-gaining capacities if the predicted seat shares lie below the line of a perfect prediction. To give a more exact measurement of the goodness of the fit, I calculate the bivariate correlation coefficients between the actual seat share (s_{actual}) and the predicted seat share ($s_{\text{predicted}}$). The model yields a perfect prediction if, in the equation $s_{\text{actual}}=\alpha+\beta s_{\text{predicted}}$, $\alpha=0$, $\beta=1$, and R-square=1. The variance and value of these coefficients disclose further the overall pattern generated by the threshold model.

The test is operationalized as follows. First, I set $\lambda=0.5$ and hence $t_1=1/2m$ to make seat share determined only by district magnitude and vote share. As explained already, this value has theoretical implications and marks the median of the possible values of λ. Most important, the reader can verify by conducting the same test that a slight adjustment of λ produces almost no change of the prediction. Second, in correspondence with Eq. (2), the seat share of party i is set to be 0 if $v_i \leq 1/2m$ and 1 if $v_i \geq \frac{m}{m+1}$. For a party in between, its vote share in a district and the magnitude of that district are plugged into Eq. (5) to render its seat share. The seat share is then multiplied by district magnitude and rounded off to the nearest integer to find the number of seats a party is expected to gain under the threshold model. To see whether the threshold model improves the cube law, I conduct the same test using Eq. (1) and setting $D=1$ for each district. The procedure is applied to nine parties in Finland and three parties in Taiwan in all electoral districts.[14]

Several remarks can be made about the testing results presented in Tables 1 and 2. For the Finnish case, both models predict considerably well, with the cube law model closer to the perfect fit. This is not a surprising result: the average district magnitude is much higher in Finland than in Taiwan, making $\lambda=0.5$ an overestimation for Finland. We can measure the actual inclusion thresholds for the Finnish elections and produce a much more accurate prediction. For the Taiwanese case, it

Table 1
Empirical tests of the threshold model and the cube law model, Finland 1999

	The threshold model	The cube law model
Intercept	0.008	−0.004
Slope	1.123	1.039
R-square (adjusted)	0.953	0.956
Number of cases	135	135
Number of correct predictions (percent)	98 (72.6%)	115 (85.2%)

[14] We can of course run the same test for the other minor parties in Finland and the independent candidates in Taiwan. But the result will not be significantly changed because their vote shares are generally too low to give them any seat.

Table 2
Empirical tests of the threshold model and the cube law model, Taiwan 1999

	The threshold model	The cube law model
Intercept	0.057	0.045
Slope	0.954	0.821
R-square (adjusted)	0.719	0.704
Number of cases	83	83
Number of correct predictions (percent)	60 (72.3%)	57 (68.7%)

is apparent that the threshold model predicts better. Although the R-squares in both models are almost identical, coefficients in the cube law model suggest that it has underestimated the seat share of leading parties. As for why both models work better for the Finnish case, two conjectures are plausible. First, due to its higher average magnitude, the Finnish system is quite proportional and produces a predictable seat share distribution. Second, political parties competing under SNTV must solve the vote division dilemma and nominate the optimal number of candidates. The threshold model has assumed rational nomination, which can be difficult to follow sometimes and bring unnecessary loss of seats. The cube law does not even make these assumptions.

It should be fair to conclude that the threshold model performs as successfully as the cube law by using much fewer variables. With the threshold model, a party can estimate its expected seat share in an electoral district by simply knowing its vote share. Other variables like λ can be fitted to yield a more accurate prediction, but the institutional variable m will prove to be more consequential. The threshold model is thus useful to the analysis of electoral system reform. Party i can compute the maximum of s_i as a function of m and v_i and determine its position on the selection of district magnitude. It is this issue that I now turn to.

8. The optimal district magnitude

With the threshold model affirmed by empirical test, we are ready to check whether it implies any optimal district magnitude for political parties. An alternative possibility could be that the expected seat share increases or decreases monotonically with the district magnitude, such that a party is either going to support the single-member district system or the ideal PR system, but nothing in between.[15]

The solution is a typical problem of optimization with constraints. Simply put, we are to examine the first and second order conditions of $s(m, v)$, i.e., to find the conditions under which m maximizes s given v and λ. The result is the following theorem:

[15] In such a case, the first (partial) derivative of s to m will never be zero.

THEOREM. *For* $m>1$ and $0<s_i<1$, $m* = \dfrac{\lambda(1-v_i) + \sqrt{\lambda^2(1-v_i)-\lambda v_i^2}}{v_i-\lambda(1-v_i)}$ *maximizes* $s_i(m, v_i)$ if $\dfrac{\lambda}{1+\lambda} < v_i < \dfrac{3\lambda}{1+4\lambda}$.

The calculation is given in Appendix A. In plain language, this theorem says that *a median-sized party will find a particular multi-member district its best choice*, if the median-sized parties are those with a vote share between $\dfrac{\lambda}{1+\lambda}$ and $\dfrac{3\lambda}{1+4\lambda}$. It can be demonstrated that s decreases monotonically with m when v is higher than $\dfrac{3\lambda}{1+4\lambda}$ and increases monotonically with m when v is lower than $\dfrac{\lambda}{1+\lambda}$. In the former case, a party should like the district magnitude to be as small as possible, while in the latter case the district magnitude should be as large as possible. When the size of a party is between the boundaries, $m*>1$ becomes its optimal district magnitude.

The theorem suggests that λ affects not only the existence and value of $m*$, but also the definition of a median-sized party. As illustrated in Fig. 1, the boundaries that define a median-sized party, to which $m*>1$ exists as its optimum, range between (0, 0) (when λ=0) and (0.5, 0.6) (when λ=1). Despite this variance, however, it is unlikely for λ to deviate from the median value (0.5) very much. Consider a typical case where λ should be *low*: suppose m=10 and t_1=0.05. Since λ/m =0.05, λ=0.5. λ is increased to 0.6 when t_1=0.06, and lowered to 0.4 when t_1=0.04.

In addition to the reasons already mentioned, other properties make λ=0.5 a case that deserves special attention. First, λ=0.5 implies that $t_1=1/2m$. This is equivalent

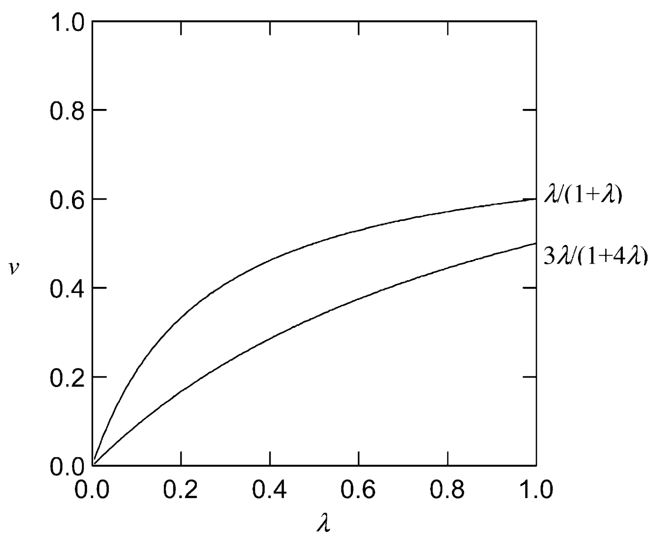

Fig. 1. The definitions of median-sized party as a function of λ.

to the ‘average threshold’ proposed by Taagepera and Shugart. According to the authors, regardless of district magnitude, the number of parties, and the allocation rules, the average of inclusion and exclusion thresholds is in most cases close to 50%/m=1/2m. Second, $\frac{\lambda}{1+\lambda}=0.33$ and $\frac{3\lambda}{1+4\lambda}=0.5$ when $\lambda=0.5$. These parameters are intuitively interesting: a ‘dominant’ party which grabs more than half of the votes will always find the single-member district its first preference, while a ‘small’ party which gains less 1/3 of the votes should want the district magnitude to be as large as possible. The ‘median-sized’ parties in between will find some ‘median-sized’ district the most favorable choice.

We can thus use $\lambda=0.5$ as a typical case to illustrate how parties of different sizes determine their optimal district magnitude (Fig. 2). For instance, with a vote share of 0.42, the expected seat share can be maximized to 0.44 when the district magnitude is 4.06 (or 4 in terms of the nearest integer). The expected seat share will be lowered to 0.41 when m is decreased to 2, and drops drastically when m is reduced to 1.

More generally, two interesting observations can be made from the theorem and Fig. 2. First, a district magnitude higher than the optimum is less harmful to the median-sized parties than one that is lower. The expected seat share approaches v when m increases, but suddenly slumps to 0 when m drops to 1. It is thus safer for a party with an unstable vote basis to run in a multi-member district than in a single-member district. Second, the preference of the median-sized parties over district magnitude tends to vacillate. According to the theorem, the optimal district magni-

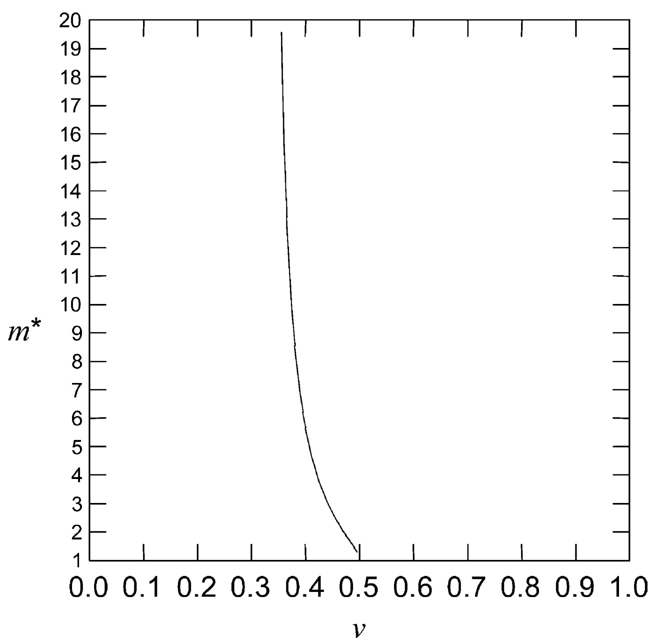

Fig. 2. Optimal district as a function of vote share (λ=0.5).

tude soars from 1 to infinity when v drops from $\frac{3\lambda}{1+4\lambda}$ to $\frac{\lambda}{1+\lambda}$. As shown in Fig. 1, the maximum difference between these two values is 0.179, when λ=0.323. A 10% vote swing can thus change a party's preference over district magnitude radically. Parties declining from a dominant position (i.e., v>0.5) will be especially sensitive to the adjustment of district magnitude. For smaller parties, it will be safer to stick to larger district magnitudes, even though a chance of upward swing exists.

9. Conclusion

This article proposes a model to link three central themes in the study of electoral systems: the seat–vote relationship, the threshold of representation, and the effects of district magnitude. My strategy is simply to build a new seat–vote equation on the basis of the existing proposals of effective thresholds. Through this model, I establish the conditions under which a party's seat share is maximized under a certain district magnitude. Fundamental to the threshold model is the assumption that a linear relationship between seat and vote share exists. I estimate the intercept and slope of this linear equation by finding the thresholds through which this line must pass. Based on the assumption of rational nomination, this threshold model can be further linked to the rational choice approach in electoral studies.

Unlike the cube law, the threshold model places more emphasis on the variance of district magnitude by taking each electoral district as the unit of analysis. The price to be paid for adopting this approach is that the threshold model can not be easily applied to comparative studies where the unit of analysis is a nation. For a similar reason, many other aspects of electoral system remain unaccounted for by this model. In particular, my definition of the exclusion threshold applies well to the SNTV system and the d'Hondt PR, but not necessarily to all multi-seat systems. These are all topics to be explored in the future.

Also on the agenda of further research are two related studies that this paper should have shed some light on. First, studies on redistricting usually ask whether redistricting reduces electoral responsiveness by protecting the incumbents or a particular party. In terms of the threshold model, the question is whether redistricting changes v_i for party i when everything else remains the same. What my model has pointed out is the *dual* effects of district magnitude adjustment: when m is changed, changes of v in the new districts must follow. Both will in turn affect the expected seat share of each party. Similarly, redistricting in multi-member district systems usually means the adjustment of district magnitude. This article thus proposes a new variable for the study of redistricting.

Second, the present study addresses an important issue for the choice of electoral systems: the relationship between the size of a party and its preference over district magnitude. The findings in this article go far beyond the common perception that larger parties prefer smaller district magnitudes. I have established two thresholds above and below which a party should find the single-member district system or a pure PR system the ideal choice. For median-sized parties, a particular district magnitude is most favorable. However, these parties are likely to be uncertain about the optimal magnitude, which shifts drastically with a slight change in a party's strength. The most interesting case to be studied is the ruling party whose typical vote share is roughly 50%. It is very likely that the vote share of a party can decline from 50 to 45% in a single election, but the optimal district magnitude for this party will be lowered from 1 to 3. When there is no dominant party (v>50%) in a single-member district system, the increase of district magnitude becomes an attractive proposal for all parties. In any event, the threshold model can be used to predict the mostly likely compromise on district magnitude as soon as the typical vote share of each party becomes stable.

Appendix A. Proof of the Theorem

Examining the first order condition in Eq. (5), we have:

$$\frac{\partial s}{\partial m} = \frac{-\lambda v m^2 - 2\lambda v m - \lambda v + \lambda m^2 - v m^2 + 2\lambda m}{(\lambda m + \lambda - m^2)^2}. \tag{A1}$$

$$\text{(i)} = 0 \Leftrightarrow \lambda m^2 + 2\lambda m - v\lambda m^2 - 2v\lambda m - \lambda v - v m^2 = 0. \tag{A2}$$

From Eq. (A2), $v = \frac{\lambda m(m+2)}{(1+\lambda)m^2 + 2\lambda m + \lambda}$. I shall call v and m satisfying this condition $v*$ and $m*$. It can be verified that $\frac{\partial v*}{\partial m*} < 0$. Since m>1, it follows that $v* < \frac{3\lambda}{1+4\lambda}$. Based on the same reason, $\lim_{m \to +\infty} v* = \frac{\lambda}{1+\lambda} < v*$.

Now examine the second order condition:

$$\frac{\partial^2 s}{\partial m^2} = \frac{-2\lambda v m^3 + 2\lambda m^3 - 2v m^3 + 6\lambda m^2 - 6\lambda v m^2 - 6\lambda v m + 2\lambda^2}{(\lambda m + \lambda - m^2)^3}. \tag{A3}$$

Since $m>1$ and $\lambda/m \leq 1/(m+1)$, $\lambda(m+1) < m < m^2$ and thus $(\lambda m + \lambda - m^2)^3 < 0$. Therefore, Eq. (A3)<0 if and only if

$$v < \frac{\lambda m^3 + 3\lambda m^2 + \lambda^2}{(1+\lambda)m^3 + 3\lambda m^2 + 3\lambda m}. \tag{A4}$$

It is easy to see that Eq. (A4) is always true as long as Eq. (A2) is satisfied. Accordingly, the test for the second order condition establishes that $s(v*, m*)$ is a local maximum with the given constraints, in so far as $\frac{\lambda}{1+\lambda} < v_i < \frac{3\lambda}{1+4\lambda}$.

With Eq. (A2), we can also write $m*$ as a function of $v*$. From Eq. (A2), we obtain two roots for $m*$. That is,

$$m* = \frac{\lambda(1-v) \pm \sqrt{\lambda^2(1-v) - \lambda v^2}}{v - \lambda(1-v)}.$$

For these two roots, it can be demonstrated that $\frac{\lambda(1-v)-\sqrt{\lambda^2(1-v)-\lambda v^2}}{v-\lambda(1-v)} < 1$. To see this, let $L = \lambda(1-v)-\sqrt{\lambda^2(1-v)-\lambda v^2}-[v-\lambda(1-v)]$, show that *maximum* $L<0$:

$$\frac{\partial L}{\partial v} = \frac{\lambda\sqrt{(\lambda + 2v)^2}-(2 + 4\lambda)\sqrt{(\lambda^2(1-v)-\lambda v^2}}{2\sqrt{\lambda^2(1-v)-\lambda v^2}}. \tag{A5}$$

If we let Eq. (A5) $=\frac{\lambda}{1 + \lambda}$, $(v)=\frac{\lambda^2-\lambda-2}{2}<0$. Since $\frac{\partial m*}{\partial v*} > 0$, $v*>\frac{\lambda}{1 + \lambda}$, and $\frac{\lambda^2-\lambda-2}{2}<0$, *maximum* $L<0$.

Consequently, $m* = \frac{\lambda(1-v) + \sqrt{\lambda^2(1-v)-\lambda v^2}}{v-\lambda(1-v)}$ is the only solution that meets the requirement that $m>1$. As demonstrated already, the relationship satisfies the constraints if and only if $\frac{\lambda}{1 + \lambda} < v_i < \frac{3\lambda}{1 + 4\lambda}$. QED.

References

Cox, G.W., 1991. SNTV and d'Hondt are 'equivalent'. Electoral Studies 10, 118–132.

Cox, G.W., 1994. Strategic voting equilibria under the single nontransferable vote. American Political Science Review 88, 608–621.

Cox, G.W., Rosenbluth, F.M., 1994. Reducing nomination errors: factional competition and party strategy in Japan. Electoral Studies 13, 4–16.

Grofman, B., 1975. A review of macro-election systems. In: Wildenmann, R. (Ed.), Sozialwissenschaftliches Jahrbuch für Politik. Gunter Olzog Verlag, Munich.

Grofman, B., 1999. SNTV, STV, and single-member-district systems: theoretical comparisons and contrasts. In: Grofman, B., Lee, S-C., Winckler, E.A., Woodall, B. (Eds.), Elections in Japan, Korea, and Taiwan under the Single Non-Transferable Vote: The Comparative Study of an Embedded Institution. The University of Michigan Press, Ann Arbor.

Lijphart, A., 1994. Electoral Systems and Party Systems: A Study of Twenty-Seven Democracies, 1945–1990. Oxford University Press, Oxford.

Lijphart, A., 1997. The difficult science of electoral systems: a commentary on the critique by Alberto Penadés. Electoral Studies 16, 73–77.

Lijphart, A., Pintor, R.L., Sone, Y., 1986. The limited vote and the single nontransferable vote: lessons from the Japanese and Spanish examples. In: Grofman, B., Lijphart, A. (Eds.), Electoral Laws and Their Political Consequences. Agathon Press, New York.

Nohlen, Dieter, 1984. Two incompatible principles of representation. In: Lihphart, Al, Grofman, B. (Eds.), Choosing an Electoral System. Praeger, New York.

Rae, D.W., 1971. The Consequences of Electoral Laws, 2nd ed. Yale University Press, New Haven.

Rae, D.W., Hanby, V., Loosemore, J., 1971. Thresholds of representation and threshold of exclusion: an analytic note on electoral systems. Comparative Political Studies 3, 479–488.

Reed, S.R., 1990. Structure and behavior: extending Duverger's law to the Japanese case. British Journal of Political Science 20, 335–356.

Reed, S.R., 1996. Seats and votes: testing Taagepera in Japan. Electoral Studies 15, 71–81.

Rokkan, S., 1968. Elections: electoral systems. In: International Encyclopedia of the Social Sciences. Crowell-Collier-Macmillan, New York.

Sankoff, D., Mellos, K., 1972. The swing ratio and game theory. American Political Science Review 66, 551–554.
Sartori, G., 1968. Representational systems. In: International Encyclopedia of the Social Sciences. Crowell-Collier-Macmillan, New York.
Taagepera, R., 1986. Reformulating the cube law for proportional representation elections. American Political Science Review 80, 489–504.
Taagepera, R., 1998. Nationwide inclusion and exclusion thresholds of representation. Electoral Studies 17, 405–417.
Taagepera, R., Shugart, M.S., 1989. Seats and Votes: The Effects and Determinants of Electoral Systems. Yale University Press, New Haven.