2 Entire solutions for discrete reaction-diffusion equations

2.1 Preliminaries

First, we define and make the notion of subsolution and supersolution of (1.1) as follows.

Definition 2.1 *A function* $\underline{u}(x,t)$ *defined on* $\mathbb{R}\times[s,S]$ *is called a subsolution of* (1.1) *if* $\underline{u}(x,t) \leq u(x,t)$ $((x,t) \in \mathbb{R} \times [s,S])$ *for any solution* $u(x,t)$ *of* (1.1) *such that* $\underline{u}(x,s) \leq u(x,s)$ $(x \in \mathbb{R})$. *We call* $\underline{u}(x,t)$ *a subsolution of* (1.1) *in* $\mathbb{R}\times(-\infty,-T]$ *for some* $T \geq 0$, *if* $\underline{u}(x,t)$ *is a subsolution of* (1.1) *defined on* $\mathbb{R} \times [s,-T]$ *for any* $s < -T$. *Similarly, a supersolution can be defined by reversing the inequalities.*

Lemma 2.2 *Let* $\phi_i(x,t)$, $i = 1,2$, *be functions satisfying* $0 < \phi_i(x,t) < 1$ *and* $(\phi_i)_t(\cdot,t) - \phi_i(\cdot+1,t) - \phi_i(\cdot-1,t) + 2\phi_i(\cdot,t) - f(\phi_i(\cdot,t)) \leq 0$ $((x,t) \in \mathbb{R}\times(-\infty,-T])$. *Then* $\underline{u}(x,t) := \max\{\phi_1(x,t), \phi_2(x,t)\}$ *is a subsolution of* (1.1) *in* $\mathbb{R} \times (-\infty,-T]$.

Proof. Given any $s < -T$. Set $\Omega := \mathbb{R} \times [s,-T]$. Let $u(x,t)$ be a solution of (1.1) in Ω with $u(x,s) \geq \underline{u}(x,s)$ for all $x \in \mathbb{R}$. Applying the strong maximum principle (see [1]) to $\omega_i(x,t) = u(x,t) - \phi_i(x,t)$, $i = 1,2$, we assert that $\omega_i(x,t) \geq 0$ in Ω, $i = 1,2$.Thus $u(x,t) \geq \phi_i(x,t)$ in Ω, $i = 1,2$, which yields the desired conclusion.$\square$

We note that a bounded function $\phi(x,t)$ of C^2 is a subsolution of (1.1) in $\mathbb{R}\times(-\infty,-T]$ if $\phi_t(\cdot,t)-\phi(\cdot+1,t)-\phi(\cdot-1,t)+2\phi(\cdot,t)-f(\phi(\cdot,t))\leq 0$ in $\mathbb{R}\times(-\infty,-T]$, while it is a supersolution if $\phi_t(\cdot,t)-\phi(\cdot+1,t)-\phi(\cdot-1,t)+2\phi(\cdot,t)-f(\phi(\cdot,t))\geq 0$ in $\mathbb{R}\times(-\infty,-T)$ (see [1]).

From now on, we alway assume $c=c_{min}$. Let λ be the larger root of the characteristic equation

$$c\lambda - e^{\lambda} - e^{-\lambda} + 2 = 0. \tag{2.1}$$

Concerning the asymptotic behaviors of the traveling wave solution $U(x)$ near $x=\pm\infty$ in [3], we have the following estimates for $x\leq 0$:

$$ke^{\lambda x}\leq U(x)\leq Ke^{\lambda x}, \tag{2.2}$$

for some positive k,K. Also, for $x\geq 0$ we have

$$\gamma e^{-\mu x}\leq 1-U(x)\leq \delta e^{-\mu x}, \tag{2.3}$$

for some positive γ, δ and μ is the unique positive root of

$$c\mu + e^{\mu} + e^{-\mu} - 3 = 0. \tag{2.4}$$

Moveover, there are positive numbers ψ_i $(i=1,2)$ such that

$$\inf_{x\leq 0}\frac{U'(x)}{U(x)}=\psi_1, \quad \inf_{x\geq 0}\frac{U'(x)}{1-U(x)}=\psi_2. \tag{2.5}$$

2.2 Existence of entire solutions

Consider the following ordinary differential equation:

$$\dot{p}(t)=c+Ne^{\alpha p(t)}, \quad (t\leq 0), \tag{2.6}$$

where N, c and α are constants with c, $\alpha > 0$. We can solve this equation easily and obtain the solution as

$$p(t) = p(0) + ct - \frac{1}{\alpha} log \left\{ 1 + \frac{N}{c} e^{\alpha p(0)} (1 - e^{c\alpha t}) \right\}. \tag{2.7}$$

If $N > 0$, it is clear that the solution $p(t)$ is monotone increasing. Let

$$\omega := p(0) - \frac{1}{\alpha} log \left(1 + \frac{N}{c} e^{\alpha p(0)} \right) . \tag{2.8}$$

Then we obtain

$$0 < p(t) - ct - \omega \leq R_0 e^{c\alpha t}, \quad (t \leq 0), \tag{2.9}$$

for some positive constant R_0. Now, we have the following lemma.

Lemma 2.3 *Let* $p(t)$ *be the solution of* (2.6) *with* $p(0) < 0$, $\alpha = \lambda$, $N > \max\{K^2/(\psi_1 k)$, $2K/(\psi_2 \gamma)\}$ *and let* ω *be defined by* (2.8). *Suppose that* $\lambda \geq \mu$. *Then*

$$\overline{u}(x,t) := U(x + p(t)) + U(-x + p(t)) \tag{2.10}$$

and

$$\underline{u}(x,t) := \max\{U(x + ct + \omega), U(-x + ct + \omega)\} \tag{2.11}$$

are a supersolution and a subsolution of (1.1) *for* $t \leq 0$, *respectively.*

Proof. First, by Lemma 2.2, we see that $\underline{u}(x,t) := \max\{U(x+ct+\omega), U(-x+ct+\omega)\}$ is a subsolution of (1.1) for $t \leq 0$. Next, we prove that $\overline{u}(x,t)$ is a supersolution. Let $U(x + p(t)) = U_1$, $U(-x + p(t)) = U_2$. Set $\mathcal{N}[\nu](x,t) := \nu_t(x,t) - \nu(x+1,t) - \nu(x-1,t) + 2\nu(x,t) - f(\nu(x,t))$. By a simple computation, we have

$$\mathcal{N}[\overline{u}] = (U_1' + U_2')(Ne^{\lambda p} - G(x,t)), \tag{2.12}$$

where

$$G(x,t) := \frac{U_1 U_2 (2 - 3U_1 - 3U_2)}{U_1' + U_2'}. \tag{2.13}$$

We also see from (2.2), (2.3) and (2.5) that

$$\begin{aligned} ke^{\lambda y} \leq & \quad U(y) & \leq Ke^{\lambda y}, & \quad (y \leq 0), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \psi_1 ke^{\lambda y} \leq & \quad \psi_1 U(y) & \leq U'(y), & \quad (y \leq 0), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \psi_2 \gamma e^{-\mu y} \leq & \quad \psi_2(1 - U(y)) & \leq U'(y), & \quad (y \geq 0). \end{aligned} \tag{2.16}$$

Note that $p(t) < 0$ for all $t \leq 0$. We divide $\mathbb{R}$ into three regions to estimate $G(x,t)$.

(1) $p \leq x \leq -p$: Using (2.14) and (2.15), we obtain

$$\begin{aligned} G(x,t) &\leq \frac{2U_1U_2}{U_1' + U_2'} \leq \frac{2K^2 e^{\lambda(x+p)} e^{\lambda(-x+p)}}{\psi_1 k (e^{\lambda(x+p)} + e^{\lambda(-x+p)})} \\ &= \frac{2K^2 e^{2\lambda p}}{\psi_1 k (e^{\lambda x} + e^{-\lambda x}) e^{\lambda p}} \leq \frac{2K^2}{2\psi_1 k} e^{\lambda p}. \end{aligned} \tag{2.17}$$

(2) $x \leq p$: It follows from (2.14)-(2.16) that

$$\begin{aligned} G(x,t) &\leq \frac{2U_1}{U_1' + U_2'} \leq \frac{2K e^{\lambda(x+p)}}{\psi_1 k e^{\lambda(x+p)} + \psi_2 \gamma e^{-\mu(-x+p)}} \\ &= \frac{2K}{\psi_1 k e^{\lambda p} + \psi_2 \gamma e^{-(\lambda-\mu)x} e^{-\mu p}} e^{\lambda p} \\ &\leq \frac{2K}{\psi_2 \gamma} e^{\lambda p}. \end{aligned} \tag{2.18}$$

(3) $-p \leq x$: By the symmetry $G(-x,t) = G(x,t)$ and (2.18), we obtain

$$G(x,t) \leq \frac{2K}{\psi_2 \gamma} e^{\lambda p}. \tag{2.19}$$

Hence we obtain

$$\mathcal{N}[\overline{u}] = (U_1' + U_2')(Ne^{\lambda p} - G(x,t)) \geq 0.$$

Therefore, $\overline{u}$ is a supersolution of (1.1) for $t \leq 0$. This proves the lemma. □

Remark 2.4 *The assumption* $\lambda \geq \mu$ *in Lemma* 2.3 *is valid provided that* $c_{min} \geq \frac{1}{2\log 2}$.

Lemma 2.5 *Let* $\overline{u}(x,t)$ *and* $\underline{u}(x,t)$ *be the supersolution and the subsolution given in Lemma* 2.3. *Suppose all the assumption of Lemma* 2.3 *holds. Then there is a positive constant* M_1 *such that*

$$0 < \overline{u}(x,t) - \underline{u}(x,t) \leq M_1 e^{c\lambda t} \quad ((x,t) \in \mathbb{R} \times (-\infty, 0]). \tag{2.20}$$

Proof. Suppose that $t \leq 0$. Since $U' > 0$, we have $U(x+ct+\omega) \geq U(-x+ct+\omega)$ for $x \geq 0$. Thus $\underline{u}(x,t) = U(x+ct+\omega)$ for $x \geq 0$ and $\underline{u}(x,t) = U(-x+ct+\omega)$ for $x \leq 0$. For $x \geq 0$, we have

$$\begin{aligned} 0 \leq \overline{u}(x,t) - \underline{u}(x,t) &= U(x+p(t)) + U(-x+p(t)) - U(x+ct+\omega) \\ &\leq \ Ke^{\lambda(-x+p(t))} + \sup_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq \ Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq M_1 e^{c\lambda t}, \end{aligned} \tag{2.21}$$

for some $M_1 > 0$. On the other hand, for $x \leq 0$, we have

$$\begin{aligned} 0 \leq \overline{u}(x,t) - \underline{u}(x,t) &= U(x+p(t)) + U(-x+p(t)) - U(-x+ct+\omega) \\ &\leq \ Ke^{\lambda(x+p(t))} + \sup_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq \ Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq M_1 e^{c\lambda t}. \end{aligned} \tag{2.22}$$

This completes the proof. □

Following [5], we have the following proposition.

Proposition 2.6 *Under the same assumptions of Lemma* 2.3*, there is an entire solution* $u^*(x,t)$ *of* (1.1) *such that*

$$\underline{u}(x,t) \le u^*(x,t) \le \overline{u}(x,t) \quad ((x,t) \in \mathbb{R} \times (-\infty, 0]), \tag{2.23}$$

where ω *is defined by* (2.8)*,* $\underline{u}(x,t)$ *and* $\overline{u}(x,t)$ *are given in Lemma* 2.3*.*

Proof. Denote by $u(x,t;\nu_0)$ a solution to (1.1) with the initial condition $u(x,0;\nu_0(\cdot)) = \nu_0(x)$. Set

$$\nu_n(x,t) = u(x,t;\underline{u}(\cdot,-n)), \qquad n = 1,2,\ldots\,.$$

Since $\underline{u}$ is a subsolution and $\underline{u}(x,-n-1+0) = u(x,0;\underline{u}(\cdot,-(n+1)))$, we have

$$\underline{u}(x,-n-1+t) \le u(x,t;\underline{u}(\cdot,-(n+1))).$$

By taking $t=1$, we obtain

$$\nu_n(x,0) = \underline{u}(x,-n) \le u(x,1;\underline{u}(\cdot,-(n+1))) = \nu_{n+1}(x,1).$$

Thus the maximum principle yields

$$\nu_n(x,n) \le \nu_{n+1}(x,n+1),$$

which implies $\{\nu_n(\cdot,n)\}$ is monotone increasing. On the other hand, since $\nu_n(x,n) \le \overline{u}(x,0)$, there is a function ν^* such that ν_n converges uniformly to ν^*. Therefore, $u^*(x,t) := u(x,t;\nu^*)$ is a solution for all $t \ge 0$.

Next, we show that $u^*(x,t)$ is defined for all $t \le 0$. Given $T \ge 0$, there is an integer n_1 such that $n_1 > T$. Then, for $n \ge n_1$, we have

$$u(x,-T;\nu_n) = u(x,-T;u(x,n;\underline{u}(\cdot,-n))) = u(x,n-T;\underline{u}(\cdot,-n)).$$

Set

$$w_n(x) = u(x, n - T; \underline{u}(\cdot, -n)). \tag{2.24}$$

Then $\nu_n(x, n) = u(x, T; w_n(x, t))$ and

$$w_{n+1}(x) = u(x, n + 1 - T; \underline{u}(\cdot, -(n + 1))) \geq u(x, n - T; \underline{u}(\cdot, -n)) = w_n(x).$$

This implies the sequence $\{w_n\}$ is monotone increasing. Applying the same argument, there is a function ν_T to which w_n converges uniformly. We see that

$$\nu^* = \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} u(x, T; w_n(x, t)) = u(x, T; \nu_T).$$

Thus we obtain

$$\nu_T = u(x, -T; \nu^*).$$

Since $T > 0$ is arbitrary, we conclude that $u^*(x, t) := u(x, t; \nu^*)$ is defined for all $t \in \mathbb{R}$.

Finally, we show that (2.23) holds. From above, we have

$$u^*(x, -T) = u(x, -T; \nu^*) = \nu_T = \lim_{n\to\infty} \omega_n \tag{2.25}$$

Since $\underline{u}$ is a subsolution and $\overline{u}(x, -n) \geq u(x, 0; \underline{u}(\cdot, -n)) = \underline{u}(x, -n)$, we have

$$\overline{u}(x, -n + t) \geq u(x, t; \underline{u}(\cdot, -n)) \geq \underline{u}(x, -n + t) \quad \forall (x, t) \in \mathbb{R} \times [0, n].$$

By taking $t = n - T$, we obtain

$$\overline{u}(x, -T) \geq \omega_n = u(x, n - T; \underline{u}(\cdot, -n)) \geq \underline{u}(x, -T). \tag{2.26}$$

Hence, it follows from (2.25) and (2.26) that $\underline{u}(x, -T) \leq u^*(x, -T) \leq \overline{u}(x, -T)$. Since $T > 0$ is arbitrary, (2.23) holds. This proves the proposition. □

Remark 2.7 *By virtue of the condition* $\lambda \geq \mu$ *we can check that the supersolution* $\overline{u}(x,t)$*, defined for* $t \leq 0$*, is bounded by 1 for large* $|t|$*. In fact, we may assume that* $K < 1/2$ *in the condition* (2.2) *by shifting appropriately. Then*

$$U(x+p(t)) + U(-x+p(t)) \leq K(e^{\lambda x} + e^{-\lambda x})e^{\lambda p} \quad (p \leq x \leq -p),$$

while

$$\begin{aligned} U(x+p) + U(-x+p) &\leq 1 - \gamma e^{-\mu(x+p)} + Ke{-}\lambda(x-p) \\ &\leq 1 - (\gamma - Ke^{(\lambda+\mu)p}e^{-(\lambda-\mu)x})e^{-\mu(x+p)} \qquad (-p \leq x), \end{aligned}$$

$$\begin{aligned} U(x+p) + U(-x+p) &\leq Ke^{\lambda(x+p)} + 1 - \gamma e^{\mu(x-p)} \\ &\leq 1 - (\gamma - Ke^{(\lambda+\mu)p}e^{(\lambda-\mu)x})e^{\mu(x-p)} \qquad (x \leq p). \end{aligned}$$

This implies $\overline{u}(x,t) \leq 1$ *for* $t < -T$ *with a large* $T > 0$*. Hence, by the strong maximum principle, we can assert that the solution* $u(x,t)$ *of Proposition* 2.6 *satisfies* $0 < u(x,t) < 1$ *for all* $(x,t) \in \mathbb{R}^2$.

Proposition 2.8 *Let* $u(x,t)$ *be an entire solution constructed in Proposition* 2.6. *Under the same assumptions of Lemma* 2.3 *and Proposition* 2.6*, there is a positive number* M_1 *such that for* $t \leq 0$,

$$\begin{aligned} 0 \leq \sup\nolimits_{x \geq 0} \{u(x,t) - U(x+ct+\omega)\} \\ + \sup\nolimits_{x \leq 0} \{u(x,t) - U(-x+ct+\omega)\} \leq M_1 e^{c\lambda t}. \end{aligned} \tag{2.27}$$

Proof. Suppose that $t \leq 0$. For $x \geq 0$,

$$\begin{aligned} 0 \ &\leq U(x+p(t)) + U(-x+p(t)) - U(x+ct+\omega) \\ &\leq \ Ke^{\lambda(-x+p(t))} + \sup\nolimits_z |U'(z)| R_0 e^{c\lambda t} \\ &\leq \ Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq \tfrac{1}{2} M_1 e^{c\lambda t}, \end{aligned} \tag{2.28}$$

for some $M_1 > 0$. Combining (2.23) and (2.28), we obtain

$$0 \leq \; u(x,t) - U(x+ct+\omega) \leq \overline{u}(x,t) - U(x+ct+\omega) \leq \frac{1}{2}M_1 e^{c\lambda t}.$$

On the other hand, for $x \leq 0$, we have

$$\begin{aligned} 0 \;\; & \leq U(x+p(t)) + U(-x+p(t)) - U(-x+ct+\omega) \\ & \leq \; Ke^{\lambda(x+p(t))} + \sup_z |U'(z)| R_0 e^{c\lambda t} \\ & \leq \; Ke^{\lambda p(t)} + M_2 e^{c\lambda t} \leq \tfrac{1}{2} M_1 e^{c\lambda t}. \end{aligned} \tag{2.29}$$

Therefore it follows from (2.23) and (2.29) that

$$0 \leq \; u(x,t) - U(-x+ct+\omega) \leq \overline{u}(x,t) - U(-x+ct+\omega) \leq \frac{1}{2}M_1 e^{c\lambda t}.$$

Hence (2.27) holds. □

Proof of Theorem 1.1: Given arbitrary θ_1, θ_2, we consider the translation and the time-shift as

$$U(x+\xi+c(t+\tau)) = U(x+ct+\xi+c\tau),$$

$$U(-x-\xi+c(t+\tau)) = U(-x+ct-\xi+c\tau).$$

Define $\widetilde{u}(x,t) := u(x+\xi, t+\tau)$ with

$$\xi := \frac{\theta_1 - \theta_2}{2}, \quad \tau := \frac{\theta_1 + \theta_2 - 2\omega}{2c},$$

where $u(x,t)$ is the entire solution of Proposition 2.6. Then we easily obtain

$$\begin{aligned} & \max\{U(x+ct+\theta_1), U(-x+ct+\theta_2)\} \\ & \qquad \leq \tilde{u}(x,t) \leq \overline{u}(x+\xi, t+\tau) \quad\;\; (t \leq -\tau). \end{aligned}$$

On the other hand, (1.4) immediately follows from (2.27). Thus we complete the proof of Theorem 1.1. □

Remark 2.9 *Entire solutions can also be constructed by using traveling wave with speed* $c > c_{min}$ *if one can find a pair of suitable supersolution and subsolution. However, we cannot find such one. Therefore we left it as an open problem.*