Chapter 3

Two-dimensional CRR Models

This chapter first devises two-dimensional CRR models for the relative price of a reference portfolio and a zero coupon bond price to fairly value the participation contract and early exercise of the surrender option. A backwards algorithm is then deveploed for contract pricing.

3.1 Construction of Two-dimensional CRR Model

A CRR (Cox, Ross & Rubinstein, 1979) model with a constant interest rate provides a simple and efficient numerical method for valuing options in situations where premature exercise may be optimal, for example American or Bermudan options. The details of the CRR model are described in Appendix C. However, given a stochastic reference portfolio and risk-free interest rate, it becomes necessary to establish a two-dimensional CRR model for a joint two-dimensional log-normal distribution of the relative price of reference portfolio and the zero coupon bond price, for efficiently calculating the fair value of the participation policy with a surrender option. Therefore, this subsection devises a two-dimensional CRR model for the prices of a reference portfolio and a zero coupon bond to approximate the two-dimensional Gaussian density.

The tree structure of the two-dimensional CRR model is constructed by following procedures. (Fig. 1) The root of the tree is the bond price $B(0,1)$.

Step1：(The first dimension) Given initial bond price $B(t-1,t)$, generate the nodes of relative price $G(t)$.

Step 2：(The second dimension) For each node of $G(t)$ derived in Step 1, generate nodes of bond price $B(t,t+1)$.

Repeating steps 1 and 2 for $t = 1, 2, ..., T$.

Notably, under the Ho and Lee model, the joint distribution of bond price $B(t,t+1)$ and relative price $G(t)$ given information set $\mathcal{F}_{t-1}$ depends only on $B(t-1,t)$. Therefore, the two-dimensional CRR model follows a Markovian process. That is, given a current value of bond price $B(t-1,t)$, future development of the bond price $B(t,t+1)$ and relative price $G(t)$ are independent of past movement. Details of proof are represented in Appendix D.

The two steps in the two-dimensional CRR model are detailed as follows. In step 1, from Eqn. (2.12), under the P^t-measure, the dynamic process of relative price $G(t)$, given bond price $B(t-1,t)$, is denoted by

$$\begin{aligned} G(t) &= \frac{1}{B(t-1,t)} \exp\left\{ -\frac{1}{2} \int_{t-1}^{t} \left((\sigma_1 - \sigma(t-s))^2 + \sigma_2^2 \right) ds + \int_{t-1}^{t} \left(\sigma_1 - \sigma(t-s) \right) dW_1^t(s) + \int_{t-1}^{t} \sigma_2 dW_2^t(s) \right\} \\ &\overset{d}{=} \frac{1}{B(t-1,t)} \exp\left\{ -\frac{1}{2} \sigma_G^2 + \sigma_G \tilde{W}_1(1) \right\} \end{aligned} \tag{3.1}$$

where $\overset{d}{=}$ indicates an equal distribution, $\tilde{W}_1$ is a new Brownian Motion and

$$\sigma_G^2 = \int_{t-1}^{t} \left((\sigma_1 - \sigma(t-s))^2 + \sigma_2^2 \right) ds = \sigma_1^2 - \sigma\sigma_1 + \frac{1}{3}\sigma^2 + \sigma_2^2 . \tag{3.2}$$

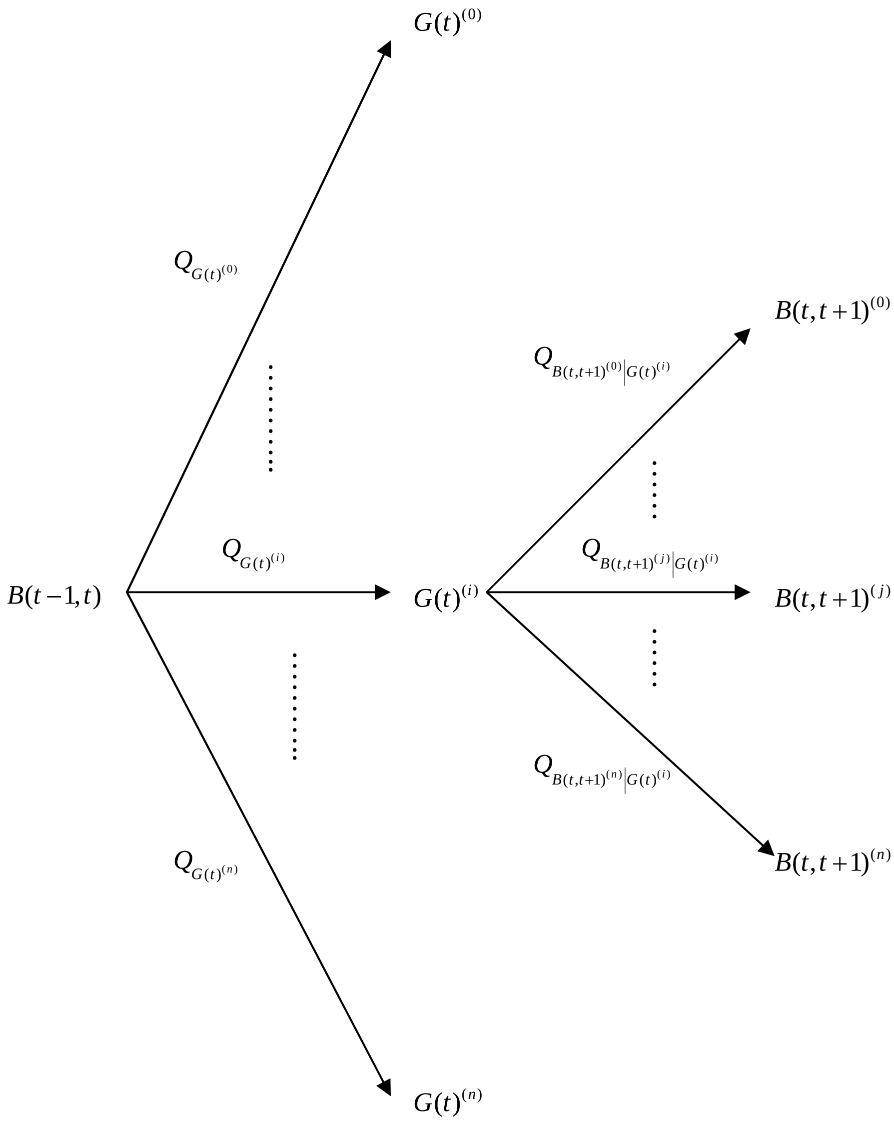

Figure 1 Two-dimensional CRR model for relative price $G(t)$ and bond price $B(t,t+1)$ during time $t-1$ to t

From the CRR model, this study divides each policy year into n sub-period with equal length, $\Delta = 1/n$. Moreover, the drift term equals 0. Furthermore, the up and down factors are

$$U_G = e^{\sigma_G \sqrt{\Delta}}, \quad D_G = 1/U_G . \tag{3.3}$$

Under the risk neutral measure, the probability of event $\{G(\tau+\Delta) = U_G G(\tau)\}$ is given by

$$Q_G = \frac{1-D_G}{U_G - D_G}, \tag{3.4}$$

whereas

$$1-Q_G = \frac{U_G - 1}{U_G - D_G}$$

denotes the probability of event $\{G(\tau+\Delta) = D_G G(\tau)\}$. Notably, given a drift term r, to prevent arbitrage opportunities, the volatility parameter is fixed so that $D_G < e^{r\Delta} < U_G$, implying a strictly positive value less than 1 for both Q_G and $1-Q_G$. In step 1, the drift term equals 0, so the arbitrage condition, $D_G < e^{r\Delta} < U_G$, clearly holds.

From Eqns. (3.3) and (3.4), let $\{G(t)^{(i)} : i = 0,1,...,n\}$ represent the possible values derived by $(n-i)$'s up factor U_G and i's down factor D_G, that is,

$$G(t)^{(i)} = \frac{1}{B(t-1,t)} U_G^{n-i} D_G^{i}, \quad i = 0,1,...,n. \tag{3.5}$$

with risk neutral probabilities

$$Q_{G(t)^{(i)}} = \binom{n}{i} Q_G^{n-i}(1-Q_G)^i, \quad i = 0,1,\ldots,n. \tag{3.6}$$

In step 2, From Eqn. (2.11), under the P^{t+1}-measure, the dynamic process of zero coupon bond price $B(t,t+1)$ is

$$B(t,t+1) = \frac{B(0,t+1)}{B(0,t)} \exp\left\{\frac{1}{2}\sigma^2 t + \sigma \cdot W_1^{t+1}(t)\right\}. \tag{3.7}$$

By changing the measure, under the P^t-measure, the bond price process $B(t,t+1)$ is

$$B(t,t+1) = \frac{B(0,t+1)}{B(0,t)} \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma \cdot W_1^t(t)\right\}. \tag{3.8}$$

Details of the computational procedure are presented in Appendix D. Dividing Eqn. (3.8) by Eqn. (2.13), yields the following result,

$$B(t,t+1) = B(t-1,t)\frac{B(0,t-1)B(0,t+1)}{B(0,t)^2} e^{-\sigma^2(t-1)} \exp\left\{-\frac{1}{2}\sigma^2 + \int_{t-1}^{t} \sigma dW_1^t(s)\right\}. \tag{3.9}$$

Applying log-transformation to Eqns. (2.12) and (3.9), $\log B(t,t+1)$ and $\log G(t)$ have a bivariate Gaussian distribution. Using linear regression, the conditional distribution of $\log B(t,t+1)$ given $\log G(t)$ is a Gaussian distribution with conditional mean μ_t and conditional volatility σ_t. Details of the computation are left in Appendix E.

Therefore, the conditional distribution of $\log B(t,t+1)$ given $\log G(t)$ is equivalent to the following process:

$$\log B(t,t+1) \overset{d}{=} \mu_t + \sigma_t \varepsilon \overset{d}{=} \log \alpha_t + r_{G(t)} - \frac{1}{2}\sigma_t^2 + \sigma_t \tilde{W}_2(1)\,, \tag{3.10}$$

where

$$\alpha_t = B(t-1,t)\frac{B(0,t+1)B(0,t)}{B(0,t)^2}\,,$$

$$r_{G(t)} = -\sigma^2(t-\frac{1}{2}) + \frac{\sigma(\sigma_1 - \frac{1}{2}\sigma)}{\sigma_G^2}(\log G(t) + \log B(t-1,t) + \frac{1}{2}\sigma_G^2) + \frac{1}{2}\sigma_t^2\,,$$

$$\sigma_t^2 = \sigma^2(1-\rho^2)\,,$$

and

$$\rho = \frac{\sigma_1 - \frac{1}{2}\sigma}{\sigma_G}\,.$$

Notably, α_t depends on bond price $B(t-1,t)$, $r_{G(t)}$ depends on relative price $G(t)$, $\varepsilon \sim N(0,1)$, and $\tilde{W}_2$ is a new Brownian Motion. To approximate the conditional distribution of $\log B(t,t+1)$ given $\log G(t)$, this work chooses α_t and r_G as the initial value and drift term, respectively, for the CRR model. Then, the up and down factors, for the first dimension CRR model, are

$$U_B = e^{\sigma_t \sqrt{\Delta}}\,,\quad D_B = 1/U_B\,. \tag{3.11}$$

and the risk neutral probability of up movement is given by

$$Q_B = \frac{e^{r_{G(t)}\Delta} - D_B}{U_B - D_B}. \tag{3.12}$$

Significantly, ruling out arbitrage requires assuming that $r_{G(t)}$ satisfies the no-arbitrage condition $D_B < e^{r_{G(t)}\Delta} < U_B$, so that both the risk-neutral probabilities, Q_B and $1 - Q_B$, are strictly positive and below 1. The no-arbitrage condition then is obtained by $\left| r_{G(t)} \right| < \sigma_B / \sqrt{\Delta}$.

Under the P^t-measure and given bond price $B(t-1,t)$ and relative price $G(t)$, the possible value of bond price $B(t,t+1)$ can be derived by

$$B(t,t+1)^{(j)} = \alpha_t U_B^{n-i} D_B^i, \quad j = 0,1,\ldots,n, \tag{3.13}$$

with corresponding probability

$$Q_{B(t,t+1)^{(j)}|G(t)} = \binom{n}{j} Q_B^{n-j}(1-Q_B)^j, \quad j = 0,1,\ldots,n. \tag{3.14}$$

Notably, this work possesses two advantages. First, using Eqn. (3.13), the initial value α_t and the moving factors U_B and D_B only depend on bond price $B(t-1,t)$, but not on the relative price $G(t)$. Hence, the value of the zero coupon bonds $B(t,t+1)$ remains unchanged regardless of the relative price $G(t)$. However, Eqns. (3.12) and (4.14) show that the corresponding risk neutral probabilities differ with the relative price $G(t)$. Second, the conditional volatilities σ_t remain constant for each time period, the tree structure of the zero coupon bond price is recombined, and the node number increases only linearly with the number of exercise dates. The proposed approach is less computationally intensive, thus making it appropriate for long term policies.

3.2 Valuation Framework

This subsection introduces the fair valuation of the policy and a recursive backwards algorithm based on the two-dimensional CRR models described above.

Let V_t and Y_t denote the optimal and continuation values at time t, respectively. Given time t, the benefit is C_{t+1} paid at time $t+1$ if the insured dies between times t and time $t+1$, otherwise, the contract value for policyholders is V_{t+1}. Consequently, using Eqn. (2.8) and the independence of financial and mortality risks, the continuation value Y_t at time t is presented as follows:

$$Y_t = B(t,t+1)\left[q_{x+t}C_{t+1} + p_{x+t}E_{p^{t+1}}\left(V_{t+1} \middle| \mathcal{F}_t\right)\right], \quad t = 0,1,2,...,T-1. \tag{3.15}$$

Specially, at time $T-1$, if the policy remains in force, the policyholder is paid by C_T at the end of the T-th year, regardless of whether the insured dies during the T-th year or survives. Therefore, the contract value at maturity date T equals C_T. From Eqn. (3.15), the continuation value at time $T-1$ is

$$Y_{T-1} = C_T B(T-1,T). \tag{3.16}$$

Comparing the continuation value with the surrender cash value, the optimal value V_t can be determined by

$$V_t = \max\left\{Y_t,\ R_t\right\}, \quad t = 0,1,2,...,T-1, \tag{3.17}$$

where R_t is calculated from Eqn. (2.4). Initializing from (3.16) and repeating the previous procedures such as Eqn. (3.15) and (3.17) from time $T-1$ to 0 recursively, the final continuation value Y_0 is the fair value of the contract.

The following lemma simplifies the calculation of the fair value of contract Y_0 and can be applied to the recursive backward algorithm for the two-dimensional CRR model.

Lemma 1 *Let* P_S *denote the fair value of the policy with basic benefit* C_1 *and maturity date* T. *Define* $\tilde{V}_T = 1$, *and*

$$\tilde{V}_t = (1+\delta_t)\max\left\{B(t,t+1)\left[q_{x+t} + p_{x+t}E_{p^{t+1}}\left(\tilde{V}_{t+1}\middle|\mathcal{F}_t\right)\right],\ kA_{x+t:\overline{T-t|}}\right\}, \qquad .(3.18)$$

for $t = T-2$ *back to* 1. *Then,*

$$P_S = C_1 B(0,1)\left[q_x + p_x E_{p^1}\left(\tilde{V}_1\right)\right]. \qquad (3.19)$$

Proof. Letting $V_t = C_t\tilde{V}_t$ in Eqns. (3.15)-(3.17), Eqns (3.18) and (3.19) can be obtained by induction. □

Based on the two-dimensional tree structure and Lemma 1, the backward recursive procedure for the fair value of the contract is as follows:

For a two-dimensional tree initiated from a node of bond price $B(t-1,t)$ (see Fig. 2), from Eqns. (3.18)-(3.19), the corresponding contract values $\tilde{V}_t$ can be calculated for each node of bond price $B(t,t+1)$, that is,

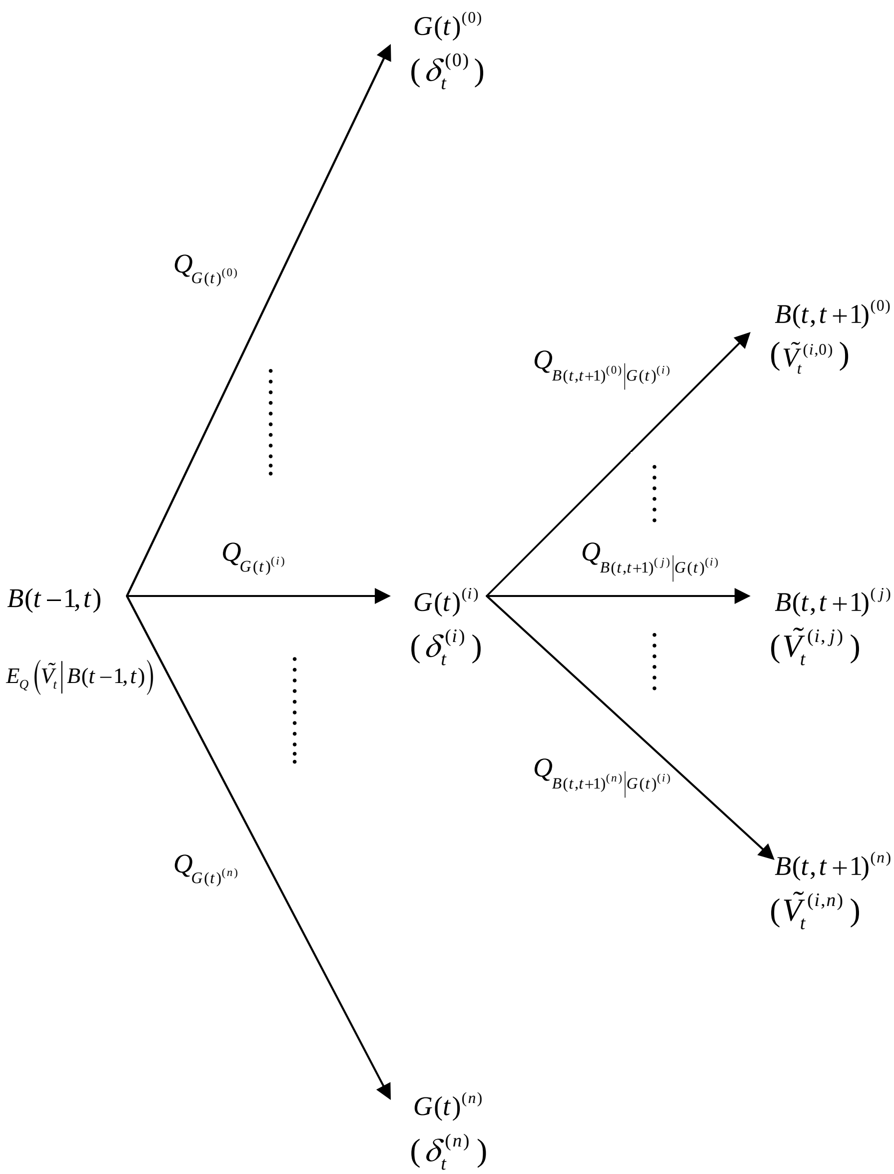

Figure 2 Cash flow in each node of a two-dimensional CRR model during time period $t-1$ to t

$$\tilde{V}_t^{(i,j)} = (1+\delta_t^{(i)}) \max\left\{ B(t,t+1)^{(j)} \left[q_{x+t} + p_{x+t} E_Q\left(\tilde{V}_{t+1} \middle| B(t,t+1)^{(j)} \right) \right],\ kA_{x+t:\overline{T-t}|} \right\} \quad (3.20)$$

and the expected optimal contract value $\tilde{V}_t$ for the root node, bond price $B(t-1,t)$, is provided by

$$E_Q\left(\tilde{V}_t \middle| B(t-1,t) \right) = \sum_{i=0}^{n} \sum_{j=0}^{n} Q_{G(t)^{(i)}} Q_{B(t,t+1)^{(j)}|G(t)^{(i)}} \cdot \tilde{V}_t^{(i,j)} , \quad (3.21)$$

where E_Q represents taking expectation with respect to the risk neutral probabilities such as those in Eqns. (3.6) and (3.14). Repeating the previous procedure from time $T-1$ back to 0 recursively, the final value $\tilde{P}_S$ for root node $B(0,1)$ is the fair contract value.

$$\tilde{P}_S = C_1 B(0,1)\left[q_x + p_x E_Q\left(\tilde{V}_1 \right) \right]$$

Notably, in Eqn. (3.20), this work derives the adjustment rate δ_t via the relative price $G(t)$ rather than the reference portfolio price $S(t)$, and then doing so without reference to the price during the previous time $S(t-1)$. Furthermore, if the continuation values are calculated based on Eqns. (3.15)-(3.17) directly rather than Lemma 1, it becomes necessary to calculate the benefits C_t or C_T based on previous path movement which leads to the initial node of bond price $B(t-1,t)$. Conversely, Eqn. (3.20) demonstrates that there are two stochastic sources: adjustment rate δ_t and bond price $B(t,t+1)$, which can be generated from the bond price $B(t-1,t)$ without reference to the previous path. Therefore, this work devises a complete recursive algorithm for contract pricing.