Chapter 3

Achievement Function

3.1 Construction

In order to transform the different measurements onto a normalized scale, we construct the achievement function μ_i for each criteria i which can be viewed as an extension of the fuzzy membership function in terms of a strictly monotonic and concave utility function as shown in Figure 3.1 (see [25], [30], [31], etc.)

We assume that the decision maker specifies requirements in aspiration and reservation levels by introducing desired and required values for several outcomes. Depending on the specified aspiration and reservation levels, a_i and r_i, respectively, we construct our achievement function of z_i as follows:

$$\mu_i(z_i) = \log_{\alpha_i} \frac{z_i}{r_i}, \quad \text{where } \alpha_i = \frac{a_i}{r_i}. \tag{3.1}$$

Formally, we define $\mu_i(\cdot)$ over the range $[0, \infty)$, with $\mu_i(0) = -\infty$ and $\mu_i'(0) = \infty$. Depending on the specified reference levels [15], this achievement function can be interpreted as a measure of the decision maker's satisfaction with the value of the i-th criteria. It is a strictly increasing function of z_i, having value 1 if $z_i = a_i$, and value 0 if $z_i = r_i$. The achievement function can map the different criteria

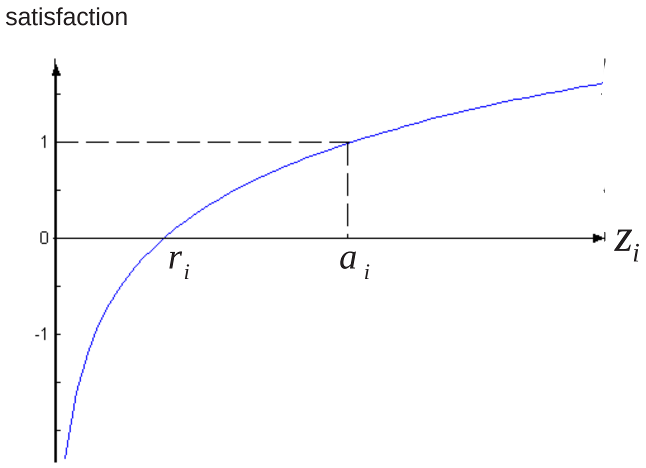

Figure 3.1: The Graph of an Achievement Function $\mu_i(z_i)$

values onto a normalized scale of the decision maker's satisfaction. Moreover, the logarithmic achievement function will be intimately associated with the concept of proportional fairness (see [18], [25], and [27].) We will formulate the mathematical model of the fair bandwidth allocation by using the achievement function.

3.2 Implication

First, we introduce the concept of majorization[1] (see [21], [22]) to provide the fairness. For any n-dimensional vector $\mathbf{s}$=($s_1, \dots, s_n$) of reals, let $s_{(1)} \leq \dots \leq s_{(n)}$ denote the components of $\mathbf{s}$ in increasing order.

Definition 3.2.1 *For* $\mathbf{s}$ *and* $\mathbf{t}$ *in* $\mathbb{R}^n$*,* $\mathbf{s} \leq_M \mathbf{t}$ *if* $\sum_{i=1}^{n} s_{(i)} = \sum_{i=1}^{n} t_{(i)}$ *and* $\sum_{i=1}^{k} s_{(i)} \geq \sum_{i=1}^{k} t_{(i)}$*, for* $k = 1, \dots, n-1$*. When* $\mathbf{s} \leq_M \mathbf{t}$ *then* $\mathbf{s}$ *is said to be* ***majorized*** *by* $\mathbf{t}$.

[1]Multiple criteria optimization defines the dominance relation by the standard vector inequality. The theory of majorization includes the results which allow us to express the relation of fair (equitable) dominance as a vector inequality on the cumulative ordered outcomes.

If $\mathbf{s} \leq_M \mathbf{t}$, then the allocation $\mathbf{s}$ is more fair than $\mathbf{t}$. Next, we have the following definition.

Definition 3.2.2 *A function $g : \mathbb{R}^n \to \mathbb{R}$ is called* ***Schur-concave****, if* $\mathbf{s} \leq_M \mathbf{t}$ *implies* $g(\mathbf{s}) \geq g(\mathbf{s})$.

Thus, we have the following theorem taken from [22].

Theorem 3.2.3 *Let h be an arbitrary real function and define $g(\mathbf{s}) = \sum_{i=1}^{n} h(s_i)$ for $\mathbf{s} \in \mathbb{R}^n$, then g is Schur-concave if and only if h is concave.*

Recall that the achievement function μ_i is a concave function. According to this definition, we know that the function $\sum \mu_i$ is Schur-concave. Next, we consider a generic resource allocation problem defined as an optimization problem with m objective functions $f_i(\mathbf{x})$:

$$\max\{\, \mathbf{f}(\mathbf{x}) : \mathbf{x} \in Q \,\}, \tag{3.2}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-function that maps the decision space $X = \mathbb{R}^n$ into the criterion space $Y = \mathbb{R}^m$, $Q \subseteq X$ denotes the feasible set, and $\mathbf{x} \in X$ denotes the vector of decision variables.

In the following, we will introduce the concept of fairness by using the fair aggregation function (see [15], [18], [22], [25], [27], [37]). Typical solution concepts for multiple criteria problems are defined by aggregation functions $g : Y \to \mathbb{R}$ to be maximized. Thus, (3.2) $\Rightarrow$

$$\max\{g(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\} \tag{3.3}$$

The simplest aggregation functions commonly used for the multiple criteria problem (3.2) are defined as the sum of outcomes

$$g(\mathbf{f}(\mathbf{x})) = \sum_{i=1}^{m} f_i(\mathbf{x}), \tag{3.4}$$

or the worst outcome

$$g(\mathbf{f}(\mathbf{x})) = \min_{i=1,\dots,m} f_i(\mathbf{x}) \tag{3.5}$$

An aggregation (3.3) is fair if it is defined by a strictly increasing and strictly Schur-concave function g.

Definition 3.2.4 *An aggregation function g satisfying all the following requirements (3.6), (3.7) and (3.8), we call the corresponding problem (3.3)* ***a fair aggregation*** *of problem (3.2). For all $i \in S = \{1, 2, \dots, m\}$*

$$g(f_1(\mathbf{x}), \dots, f_{i-1}(\mathbf{x}), f_i'(\mathbf{x}), f_{i+1}(\mathbf{x}), \dots, f_m(\mathbf{x})) < g(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})), \tag{3.6}$$

whenever $f_i'(\mathbf{x}) < f_i(\mathbf{x})$. For any permutation π of S,

$$g(f_{\pi(1)}(\mathbf{x}), f_{\pi(2)}(\mathbf{x}), \dots, f_{\pi(m)}(\mathbf{x})) = g(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})) \tag{3.7}$$

For any $0 < \epsilon < z_{i'} - z_{i''}$, we have

$$g(f_1(\mathbf{x}), \dots, f_{i'}(\mathbf{x}) - \epsilon, \dots, f_{i''}(\mathbf{x}) + \epsilon, \dots, f_m(\mathbf{x})) > g(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})) \tag{3.8}$$

We know that $\sum \mu_i$ is a fair aggregation function according to this definition. Every optimal solution to the fair aggregation (3.3) of a resource allocation problem (3.2) defines some fair allocation scheme. In order to guarantee the consistency of the aggregated problem (3.3) with the maximization all individual objective functions in the original multiple criteria problem, the aggregation function must be strictly increasing with respect to every coordinate, i.e. (3.6). In order to guarantee the fairness of the solution concept, the aggregation function must be additionally symmetric (impartial), i.e. (3.7). Symmetric functions satisfying the requirement (3.8), are called strictly Schur-concave functions. Next, we have the following two theorems taken from [22].

Theorem 3.2.5 *For a strictly concave, increasing function $\mu_i : \mathbb{R} \to \mathbb{R}$, the function*

$$g(\boldsymbol{f}(\mathbf{x})) = \sum_{i=1}^{m} \mu_i(f_i(\mathbf{x})) \tag{3.9}$$

is a strictly monotonic and strictly Schur-concave function.

Theorem 3.2.6 *For a strictly concave, increasing function $\mu_i : \mathbb{R} \to \mathbb{R}$, the optimal solution of the problem*

$$\max\{\sum_{i=1}^{m} \mu_i(f_i(\boldsymbol{x})) : \boldsymbol{x} \in Q\} \tag{3.10}$$

is a fair solution for resource allocation problem (3.2).