Appendix

Appendix A Change of measure from P^*-measure to P^T-measure

P^*-measure can by changed into P^T-measure by the following procedure,

$$\left(dW_1^t(s),\ dW_2^t(s)\right) = \left(dW_1^*(s) - \sigma(s,t)ds,\ dW_2^*(s)\right).$$

By Girsanov's theorem, the Randon-Nikodym derivative of P^T with respect to P^* is

$$\frac{dP^T}{dP^*}\bigg|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_t^T \sigma(s,T)^2 ds - \int_t^T \sigma(s,T) dW_1^T(s)\right\}$$

Applying $It\hat{o}$'s lemma to Eqn. (2.5),

$$d\log B(s,T) = \left(r(s) - \frac{1}{2}\sigma(s,T)^2\right)ds + \sigma(s,T)dW_1^*(s).$$

Taking integration from time t to T,

$$\log\frac{B(T,T)}{B(t,T)} = \int_t^T\left(r(s) - \frac{1}{2}\sigma(s,T)^2\right)ds + \int_t^T \sigma(s,T)dW_1^*(s).$$

Since $B(T,T)=1$, taking exponential transformation to previous equation,

$$\begin{aligned} e^{-\int_t^T r(s)ds} &= B(t,T)\cdot\exp\left\{-\frac{1}{2}\int_t^T \sigma(s,T)^2 ds + \int_t^T \sigma(s,T)dW_1^*(s)\right\} \\ &= B(t,T)\cdot\frac{dP^T}{dP^*} \end{aligned}$$

Consequently,

$$
\begin{aligned}
\pi_{t-1}(X) &= E_{P^*}\left(e^{-\int_{t-1}^{t} r(s)ds} X \,\middle|\, \mathcal{F}_{t-1} \right) \\
&= E_{P^*}\left(B(t-1,t) \cdot \frac{dP^T}{dP^*} \,\middle|\, \mathcal{F}_{t-1} \right). \\
&= B(t-1,t) E_{P^t}\left(X \,\middle|\, \mathcal{F}_{t-1} \right)
\end{aligned}
$$

Appendix B Dynamics under forward measure

(1) Under P^T -measure, the closed form for bond price $B(t,T)$ can be derived by the following procedure: Applying $It\hat{o}$'s lemma to Eqn. (2.5),

$$d\log B(s,T)=\left(r(s)-\frac{1}{2}\sigma(s,T)^2\right)ds+\sigma(s,T)dW_1^*(s)\ ,$$

and

$$d\log B(s,t')=\left(r(s)-\frac{1}{2}\sigma(s,t')^2\right)ds+\sigma(s,t')dW_1^*(s)\ ,$$

where $s<t'<T$. Subtracting the first process from the second one,

$$d\log\frac{B(s,t')}{B(s,T)}=-\frac{1}{2}\left(\sigma(s,t')^2-\sigma(s,T)^2\right)ds+\left(\sigma(s,t')-\sigma(s,T)\right)dW_1^*(s)\ .$$

By Eqn. (2.7), under P^T -measure,

$$d\log\frac{B(s,t')}{B(s,T)}=-\frac{1}{2}\left(\sigma(s,t')-\sigma(s,T)\right)^2ds+\left(\sigma(s,t')-\sigma(s,T)\right)dW_1^T(s)\ .$$

The closed form of $B(t,T)$ process is

$$B(t,T)=\frac{B(0,T)}{B(0,t)}\exp\left\{\frac{1}{2}\int_0^t\left(\sigma(s,t)-\sigma(s,T)\right)^2ds-\int_0^t\left(\sigma(s,t)-\sigma(s,T)\right)dW_1^T(s)\right\}$$

(2) Under P^T -measure, the closed form for reference portfolio $S(t)$ can be derived by the following procedure:

Applying $It\hat{o}$'s lemma to Eqns. (2.5) and (2.6) with log-transformation, the result can be presented as

$$S(t) = S(0)\exp\left\{\int_0^t \left(r(s) - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)\right) ds + \int_0^t \sigma_1 dW_1^*(s) + \int_0^t \sigma_2 dW_2^*(s)\right\},$$

and

$$B(t,T) = B(0,T)\exp\left\{\int_0^t \left(r(s) - \frac{1}{2}\sigma(s,T)^2 \right) ds + \int_0^t \sigma(s,T) dW_1^*(s)\right\}.$$

Since risk-free interest rate $r(t)$ is stochastic, the $r(t)$ process can be eliminated by dividing the $S(t)$ process with $B(t,T)$ process, namely,

$$\frac{S(t)}{B(t,T)} = \frac{S(0)}{B(0,T)}\exp\left\{\begin{array}{l} \frac{1}{2}\int_0^t \sigma(s,T)^2 ds - \frac{1}{2}\int_0^t \left(\sigma_1^2 + \sigma_2^2\right) ds \\ +\int_0^t \left(\sigma_1 - \sigma(s,T)\right) dW_1^*(s) + \int_0^t \sigma_2 dW_2^*(s) \end{array}\right\}$$

By Eqn. (2.7), under P^T -measure,

$$\frac{S(t)}{B(t,T)} = \frac{S(0)}{B(0,T)}\exp\left\{-\frac{1}{2}\int_0^t \left(\left(\sigma_1 - \sigma(s,T)\right)^2 + \sigma_2^2\right) ds + \int_0^t \left(\sigma_1 - \sigma(s,T)\right) dW_1^T(s) + \int_0^t \sigma_2 dW_2^T(s)\right\}$$

Dividing the previous process by the same process with $t = T$, the relative price process $\frac{S(T)}{S(t)}$ can be derived from previous equation:

$$\frac{S(T)}{S(t)} = \frac{1}{B(t,T)}\exp\left\{-\frac{1}{2}\int_t^T \left(\left(\sigma_1 - \sigma(s,T)\right)^2 + \sigma_2^2\right) ds + \int_t^T \left(\sigma_1 - \sigma(s,T)\right) dW_1^T(s) + \int_t^T \sigma_2 dW_2^T(s)\right\}$$

Appendix C Reviews of CRR models

Under the framework of CRR model, the log-normal process for the price of underlying asset S_t can be expressed by

$$\log\frac{S_t}{S_0}=\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\cdot W(t)$$

where $r\cdot t$ denotes the drift term during time period $[0,t]$ and σ denotes the volatility of the value of reference portfolio. A CRR model divides the interval $[0,t]$ into n equal subintervals of length $\Delta=t/n$. Under the no-arbitrage condition $d<e^{r\Delta}<u$, the movement factors u, d and the risk-neutral probability q can be specified as follows:

$$u=e^{\sigma\sqrt{\Delta}},\quad d=e^{-\sigma\sqrt{\Delta}},\quad q=\frac{e^{r\sqrt{\Delta}}-d}{u-d}.$$

This log-normal process can be approximated by discrete binomial process $\sum_{i=0}^{n}Z_i$, where Z_i s are *i.i.d* random variables with move-up factor u, move-down factor d, and the probability law:

$$Z_i=\begin{cases} u, & w.p.\ \ p \\ d, & w.p.\ \ 1-p \end{cases}.$$

A CRR model not only matches the true mean and variance of log-normal process, but also converges in distribution to log-normal process when n goes to infinity. Besides, CRR model also guarantees the convergence under Black-Scholes option pricing formula.

Take $n=3$ for example, the underlying binomial tree structure could be represented with the following diagram:

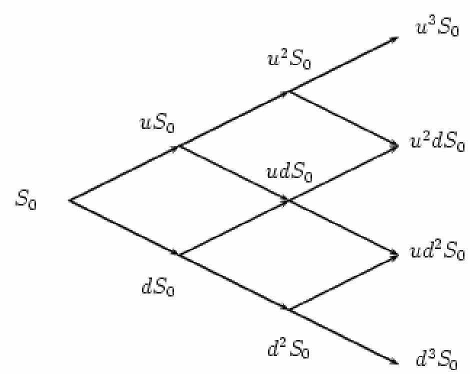

Figure B.1 The underlying asset price movement under CRR model

The risk-neutral expected value is obtained by discounting from the time of payment to zero. If the option is American type, it is necessary to determine the optimal value by checking at each node to see whether exercise is preferable to hold the option to next time period. Eventually, by working through all nodes backward to time zero, the value of option is obtained.

Appendix D The Markovian property of two-dimensional CRR models

(i) Changing of measure from P^{t+1}-measure to P^t-measure:

Since the pricing for contingent claim is performed under P^t-measure, from Eqn. (2.7), for $s < t$,

$$dW_1^{t+1}(s) = dW_1^*(s) - \sigma(t+1-s)ds,$$

and

$$dW_1^t(s) = dW_1^*(s) - \sigma(t-s)ds.$$

The following equation can be obtained by subtracting the second equations from the first one; that is,

$$dW_1^{t+1}(s) = dW_1^t(s) - \sigma ds.$$

From Eqn. (3.7), under the P^t-measure, the bond price process $B(t,t+1)$ is

$$B(t,t+1) = \frac{B(0,t+1)}{B(0,t)} \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma \cdot W_1^t(t)\right\}.$$

(ii) Markovian Property

Let $\mathcal{L}_t(X)$ denote the conditional distribution of random variable X given information set $\mathcal{F}_t = \sigma\left\{\left(W_1^t(s),\ W_2^t(s)\right): s \le t\right\}$. From Eqns. (3.1) and (3.9), under the P^t-measure, the conditional distribution of $\left(\log G(t),\ \log B(t,t+1)\right)$ given information set $\mathcal{F}_{t-1}$ is

$$\mathcal{L}_{t-1}\begin{pmatrix}\log G(t)\\ \log B(t,t+1)\end{pmatrix}=\mathcal{N}_2\left[\begin{pmatrix}-\log B(t-1,t)-\frac{1}{2}\sigma_G^2\\ \log B(t-1,t)\frac{B(0,t-1)B(0,t+1)}{B(0,t)^2}-\sigma^2(t-\frac{1}{2})\end{pmatrix},\begin{pmatrix}\sigma_G^2 & \sigma(\sigma_1-\frac{1}{2}\sigma)\\ \sigma(\sigma_1-\frac{1}{2}\sigma) & \sigma^2\end{pmatrix}\right]$$

(D.1)

where

$$Cov\left(\log G(t),\log B(t,t+1)\right)=\int_{t-1}^{t}\sigma(\sigma_1-\sigma(t-s))ds=\sigma(\sigma_1-\frac{1}{2}\sigma)\,.$$

Notably, the conditional distribution of $\left(\log G(t),\ \log B(t,t+1)\right)$ given information set $\mathcal{F}_{t-1}$ depends on bond price $B(t-1,t)$ which is $\mathcal{F}_{t-1}$-measurable. Consequently, the dynamics of bond price $B(t,t+1)$ and relative price $G(t)$ are Markovian stochastic processes

Appendix E The parameters for the second dimension CRR models

From Eqn. (D.1), the conditional distribution of bond price $B(t,t+1)$ given information set $\mathcal{F}_{t-1}$ (or bond price $B(t-1,t)$) and relative price $G(t)$ is

$$\mathcal{L}_{t-1}\left(\log B(t,t+1)\middle|G(t)\right)=\mathcal{N}\left(\mu_t,\ \sigma_t^2\right)\overset{d}{=}\log\alpha_t+r_{G(t)}-\frac{1}{2}\sigma_t^2+\sigma_t\tilde{W}_2(1)\,.$$

The conditional mean μ_t is decomposed into three components, that is,

$$\begin{aligned}\mu_t&=E\left(\log B(t,t+1)\right)+\frac{Cov\left(\log G(t),\log B(t,t+1)\right)}{Var\left(\log G(t)\right)}\left[\log G(t)-E\left(\log G(t)\right)\right]\\&=\log B(t-1,t)\frac{B(t-1)B(0,t+1)}{B(0,t)^2}-\sigma^2(t-\frac{1}{2})+\frac{\sigma(\sigma_1-\frac{1}{2}\sigma)}{\sigma_G^2}\left[\log G(t)+\log B(t-1,t)+\frac{1}{2}\sigma_G^2\right]\\&=\log\alpha_t+r_{G(t)}-\frac{1}{2}\sigma_t^2\end{aligned}$$

where

$$\alpha_t=B(t-1,t)\frac{B(0,t+1)B(0,t)}{B(0,t)^2}\,, \tag{E.1}$$

$$r_{G(t)}=-\sigma^2(t-\frac{1}{2})+\frac{\sigma(\sigma_1-\frac{1}{2}\sigma)}{\sigma_G^2}(\log G(t)+\log B(t-1,t)+\frac{1}{2}\sigma_G^2)+\frac{1}{2}\sigma_t^2\,, \tag{E.2}$$

$$\sigma_t^2=\sigma^2(1-\rho^2)\,, \tag{E.3}$$

and

$$\rho=\frac{\sigma_1-\frac{1}{2}\sigma}{\sigma_G}\,. \tag{E.4}$$

Appendix F Monte Carlo simulation for $T=3$

This Appendix shows the computation of

$$E_{p^2}\left((1+\delta_2)\max\left\{B(2,3),\ kA_{x+2:\overline{1}|}\right\}\middle|\mathcal{F}_1\right)$$

which is used in Monte Carlo simulation method for $T=3$. The above expectation value can be divided into four parts; that is,

$$\begin{aligned}&E_{p^2}\left((1+\delta_2)\max\left\{B(2,3),\ kA_{x+2:\overline{1}|}\right\}\middle|\mathcal{F}_1\right)\\&=E_{p^2}\left(\max\left\{B(2,3),\ kA_{x+2:\overline{1}|}\right\}\middle|\mathcal{F}_1\right)+E_{p^2}\left(\delta_2\max\left\{B(2,3),\ kA_{x+2:\overline{1}|}\right\}\middle|\mathcal{F}_1\right)\\&=A_2+E_{p^2}\left((B(2,3)-A_2)^+\middle|\mathcal{F}_1\right)+A_2E_{p^2}\left(\delta_2\middle|\mathcal{F}_1\right)+E_{p^2}\left(\delta_2(B(2,3)-A_2)^+\middle|\mathcal{F}_1\right)\end{aligned},$$

where $A_2=kA_{x+2:\overline{1}|}$. Lemmas 2, 3, and 4 are applied to the second, third and fourth terms, respectively. Let Φ denote the cumulative distribution function of standard gaussian distribution.

Lemma 2 *Consider* $T=3$, *given* $B(1,2)$, *let* $A_2=kA_{x+2:\overline{1}|}$ *, under* P^2*-measure,*

$$E_{p^2}\left((B(2,3)-A_2)^+\middle|\mathcal{F}_1\right)=B(1,2)\frac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\Phi(d_1)-A_2\Phi(d_2),$$

where $d_1=\dfrac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}-\log A_2}{\sigma}+\dfrac{1}{2}\sigma$ *, and* $d_2=d_1-\sigma$.

Proof. From Eqn. (3.9), bond price $B(2,3)$ given $B(1,2)$ under P^2-measure is

$$B(2,3) = B(1,2)\frac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\frac{1}{2}\sigma^2 + \int_1^2 \sigma dW_1^2(s)\right\}.$$

Applying the Black-Scholes method to calculate the call option price of $B(2,3)$ with strike price A_2, then the results of Lemma 2 is obtained. □

Lemma 3 *Consider* $T = 3$, *given* $B(1,2)$, *under* P^2*-measure,*

$$E_{p^2}\left(\delta_2 \middle| \mathcal{F}_1\right) = \frac{\eta}{1+i_g}\left[\frac{1}{B(1,2)}\Phi(d_1) - \left(1+\frac{i_g}{\eta}\right)\Phi(d_2)\right],$$

where $d_1 = \dfrac{-\log B(1,2) - \log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G} + \dfrac{1}{2}\sigma_G$, *and* $d_2 = d_1 - \sigma_G$.

Proof. From Eqn. (2.12), the process $G(2)$ given $B(1,2)$ under P^2-measure is

$$G(2) = \frac{1}{B(1,2)}\exp\left\{-\frac{1}{2}\int_1^2\left((\sigma_1 - \sigma(2-s))^2 + \sigma_2^2\right)ds + \int_1^2\left(\sigma_1 - \sigma(2-s)\right)dW_1^2(s) + \int_1^2 \sigma_2 dW_2^2(s)\right\}$$

From Eqn. (2.1),

$$E_{p^2}\left(\delta_2 \middle| \mathcal{F}_1\right) = \frac{\eta}{1+i_g}E_{p^2}\left[\left(G(2) - (1+\frac{i_g}{\eta})\right)^+ \middle| \mathcal{F}_1\right].$$

Applying the Black-Scholes method to calculate the call option price of $G(2)$ with strike price $1 + i_g/\eta$, then the results of Lemma 3 is obtained. □

Lemma 4 *Consider* $T=3$, *given* $B(1,2)$, *let* $A_2 = kA_{x+2:1}$ *, under* P^2 *-measure,*

$$E_{p^2}\left(\delta_2(B(2,3)-A_2)^+\middle|\mathcal{F}_1\right)=\frac{\eta}{1+i_g}\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}B(1,2)^{i-j}\left(\frac{B(0,1)B(0,3)}{B(0,2)^2}\right)^i A_2^{1-i}\left(1+\frac{i_g}{\eta}\right)^{1-j}$$
$$\cdot e^{-i\sigma^2+ij\sigma(\sigma_1-\frac{1}{2}\sigma)}\Phi(x_1^{i,j},x_2^{i,j};\rho),$$

where

$$x_1^{i,j}=\frac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}-\log A_2}{\sigma}-\frac{1}{2}\sigma+j\left(\sigma_1-\frac{1}{2}\sigma\right)-(1-i)\sigma,$$

and

$$x_2^{i,j}=\frac{-\log B(1,2)-\log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G}+\frac{1}{2}\sigma_G+i\frac{\sigma\left(\sigma_1-\dfrac{1}{2}\sigma\right)}{\sigma_G}-(1-j)\sigma_G.$$

σ_G *equals Eqn. (3.2) and* ρ *equals Eqn. (E.4).*

Proof. From Eqn. (2.1),

$$E_{p^2}\left(\delta_2(B(2,3)-A_2)^+\middle|\mathcal{F}_1\right)=\frac{\eta}{1+i_g}E_{p^2}\left[\left(G(2)-(1+\frac{i_g}{\eta})\right)^+\left(B(2,3)-A_2\right)^+\middle|\mathcal{F}_1\right]$$
$$=\frac{\eta}{1+i_g}\left\{\begin{array}{l}E_{p^2}\left[G(2)B(2,3)I_A\middle|\mathcal{F}_1\right]-\left(1+\dfrac{i_g}{\eta}\right)E_{p^2}\left[B(2,3)I_A\middle|\mathcal{F}_1\right]\\-A_2E_{p^2}\left[G(2)I_A\middle|\mathcal{F}_1\right]+\left(1+\dfrac{i_g}{\eta}\right)A_2E_{p^2}\left[I_A\middle|\mathcal{F}_1\right]\end{array}\right\},\qquad\text{(F.1)}$$

where $A = \left\{ G(2) > (1+\frac{i_g}{\eta}) \right\} \cap \left\{ B(2,3) > A_2 \right\}$.

(i) The first term in Eqn. (F.1) is derived as follows. By Eqns. (2.12) and (3.9),

$$G(2)B(2,3)$$
$$= \frac{B(0,1)B(0,3)}{B(0,2)^2} e^{-\sigma^2} \exp\left\{ \begin{array}{l} -\frac{1}{2}\sigma^2 - \frac{1}{2}\int_1^2 \left((\sigma_1 - \sigma(2-s))^2 + \sigma_2^2 \right) ds \\ + \int_1^2 \sigma dW_1^2(s) + \int_1^2 \left(\sigma_1 - \sigma(2-s) \right) dW_1^2(s) + \int_1^2 \sigma_2 dW_2^2(s) \end{array} \right\}$$

Letting

$$\begin{array}{l} d\tilde{W}_1 = dW_1^2 - \left(\sigma_1 - \sigma(2-s) + \sigma \right) ds \\ d\tilde{W}_2 = dW_2^2 - \sigma_2 ds \end{array}, \qquad \text{(F.2)}$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are two independent standard Wiener processes under $\tilde{Q}$-measure.

$$\frac{d\tilde{Q}}{dP^2} = \exp\left\{ -\frac{1}{2}\int_1^2 \left[\left(\sigma_1 - \sigma(2-s) + \sigma \right)^2 + \sigma_2^2 \right] ds + \int_1^2 \left(\sigma_1 - \sigma(2-s) + \sigma \right) dW_1^2 + \sigma_2 dW_2^2 \right\}$$
$$= \exp\left\{ \begin{array}{l} -\int_1^2 \sigma\left(\sigma_1 - \sigma(2-s) \right) ds - \frac{1}{2}\sigma^2 - \frac{1}{2}\int_1^2 \left[\left(\sigma_1 - \sigma(2-s) \right)^2 + \sigma_2^2 \right] ds \\ + \int_1^2 \left(\sigma_1 - \sigma(2-s) + \sigma \right) dW_1^2 + \sigma_2 dW_2^2 \end{array} \right\}$$

By changing of measure,

$$E_{p^2}\left[G(2)B(2,3)I_A \middle| \mathcal{F}_1 \right] = \frac{B(0,1)B(0,3)}{B(0,2)^2} e^{-\sigma^2} E_{p^2}\left[e^{\sigma(\sigma_1 - \frac{1}{2}\sigma)} \frac{d\tilde{Q}}{dP^2} I_A \middle| \mathcal{F}_1 \right] = \frac{B(0,1)B(0,3)}{B(0,2)^2} e^{-\sigma^2 + \sigma(\sigma_1 - \frac{1}{2}\sigma)} P^{\tilde{Q}}(A)$$

From Eqn. (F.2),

$$P^{\tilde{Q}}(A) = P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\dfrac{1}{2}\sigma^2+\displaystyle\int_1^2 \sigma dW_1^2(s)\right\} > A_2 \\ \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l} -\dfrac{1}{2}\displaystyle\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds \\ +\displaystyle\int_1^2\left(\sigma_1-\sigma(2-s)\right)dW_1^2(s)+\int_1^2\sigma_2 dW_2^2(s)\end{array}\right\} > (1+\dfrac{i_g}{\eta}) \end{array}\right)$$

$$= P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{\dfrac{1}{2}\sigma^2+\sigma\left(\sigma_1-\dfrac{1}{2}\sigma\right)+\displaystyle\int_1^2 \sigma d\tilde{W}_1(s)\right\} > A_2 \\ \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l} \dfrac{1}{2}\displaystyle\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds \\ +\displaystyle\int_1^2\left(\sigma_1-\sigma(2-s)\right)d\tilde{W}_1(s)+\int_1^2\sigma_2 d\tilde{W}_2(s)\end{array}\right\} > (1+\dfrac{i_g}{\eta}) \end{array}\right)$$

$$= \Phi(x_1^{1,1}, x_2^{1,1}; \rho)$$

where

$$x_1^{1,1} = \frac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2} - \log A_2}{\sigma} - \frac{1}{2}\sigma + \left(\sigma_1 - \frac{1}{2}\sigma\right),$$

$$x_2^{1,1} = \frac{-\log B(1,2) - \log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G} + \frac{1}{2}\sigma_G + \frac{\sigma\left(\sigma_1 - \dfrac{1}{2}\sigma\right)}{\sigma_G},$$

and

$$\rho = \int_1^2 \sigma\left(\sigma_1 - \sigma(t-s)\right)ds = \sigma\left(\sigma_1 - \frac{1}{2}\sigma\right).$$

(ii) The second term in Eqn. (F.1) is derived as follows. By Eqn. (3.9),

$$B(2,3) = B(1,2)\frac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\frac{1}{2}\sigma^2 + \int_1^2 \sigma dW_1^2(s)\right\}.$$

Letting

$$\begin{aligned} d\tilde{W}_1 &= dW_1^2 - \sigma ds \\ d\tilde{W}_2 &= dW_2^2 \end{aligned}, \qquad \text{(F.3)}$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are two independent standard Wiener processes under $\tilde{Q}$-measure.

$$\frac{d\tilde{Q}}{dP^2} = \exp\left\{-\frac{1}{2}\int_1^2 \sigma^2 ds + \int_1^2 \sigma dW_1^2\right\}.$$

By changing of measure,

$$E_{p^2}\left[B(2,3)I_A \middle| \mathcal{F}_1\right] = B(1,2)\frac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}E_{p^2}\left[\frac{d\tilde{Q}}{dP^2}I_A \middle| \mathcal{F}_1\right] = B(1,2)\frac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}P^{\tilde{Q}}(A)$$

From Eqn. (F.3),

$$P^{\tilde{Q}}(A) = P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\dfrac{1}{2}\sigma^2 + \displaystyle\int_1^2 \sigma dW_1^2(s)\right\} > A_2 \\ \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l} -\dfrac{1}{2}\displaystyle\int_1^2 \left((\sigma_1 - \sigma(2-s))^2 + \sigma_2^2\right)ds \\ +\displaystyle\int_1^2 \left(\sigma_1 - \sigma(2-s)\right)dW_1^2(s) + \int_1^2 \sigma_2 dW_2^2(s) \end{array}\right\} > (1+\dfrac{i_g}{\eta}) \end{array}\right)$$

$$
= P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{\dfrac{1}{2}\sigma^2+\int_1^2\sigma d\tilde{W}_1(s)\right\}>A_2 \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l}\sigma\left(\sigma_1-\dfrac{1}{2}\sigma\right)+\dfrac{1}{2}\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds \\ +\int_1^2\left(\sigma_1-\sigma(2-s)\right)d\tilde{W}_1(s)+\int_1^2\sigma_2 d\tilde{W}_2(s)\end{array}\right\}>(1+\dfrac{i_g}{\eta})\end{array}\right)
$$

$$
=\Phi(x_1^{1,0},x_2^{1,0};\rho)
$$

where

$$
x_1^{1,0}=\frac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}-\log A_2}{\sigma}-\frac{1}{2}\sigma\,,
$$

$$
x_2^{1,0}=\frac{-\log B(1,2)-\log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G}+\frac{1}{2}\sigma_G+\frac{\sigma\left(\sigma_1-\dfrac{1}{2}\sigma\right)}{\sigma_G}-\sigma_G\,,
$$

and

$$
\rho=\int_1^2\sigma\left(\sigma_1-\sigma(t-s)\right)ds=\sigma\left(\sigma_1-\frac{1}{2}\sigma\right).
$$

(iii) The third term in Eqn. (F.1) is derived as follows. By Eqn. (2.12),

$$
G(2)=\frac{1}{B(1,2)}\exp\left\{-\frac{1}{2}\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds+\int_1^2\left(\sigma_1-\sigma(2-s)\right)dW_1^2(s)+\int_1^2\sigma_2 dW_2^2(s)\right\}
$$

Letting

$$\begin{aligned} d\tilde{W}_1 &= dW_1^2 - \left(\sigma_1 - \sigma(2-s)\right) ds \\ d\tilde{W}_2 &= dW_2^2 - \sigma_2 ds \end{aligned} , \tag{F.4}$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are two independent standard Wiener processes under $\tilde{Q}$-measure.

$$\frac{d\tilde{Q}}{dP^2} = \exp\left\{ -\frac{1}{2}\int_1^2 \left[\left(\sigma_1 - \sigma(2-s)\right)^2 + \sigma_2^2 \right] ds + \int_1^2 \left(\sigma_1 - \sigma(2-s)\right) dW_1^2 + \sigma_2 dW_2^2 \right\}$$

By changing of measure,

$$E_{p^2}\left[G(2) I_A \middle| \mathcal{F}_1 \right] = \frac{1}{B(1,2)} E_{p^2}\left[\frac{d\tilde{Q}}{dP^2} I_A \middle| \mathcal{F}_1 \right] = \frac{1}{B(1,2)} P^{\tilde{Q}}(A)$$

From Eqn. (F.4),

$$P^{\tilde{Q}}(A) = P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2} e^{-\sigma^2} \exp\left\{ -\dfrac{1}{2}\sigma^2 + \displaystyle\int_1^2 \sigma dW_1^2(s) \right\} > A_2 \\[2em] \dfrac{1}{B(1,2)} \exp\left\{ \begin{array}{l} -\dfrac{1}{2}\displaystyle\int_1^2 \left((\sigma_1 - \sigma(2-s))^2 + \sigma_2^2 \right) ds \\ +\displaystyle\int_1^2 \left(\sigma_1 - \sigma(t-s)\right) dW_1^2(s) + \int_1^2 \sigma_2 dW_2^2(s) \end{array} \right\} > (1 + \dfrac{i_g}{\eta}) \end{array} \right)$$

$$= P^{\tilde{Q}}\left(\begin{array}{l} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\dfrac{1}{2}\sigma^2+\sigma\left(\sigma_1-\dfrac{1}{2}\sigma\right)+\displaystyle\int_1^2\sigma d\tilde{W}_1(s)\right\}>A_2 \\ \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l}\dfrac{1}{2}\displaystyle\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds \\ +\displaystyle\int_1^2\left(\sigma_1-\sigma(t-s)\right)d\tilde{W}_1(s)+\displaystyle\int_1^2\sigma_2 d\tilde{W}_2(s)\end{array}\right\}>(1+\dfrac{i_g}{\eta}) \end{array}\right)$$

$$= \Phi(x_1^{0,1}, x_2^{0,1}; \rho)$$

where

$$x_1^{0,1} = \frac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2} - \log A_2}{\sigma} - \frac{1}{2}\sigma + \left(\sigma_1 - \frac{1}{2}\sigma\right) - \sigma \,,$$

$$x_2^{0,1} = \frac{-\log B(1,2) - \log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G} + \frac{1}{2}\sigma_G \,,$$

and

$$\rho = \int_1^2 \sigma\left(\sigma_1 - \sigma(t-s)\right)ds = \sigma\left(\sigma_1 - \frac{1}{2}\sigma\right).$$

(iv) The forth term in Eqn. (F.1) is derived as follows.

$$
P^{\tilde{Q}}(A) = P^{\tilde{Q}}\left(\begin{array}{c} B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2}e^{-\sigma^2}\exp\left\{-\dfrac{1}{2}\sigma^2+\displaystyle\int_1^2 \sigma dW_1^2(s)\right\} > A_2 \\ \\ \dfrac{1}{B(1,2)}\exp\left\{\begin{array}{l} -\dfrac{1}{2}\displaystyle\int_1^2\left((\sigma_1-\sigma(2-s))^2+\sigma_2^2\right)ds \\ +\displaystyle\int_1^2\left(\sigma_1-\sigma(t-s)\right)dW_1^2(s)+\displaystyle\int_1^2\sigma_2 dW_2^2(s) \end{array}\right\} > (1+\dfrac{i_g}{\eta}) \end{array}\right)
$$

$$
=\Phi(x_1^{0,0}, x_2^{0,0};\rho)
$$

where

$$
x_1^{0,0} = \frac{\log B(1,2)\dfrac{B(0,1)B(0,3)}{B(0,2)^2} - \log A_2}{\sigma} - \frac{1}{2}\sigma - \sigma \,,
$$

$$
x_2^{0,0} = \frac{-\log B(1,2) - \log\left(1+\dfrac{i_g}{\eta}\right)}{\sigma_G} + \frac{1}{2}\sigma_G - \sigma_G \,,
$$

and

$$
\rho = \int_1^2 \sigma\left(\sigma_1 - \sigma(t-s)\right)ds = \sigma\left(\sigma_1 - \frac{1}{2}\sigma\right).
$$

Combining the formulas of (i)-(iv), then, the results of Lemma 4 are obtained. □