Appendix A

$h_1$ is the solution to the following optimization problem

\[
\max_{x \in A} E[\log X_T] \\
\text{st.} \quad dX_t = x_t' dR_t + (1 - x_t') r_t dt
\]

Since the log utility is myopic, $h_1$ is also the solution to the optimization problem

\[
\max_{x \in A} E[\log(X_t + dX_t)] \Leftrightarrow \max_{x \in A} E[\log(1 + \frac{dX_t}{X_t})] \\
\Leftrightarrow \max_{x \in A} E[\frac{dX_t}{X_t} - \frac{1}{2} \left( \frac{dX_t}{X_t} \right)^2] \\
\Leftrightarrow \max_{x \in A} E\left( \left[ \frac{x_t'}{(\mu_r - r_t 1)} - \frac{1}{2} x_t' \Sigma x_t \right] dt \right)
\]

First order condition:

\[
(\mu_r - r_t 1) - \frac{1}{2} (\Sigma + \Sigma') x_t = 0
\]

which yields $x_t^* = h_1 = (\Sigma')^{-1} \Sigma^{-1} (\mu_r - r_t 1_{N+1}) = (\Sigma')^{-1} \lambda(t)$

Consequently, the value of the numeraire portfolio at time $t$, $H_t$, is equal to

\[
\frac{dH_t}{H_t} = h_t' dR_t + (1 - h_t') r_t dt = [h_t' (\mu - r_t 1) + r_t] dt + h_t' \Sigma dw_t = [\lambda^2 + r_t] dt + \lambda' dw_t
\]

Hence, $H_t = \exp\left( \int_0^t (r_s + \frac{1}{2} \lambda_s^2) ds + \int_0^t \lambda_s' dw_s \right)$
Appendix B

Given the assets evolutions, we try to find out the minimum norm portfolios $h^m_t$ such that the minimum norm return $H^m_T = \frac{1}{H_T E(\frac{1}{H_T})}$.

Since $\frac{H^m_T}{H_T}$ is martingale, we have

$$E[\frac{H^m_T}{H_T} = E[\frac{1}{H_T^2}] / E[\frac{1}{H_T^2}]$$

$$H^m_T = H_T [\frac{1}{H_T} \exp(-2(\int_t^T (r_s + \frac{1}{2} \|H_s\|^2) ds + \int_t^T \lambda dW_s))] / E(\frac{1}{H_T^2})$$

$$= \frac{1}{H_T} \Psi(t, r_T) / E[\frac{1}{H_T^2}]$$

By Ito’s lemma,

$$\frac{dH^m_T}{H^m_T} = \frac{d\Psi(t, r_T)}{\Psi(t, r_T)} - \frac{dH_T}{H_T} + \sigma(t, r_T)dW_T$$

$$= \frac{d\Psi(t, r_T)}{\Psi(t, r_T)} \sigma(t, r_T)dW_T - \frac{dH_T}{H_T} + \sigma(t, r_T)dW_T + \sigma(t, r_T)dW_T$$

Comparing the diffusion term of (1) and (2), we obtain

$$\frac{dH^m_T}{H^m_T} = h^m_t \cdot dR_T + (1 - h^m_t)dt$$

$$\frac{dH^m_T}{H^m_T} = h^m_t \cdot \sigma dW_T + (1 - h^m_t)dt$$

Since $h^m_t$ is self-financing, the dynamic process of minimum norm portfolio follows

$$\frac{dH^m_T}{H^m_T} = h^m_t \cdot dR_T + (1 - h^m_t)dt$$

Comparing the diffusion term of (1) and (2), we obtain

$$h^m_t = \frac{\frac{\partial \Psi(t, r_T)}{\partial r_T} \Sigma^{-1} \sigma(t, r_T)}{-h^m_t}$$

If instantaneous interest rate follows generalized Ornstein-Uhlenbeck process, $h^m_t$ can be reduced to $-h^m_t + 2e^{\lambda m - \Lambda m}$.
Appendix C

In order to find out the Lagrange multipliers, we need to complete the parameters $E(1_A)$, $E(I_T1_A)$, $E(1_A)$ and $E(I_T^21_A)$. Here we take the advantage of $rac{1}{B(0,T)H_T}$ which is the Radon-Nikodym derivative defining changing the historical probability into forward probability.

under real measure $P$, we have

$$\frac{dH_t}{H_t} = \left[ r_t + \left\| \lambda \right\|^2 \right] dt + \lambda^t dw_t$$

$$I_T = \frac{1}{H_T} \exp \left[ \int_0^T \left( r_t + \frac{1}{2} \left\| \lambda \right\|^2 \right) dt + \int_0^T \lambda^t dw_t \right]$$

under risk-neutral measure, we have

$$\frac{dH_t}{H_t} = r_t dt + \lambda^t dw_t^Q$$

$$I_T = \frac{1}{H_T} \exp \left[ \int_0^T \left( r_t - \frac{1}{2} \left\| \lambda \right\|^2 \right) dt + \int_0^T \lambda^t dw_t^Q \right]$$

and under forward measure, we have

$$\frac{dH_t}{H_t} = (r_t + \lambda \sum_{N+1} ) dt + \lambda^t dw_t^{QT}$$

$$I_T = \frac{1}{H_T} \exp \left[ \int_0^T \left( r_t + \lambda \left( \sum_{N+1} \right) - \frac{1}{2} \lambda \right) dt + \int_0^T \lambda^t dw_t^{QT} \right]$$

Also, under risk-neutral measure, we have

$$\frac{dI_t}{I_t} = \left( -h_t + 2\epsilon_{N+1} \right) \frac{dR_t}{R_t} + (h_t \cdot 1_{N+1} - 1) r_t dt$$

$$= \frac{dH_t}{H_t} + 2 dB(t,T)$$

$$= r_t dt - (\lambda^T - 2\Sigma_{N+1}) dw_t^Q$$

$$= r_t dt - \sigma_t dw_t^Q$$

and under forward measure, we have

$$\frac{dI_t}{I_t} = \left( r_t - \sigma_t \Sigma_{N+1} \right) dt - \sigma_t dw_t^{QT}$$

The forward price dynamics is

$$\dot{F} = -[\sigma_t + \Sigma_{N+1}] dw_t^{QT}$$
\[ E(1_A) = P(I_T < \frac{\alpha - \lambda}{\lambda_2}) = P(\ln I_T < \ln \frac{\alpha - \lambda}{\lambda_2}) \]
\[ = P(Z < \frac{\ln \frac{\alpha - \lambda}{\lambda_2} + Y(T) + \int_0^1 \frac{1}{2} \|Z\|^2 ds}{s_1}) = N(j_1) \quad , \quad Y(T-t) = b(T-t) + (r_t - b) \frac{1 - e^{-\alpha(T-t)}}{a} \]

\[ E(I_T 1_A) = E\left(\frac{dQ_T}{d\mathbb{P}} B(0,T) 1_A\right) = B(0,T)E^{QR}(1_A) = B(0,T)P^{QR}(I_T < \frac{\alpha - \lambda}{\lambda_2}) \]
\[ = B(0,T)P^{QR} \left( \frac{I_T}{B(T,T)} < \frac{\alpha - \lambda}{\lambda_2} \right) \]
\[ = B(0,T)P^{QR}(F_s(0,T)) \exp\left( \int_0^T \frac{1}{2} \|\gamma_s\|^2 dt - \int_0^T \gamma_s dW_s^{Q_T} < \frac{\alpha - \lambda}{\lambda_2} \right) \]
where \( \gamma_s = \sigma_s + \Sigma_{N+1} \)
\[ = B(0,T)P^{QR}(Z < j_{20}) \]
\[ = \frac{\lambda - \alpha}{\lambda_2} B(0,T) \frac{\ln \frac{I_0}{\lambda_2} + \frac{1}{2} s_{20}^2}{s_{20}} \quad , \quad s_{20}^2 = \int_0^T \|\gamma_s\|^2 ds \]

\[ E(I_T 1_A) = B(0,T)E^{QR}(I_T 1_A) \]
\[ = B(0,T)F(0,T)E^{QR}(1_A) = I_0 E^{Q}(1_A) = I_0 P^{Q}(I_T < \frac{\alpha - \lambda}{\lambda_2}) \]
\[ = I_0 N(j_2 - s_2) \]

where the Radon-Nikodym derivative defining the \( \hat{Q} \) measure
\[ d\hat{Q} = \epsilon_T (-\gamma \cdot w_{Q_T}) \]
\[ I_0 = E(I_T^2) \]
\[ = \exp(-2Y(T) + \|\lambda\|^2 T + 2V^2(T) + 4 \frac{\alpha \gamma_s \lambda}{a} (T - \frac{1 - e^{-\alpha(T)}}{a})) \]
where \( V^2(T-t) = \frac{\gamma_s^2}{a^2} [T - t - \frac{1 - e^{-\alpha(T-t)}}{a} + \frac{1 - e^{-2\alpha(T-t)}}{2a}] \)