Chapter 5

A-Optimality of Completely Randomized Designs

5.1. Optimal Designs for p = 2, 3

The diallel cross experiments involving $p(p+1)/2$ distinct crosses for test line versus control comparisons under the completely randomized design model are considered. Let d be a completely randomized design for a diallel cross experiment with p test lines, one control line, and n denote the total number of crosses in d. Let s_{di} denote the number of times line i occurs in crosses in d, i = 0, 1, … , p, and $g_{dii'}$ denote the number of times the cross (i, i') appears in d, $\forall\ i \neq i'$, i, i' = 0, 1, … , p. The model for design d is then assumed to be

$$\vec{Y}_d = \mu\vec{1}_n + \Delta_d\vec{\tau} + \vec{\varepsilon}\ ,$$

where $\vec{Y}_d$ is the $n\times1$ vector of observed responses, μ is the overall mean, $\vec{1}_n$ denotes the $n\times1$ vector of 1's, $\vec{\tau} = (\tau_0, \tau_1, \cdots, \tau_p)'$ is the vector of p + 1 general combining ability effects, Δ_d is the corresponding design matrix, that is, the (s,h)th elememt of Δ_d is 1 if the sth observation pertains to line h, and is zero, otherwise; $\vec{\varepsilon}$ is the $n\times1$ vector of uncorrelated random errors with mean zero and constant variance σ^2. The coefficient matrix of the normal equation for estimating $\vec{\tau}$ is

$$C_d = G_d - (1/n)s_d s_d',$$

where $G_d = \Delta_d' \Delta_d = (g_{dii'})$, $g_{dii} = s_{di}$, and $s_d' = (s_{d0}, s_{d1}, \ldots, s_{dp})'$. Note that the row sums and column sums of C_d are all zero. Our focus is on the estimation of the test line versus control contrasts $(\tau_1 - \tau_0, \cdots, \tau_p - \tau_0)'$, and by Bechhofer and Tamhane (1981), and Das (2002) the information matrix, $M_d = (m_{dii'})$, for the estimation of $(\tau_1 - \tau_0, \cdots, \tau_p - \tau_0)'$, is obtained by deleting the first row and first column of C_d, and

$$M_d = \begin{cases} s_{di} - s_{di}^2 / n, & \text{for } i = i' \\ g_{dii'} - s_{di}s_{di'} / n, & \text{for } i \neq i' \end{cases}.$$

Let $D(p+1, n)$ be a collection of all connected designs with p test lines, one control line, and n crosses. A design $d^* \in D(p+1, n)$ is said to be A-optimal if it minimizes $\sum_{i=1}^{p} Var(\hat{\tau}_i - \hat{\tau}_0)$, where $\hat{\tau}_i - \hat{\tau}_0$ is the best linear unbiased estimator (BLUE) of $\tau_i - \tau_0$, i = 1, … , p, over all designs in $D(p+1, n)$, that is, d^* satisfies $trM_{d^*}^{-1} = \min_{d \in D(p+1,b,k)} trM_d^{-1}$.

For a design $d \in D(p+1, n)$, applying the averaging technique in Kiefer (1975), Majumdar and Notz (1983), and Jacroux and Majumdar (1989), one can show that

$$trM_d^{-1} \geq tr\overline{M}_d^{-1},$$

where $\overline{M}_d = (1/p!)\sum_{\pi} \pi \overline{M}_d \pi'$, is the average of all possible permutations of the

p test lines on M_d, and π is the corresponding $p \times p$ permutation matrix. We should note that $\overline{M}_d$ is completely symmetric, that is, $\overline{M}_d = aI_p + bJ_{p,p}$, where I_p is the $p \times p$ identity matrix, and $J_{p,p}$ is a $p \times p$ matrix of 1's. Among all designs in $D(p+1,n)$, a group of designs having completely symmetric information matrices is called a type S design by Choi, Gupta, and Kageyama (2002).

Definition 5.1. (Choi, Gupta, and Kageyama (2002)) A design $d \in D(p+1,n)$ is called a type S design, denoted as $S(p, g_0, g_1)$, if there are positive integers g_0 and g_1, such that $\forall\ i \neq i' = 1, \ldots, p,\ g_{d0i} = g_0,\ g_{dii'} = g_1$.

For p = 2, let the number of the crosses (0, 1), (0, 2), and (1, 2) appear in d are n_1, n_2, and n_3, respectively, then $n = n_1 + n_2 + n_3$, $g_{d01} = n_1$, $g_{d02} = n_2$, $g_{d12} = n_3$, $s_{d0} = n_1 + n_2$, $s_{d1} = n_1 + n_3$, and $s_{d2} = n_2 + n_3$. Hence the information matrix, M_d, for the estimation of $(\tau_1 - \tau_0, \tau_2 - \tau_0)'$ is, after straightforward calculation,

$$M_d = \frac{1}{n}\begin{bmatrix} n_2(n-n_2) & -n_1n_2 \\ -n_1n_2 & n_1(n-n_1) \end{bmatrix} = \frac{1}{n} M_{1d}\text{, say.}$$

Suppose that the eigenvalues of M_{1d} are λ_1 and λ_2. Then by solving the equation $(n_2(n-n_2)-\lambda)(n_1(n-n_1)-\lambda) - n_1^2 n_2^2 = 0$, which is equal to $\lambda^2 - (n_1(n-n_1) + n_2(n-n_2))\lambda + nn_1n_2(n-n_1-n_2) = 0$, one has $\lambda_1 + \lambda_2 = n_1(n-n_1) + n_2(n-n_2)$ and $\lambda_1\lambda_2 = nn_1n_2(n-n_1-n_2)$. Thus,

$$trM_d^{-1} = n\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = n\left(\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\right) = \frac{n(n_1 + n_2) - (n_1^2 + n_2^2)}{n_1 n_2 (n - n_1 - n_2)}.$$

For fixed value of $n_1 + n_2$, say s, where $n - s \geq 1$, and without loss of generality, we can assume that $n_1 \geq n_2$. If $n_1 - n_2 \geq 2$, then there exists $d^* \in D(p+1, n)$ with $n_1^* = n_1 - 1$ and $n_2^* = n_2 + 1$ such that $n_1^* + n_2^* = s$ and

$$trM_{d^*}^{-1} = \frac{ns - (n_1^{*2} + n_2^{*2})}{n_1^* n_2^* (n - s)} < \frac{ns - (n_1^2 + n_2^2)}{n_1 n_2 (n - s)} = trM_d^{-1}$$

since

$$n_1 n_2 ns - n_1 n_2 \left(n_1^2 + n_2^2 - 2(n_1 - n_2 - 1)\right) < (n_1 - 1)(n_2 + 1)ns - (n_1 - 1)(n_2 + 1)(n_1^2 + n_2^2)$$

if and only if

$$n_1 n_2 \left(ns - n_1^2 - n_2^2 + 2(n_1 - n_2 - 1)\right) < (n_1 n_2 + n_1 - n_2 - 1)(ns - n_1^2 - n_2^2)$$

if and only if

$$2n_1 n_2 (n_1 - n_2 - 1) < (n_1 - n_2 - 1)(ns - n_1^2 - n_2^2)$$

if and only if

$$n_1^2 + n_2^2 + 2n_1 n_2 - ns < 0,$$

and the inequality holds if and only if $s(s-n) < 0$. Hence trM_d^{-1} is minimized when n_1 and n_2 are as equal as possible for fixed value of $n_1 + n_2$.

Table 5.1 is a direct consequence for $3 \leq n \leq 30$ by using a computer. One can see that, when $n = 8$, the following two cases $n_1 = 2$, $n_2 = 2$, and $n_1 = 2$, $n_2 = 3$ both are A-optimal designs. Note that n_1 and n_2 can be exchanged without loss of generality.

Table 5.1. A Catalog of A-Optimal Designs with $p = 2$, and $3 \le n \le 30$

n	n_1	n_2	trM_d^{-1}
3	1	1	4
4	1	1	3
5	1	2	2.5
6	2	2	2
7	2	2	1.667
8	***2***	***2***	***1.5***
8	***2***	***3***	***1.5***
9	2	3	1.333
10	3	3	1.167
11	3	3	1.067
12	3	4	0.983
13	4	4	0.9
14	4	4	0.833
15	4	5	0.783
16	5	5	0.733
17	5	5	0.686
18	5	5	0.65
19	5	6	0.617
20	6	6	0.583
21	6	6	0.556
22	6	7	0.532
23	7	7	0.508
24	7	7	0.486
25	7	7	0.468
26	7	8	0.45
27	8	8	0.432
28	8	8	0.417
29	8	9	0.403
30	9	9	0.389

Example 5.1. For n = 8, the following three designs d_1, d_2, and d_3 all are A-optimal designs in $D(3+1,8)$.

$$d_1: \quad (0,1) \quad (0,1) \quad (0,2) \quad (0,2) \quad (1,2) \quad (1,2) \quad (1,2) \quad (1,2),$$

$$d_2: \quad (0,1) \quad (0,1) \quad (0,2) \quad (0,2) \quad (0,2) \quad (1,2) \quad (1,2) \quad (1,2),$$

$$d_3: \quad (0,1) \quad (0,1) \quad (0,1) \quad (0,2) \quad (0,2) \quad (1,2) \quad (1,2) \quad (1,2).$$

For $p = 3$, from equation (2.8) of Das (2002), one has

$$g(s_{d0};n,p) = \frac{np}{s_{d0}(n-s_{d0})} + \frac{np(p-1)^2}{np(2n-s_{d0}) - ph(s_{d0}) - s_{d0}(n-s_{d0})},$$

where $h(s_{d0}) = py^2 + (2n - s_{d0} - py)(2y+1)$ and $y = [(2n - s_{d0})/p]$. To find families of optimal designs, we derive the following inequality

$$g(s_{d0};n,p) \geq p\left(\frac{n}{s_{d0}(n-s_{d0})} + \frac{(p-1)^2}{2n(p-2) - s_{d0}(p-3)}\right)$$

$$= p(g*(s_{d0};n,p)), \text{ say,}$$

and the equality holds when $(2n - s_{d0})/p$ is an integer.

For p = 3, $g*(s_{d0};n,3) = (ns_{d0} - s_{d0}^2)^{-1} + 2n^{-1}$, and by taking the derivative of $g*(s_{d0};n,3)$ with respect to s_{d0}, the minimum value of $g*(s_{d0};n,3)$ is achieved at $s_{d0} = s_0 = n/2$, and $g*(n/2;n,3) = 6/n$. In the following, the problem of finding and constructing families of A-optimal type S design having $s_0 = n/2$ are investigated.

A type S design $S(3, g_0, g_1)$ with $s_0 = n/2$, has the following values for s_1, g_0, g_1, and

$$s_1 = (2n - s_0)/p = n/2,$$

$$g_0 = s_0/p = n/6, \quad g_1 = (s_1 - g_0)/(p-1) = n/6.$$

For these designs to exist, s_0, s_1, g_0, g_1 must all be integers, and the possible combination of the value of such *n* is $n = 0 \pmod 6$, or $n = 6u$, where $u \geq 1$ is an integer. Since $(2n - s_{d0})/3 = n/2 = 3u$ is an integer, hence, the minimum values of $g^*(s_{d0};3,b,k)$ or $g(s_{d0};3,b,k)$ can be achieved by the corresponding type *S* designs. Then by Theorem 5.1 listed in the following, these designs are A-optimal.

Theorem 5.1. (Das (2002)) Suppose s_0 is the value of the integer s_{d0}, $1 \leq s_{d0} \leq n-1$, which minimizes $g(s_{d0};n,p)$. Also suppose $d \in D(p+1,n)$ is a type *S* design such that $s_{d0} = s_0$. Then *d* is A-optimal over $D(p+1,n)$.

Lemma 5.2. For $n = 0 \pmod 6$, that is, $n = 6u$, where $u \geq 1$ is an integer, a type *S* design $S(3,u,u)$ exists, and is A-optimal in $D(3+1,6u)$.

Example 5.2. For *n* = 6, that is, *u* = 1, the following type *S* design *S* (3, 1, 1) is A-optimal in $D(3+1,6)$.

(0,1)   (0,2)   (0,3)   (1,2)   (1,3)   (2,3)

5.2. Optimal Designs for $4 \le p \le 6$

For $p \ge 4$, and recall that

$$g(s_{d0};n,p)=\frac{np}{s_{d0}(n-s_{d0})}+\frac{np(p-1)^2}{np(2n-s_{d0})-ph(s_{d0})-s_{d0}(n-s_{d0})},$$

where $h(s_{d0})=py^2+(2n-s_{d0}-py)(2y+1)$ and $y=\left[(2n-s_{d0})/p\right]$, then after straightforward calculation, the searching range of s_{d0} for finding A-optimal designs can be reduced, and list the consequence in the following Lemma 5.2.

Lemma 5.3. For given value of n, and p, suppose $d\in D(p+1,n)$ has $s_{d0}>[n/2]$, then there exists $d^*\in D(p+1,n)$ having $1\le s_{d0}\le[n/2]$, and satisfying $g(s_{d^*0};n,p)\le g(s_{d0};n,p)$.

Proof: Let $\phi_1(s_{d0};n,p)=np(2n-s_{d0})-ph(s_{d0})-s_{d0}(n-s_{d0})$ $=s_{d0}^2-\left((n-2y-1)p+n\right)s_{d0}+2n(n-2y-1)p+p^2y(y+1)$. For fixed $y=v$, that is, $v\le(2n-s_{d0})/p<v+1$, then $2n-(v+1)p<s_{d0}\le 2n-vp$ and by taking the derivative of $\phi_1(s_{d0};n,p)$ with respective to s_{d0}, one has

$$\frac{\partial}{\partial s_{d0}}\phi_1(s_{d0};n,p)=2s_{d0}-(n-2v-1)p-n$$

$$\le 2s_{d0}-(n-2(2n-s_{d0})/p-1)p-n$$

$$=3n-(n-1)p\le 0 \quad \text{for} \quad p\ge 4.$$

Hence $\phi_1(s_{d0};n,p)$ is a decreasing function in s_{d0} for $2n-(v+1)p<s_{d0}$ $\le 2n-vp$. We further consider the border points, that is, $s_{d0}=2n-(v+1)p$

$= s'$, say, and $s_{d0} = 2n - (v+1)p + 1 = s' + 1$, say, then

$$\phi_1(s'+1; n, p) - \phi_1(s'; n, p) = n(3-p) - p + 1 < 0 \quad \text{for} \quad p \geq 3 .$$

Hence $\phi_1(s_{d0}; n, p)$ is a decreasing function in s_{d0} for $p \geq 4$. Moreover, $s_{d0}(n - s_{d0})$ is a concave function, and is increasing in s_{d0} for $s_{d0} \leq [n/2]$. Hence if $s_{d0} > [n/2]$, there exists a design *d** having $1 \leq s_{d0} \leq [n/2]$, and satisfying $g(s_{d^*0}; n, p) \leq g(s_{d0}; n, p)$.

Theorem 5.4. For given n, $4 \leq p \leq 6$, and $v_1 = [3n/2p]$. Suppose that $v_2 < (n - 2v_1 - 1)/2(p-2) \leq v_2 + 1$, then a type *S* design $S(p, g_0, g_1)$, if exists, is A-optimal in *D*(*p* + 1, *n*) where $s_{d0} = s_0$ is obtained by

$$g(s_0; n, p) = \min\big(g([n/2]; n, p), \ldots, g([n/2] - v_2 - 1; n, p)\big).$$

Proof: For fixed $y = v$, then

$$g(s_{d0}; n, p) = \frac{np}{s_{d0}(n - s_{d0})} + \frac{np(p-1)^2}{np(2n - s_{d0}) - ph(s_{d0}) - s_{d0}(n - s_{d0})}$$

$$= np\big(\phi_2(s_{d0}; n, p)/\phi_3(s_{d0}; n, p)\big), \text{ say,}$$

where

$$\phi_2(s_{d0}; n, p) = -p(p-2)s_{d0}^2 + \big(p(p-2)n - p(n - 2v - 1)\big)s_{d0}$$

$$+ 2n(n-1)p - vp(4n - p - vp), \text{ and}$$

$$\phi_3(s_{d0}; n, p) = -s_{d0}^4 + \big(p(n - 2v - 1) + 2n\big)s_{d0}^3 - \big(n(n - 2v - 1)p + n^2$$

$$+ 2n(n-1)p - vp(4n - p - vp)\big)s_{d0}^2$$

$$+ n\big(2n(n-1)p - vp(4n - p - vp)\big)s_{d0} .$$

Denote $\upsilon_1 = 2n(n-1)p - vp(4n - p - vp)$ and $\upsilon_2 = (n - 2v - 1)p$, then

$$\begin{aligned}\upsilon_1 &= 2n(n-1)p - vp(4n - p - vp) \\ &> 2n(n-1)p - (2n - s_{d0})\big(4n - p - (2n - s_{d0}) + p\big) \\ &= 2n\big((n-2)p - p\big) + s_{d0}^2 > 0 \quad \text{for} \quad p \ge 4,\end{aligned}$$

$$\begin{aligned}\upsilon_2 &= (n - 2v - 1)p \\ &> \big(n - 2(2n - s_{d0})/p + 1\big)p \\ &= n(p-4) + 2s_{d0} + 1/p > 0 \quad \text{for} \quad p \ge 4,\end{aligned}$$

and

$$\phi_2(s_{d0}; n, p) = -p(p-2)s_{d0}^2 + \big(p(p-2)n - \upsilon_2\big)s_{d0} + \upsilon_1,$$

$$\phi_3(s_{d0}; n, p) = -s_{d0}^4 + (\upsilon_2 + 2n)s_{d0}^3 - (\upsilon_1 + \upsilon_2 n + n^2)s_{d0}^2 + \upsilon_1 n s_{d0}.$$

One has $\phi_2(s_{d0}; n, p)$ is increasing in s_{d0} when $0 < s_{d0} < n/2 - \upsilon_2 / 2p(p-2)$, and is decreasing in s_{d0} when $n/2 - \upsilon_2/2p(p-2) < s_{d0} < n/2$. (5.1)

Since

$$\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0}; n, p) = -4s_{d0}^3 + 3(\upsilon_2 + 2n)s_{d0}^2 - 2(\upsilon_1 + \upsilon_2 n + n^2)s_{d0} + \upsilon_1 n,$$

$$\frac{\partial^2}{\partial s_{d0}^2}\phi_3(s_{d0}; n, p) = -12s_{d0}^2 + 6(\upsilon_2 + 2n)s_{d0} - 2(\upsilon_1 + \upsilon_2 n + n^2),$$

one can see that $\partial^2\phi_3(s_{d0}; n, p)/\partial s_{d0}^2$ is increasing in s_{d0} when $s_{d0} < (\upsilon_2 + 2n)/4$ and is decreasing in s_{d0} when $s_{d0} > (\upsilon_2 + 2n)/4$, hence $\partial^2\phi_3(s_{d0}; n, p)/\partial s_{d0}^2$ is increasing in s_{d0} when $0 < s_{d0} \le n/2$ since $(\upsilon_2 + 2n)/4 > n/2$. Moreover,

$$\left.\frac{\partial^2}{\partial s_{d0}^2}\phi_3(s_{d0};n,p)\right|_{s_{d0}=n/2} = -3n^2 + 3(\upsilon_2 + 2n)n - 2(\upsilon_1 + \upsilon_2 n + n^2)$$

$$= n\big((1-3p)n + 3p(1+2v)\big) - 2v(1+v)p^2$$

$$\le n\big((10-3p)n + 3p\big) - 2v(1+v)p^2 < 0 \quad \text{for} \quad p \ge 4 .$$

Therefore, $\partial^2\phi_3(s_{d0};n,p)\big/\partial s_{d0}^2 < 0$ for $0 < s_{d0} \le n/2$, that is, $\partial\phi_3(s_{d0};n,p)\big/\partial s_{d0}$ is decreasing in s_{d0} when $0 < s_{d0} \le n/2$. Since

$$\left.\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0};n,p)\right|_{s_{d0}=0} = \upsilon_1 n > 0 \text{, and}$$

$$\left.\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0};n,p)\right|_{s_{d0}=n/2} = -n^3/2 + (3/4)(\upsilon_2 + 2n)n^2 - 2(\upsilon_1 + \upsilon_2 n + n^2)n + \upsilon_1 n$$

$$= -n^3 - (5/4)\upsilon_2 n^2 - \upsilon_1 n < 0 ,$$

there exists a $0 < s_{d0}^* < n/2$ such that $\partial\phi_3(s_{d0};n,p)\big/\partial s_{d0} > 0$ when $0 < s_{d0} < s_{d0}^*$ and $\partial\phi_3(s_{d0};n,p)\big/\partial s_{d0} < 0$ when $s_{d0}^* < s_{d0} \le n/2$, that is, $\phi_3(s_{d0};n,p)$ is increasing in s_{d0} when $0 < s_{d0} < s_{d0}^*$ and is decreasing in s_{d0} when $s_{d0}^* < s_{d0} \le n/2$.

Furthermore,

$$\left.\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0};n,p)\right|_{s_{d0}=\frac{n}{2}-\frac{\upsilon_2}{2p(p-2)}}$$

$$= \frac{1}{2}\left(\frac{n}{2} - \frac{\upsilon_2}{2p(p-2)}\right)\left(\left(\frac{n}{2} - \frac{\upsilon_2}{2p(p-2)}\right)\left(-n + \frac{\upsilon_2}{2p(p-2)} + \frac{3}{2}(\upsilon_2 + 2n)\right) - 2n(\upsilon_2 + n)\right)$$

$$+ \frac{\upsilon_1\upsilon_2}{p(p-2)}$$

$$= \frac{\upsilon_2}{2p(p-2)}\left(-\left(p(p-2)n-\upsilon_2\right)\left(\frac{1}{p(p-2)}\left(n+\frac{\upsilon_2}{p(p-2)} \right) + \frac{1}{2}\left(n+\frac{3\upsilon_2}{p(p-2)} \right) \right) + 2\upsilon_1 \right)$$

$$= \frac{\upsilon_2}{2p(p-2)}\left(-\left(n-\frac{\upsilon_2}{p(p-2)} \right)\left(\frac{p^2-2p+2}{2}\left(n+\frac{\upsilon_2}{p(p-2)} \right) - \upsilon_2 \right) + 2\upsilon_1 \right)$$

$$= \frac{\upsilon_2}{2p(p-2)}\left(\frac{-p^2+2p-2}{2}\left(n^2-\frac{\upsilon_2^2}{p^2(p-2)^2} \right) - \left(n\upsilon_2-\frac{\upsilon_2^2}{p(p-2)} \right) + 2\upsilon_1 \right).$$

Now since

$$\upsilon_1-(p^2-2p+2)n^2/2 = (-p^2/2+3p-1)n^2-2(1+2v)np+v(1+v)p^2,$$

$$\upsilon_1-\upsilon_2 n = pn^2-(1+2v)np+v(1+v)p^2,$$

$$\upsilon_2^2/p(p-2) = \left(p/(p-2)\right)\left(n^2-2(1+2v)n+(1+2v)^2\right),$$

$$\frac{(p^2-2p+2)\upsilon_2{}^2}{2p^2(p-2)^2} = \frac{(p^2-2p+2)}{2p^2(p-2)^2}\left(n^2-2(1+2v)n+(1+2v)^2\right)$$

$$> \frac{1}{2}\left(n^2-2(1+2v)n+(1+2v)^2\right),$$

one has

$$\left.\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0};n,p)\right|_{s_{d0}=\frac{n}{2}-\frac{\upsilon_2}{2p(p-2)}}$$

$$> \left(-p^2/2+4p+p/(p-2)-1/2\right)n^2-(1+2v)\left(3p+2p/(p-2)+1\right)n$$

$$+\left(1+p/(p-2)\right)(1+2v)^2+2v(1+v)p^2>0 \quad \text{when} \quad p\le 6, \qquad (5.2)$$

since for $p=6$,

$$\left.\frac{\partial}{\partial s_{d0}}\phi_3(s_{d0};n,6)\right|_{s_{d0}=\frac{n}{2}-\frac{\upsilon_2}{48}}$$

$> 7n^2 - 22(1+2v)n + 5(1+2v)^2/2 + 72v(1+v)$

$> 7n^2 - (22/3)(2n - s_{d0} + 3)n + (5/18)(2n - s_{d0} + 3)^2 + 12(2n - s) + 2(2n - s_{d0})^2$

$= n(13n/9 - 16s_{d0}/9 + 16/3) + (41/3)s_{d0}(s_{d0}/6 - 1) + 5/2 > 0$,

and $\left.\frac{\partial}{\partial s_{d0}} \phi_3(s_{d0}; n, p)\right|_{s_{d0} = \frac{n}{2} - \frac{\upsilon_2}{2p(p-2)}}$ is decreasing in *p*.

Therefore, by (5.1) and (5,2), the minimum value of $g(s_{d0}; n, p)$ happens when $s_{d0} = s_0$ is between $[n/2 - (n - 2v - 1)/2(p-2)]$ and $[n/2]$ for $4 \le p \le 6$. Moreover, the minimum value of v is $[3n/2p]$, say v_1, and denote v_2 such that $v_2 < (n - 2v_1 - 1)/2(p-2) \le v_2 + 1$, the theorem is thus proved.

Example 5.3. For *p* = 4 and *n* = 24, then $s_0 = 12$, and the following type *S* design *S*(4, 3, 2) is A-optimal in $D(4+1, 24)$.

(0,1)	(0,2)	(0,3)	(0,4)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)
(2,4)	(3,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)

Example 5.4. For *p* = 4 and *n* = 34, then $s_0 = 16$, and the following type *S* design *S*(4, 4, 3) is A-optimal in $D(4+1, 34)$.

(0,1)	(0,2)	(0,3)	(0,4)	(0,1)	(0,2)	(0,3)	(0,4)
(0,1)	(0,2)	(0,3)	(0,4)	(0,1)	(0,2)	(0,3)	(0,4)
(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)	(1,2)	(1,3)
(1,4)	(2,3)	(2,4)	(3,4)	(1,2)	(1,3)	(1,4)	(2,3)
(2,4)	(3,4)						

Example 5.5. For p = 5 and n = 55, then $s_0 = 25$, and the following type S design S(5, 5, 3) is A-optimal in $D(5+1,55)$.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)
(2,4)	(2,5)	(3,4)	(3,5)	(4,5)	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)
(2,4)	(2,5)	(3,4)	(3,5)	(4,5)	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)
(2,4)	(2,5)	(3,4)	(3,5)	(4,5)					

Example 5.6. For p = 6 and n = 27, then $s_0 = 12$, and the following type S design S(6, 2, 1) is A-optimal in $D(6+1,27)$.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,3)	(2,4)	(2,5)	(2,6)	(3,4)	(3,5)	(3,6)
(4,5)	(4,6)	(5,6)									

Example 5.7. For p = 6 and n = 54, then $s_0 = 24$, and the following type S design S(6, 4, 2) is A-optimal in $D(6+1,54)$.

(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,3)	(2,4)	(2,5)	(2,6)	(3,4)	(3,5)	(3,6)
(4,5)	(4,6)	(5,6)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,3)	(2,4)	(2,5)	(2,6)
(3,4)	(3,5)	(3,6)	(4,5)	(4,6)	(5,6)						