The Continuous-Time Portfolio Problem

A. The Problems and Approaches

There are two main approaches to solving the continuous-time problem. Stochastic control approach developed by Merton (1969,1971) was based on the standard stochastic control theory. The optimal solution is computed by solving so called HJB (Hamilton-Jacobi-Bellman) Equation in two steps. The first step is looking for the optimal strategies as the function of (unknown) optimal expected utility. Inserting the optimal investment (and consumption) strategies into the Hamilton-Jacobi-Bellman Equation results in a non-linear partial differential equation, whose solution forms the second step. In the special case of Black-Scholes model and HARA (Hyperbolic absolute risk aversion) utility functions, Merton was able to find the solutions for the optimization problem. In general, however, it is very hard to find the solutions to HJB Equation. Even the numerical tractability is very limited. The other approach is martingale approach. It has several advantages. For instance it release the assumptions of constant market coefficients and special utility function. Besides, if there are no closed form solutions for the continuous time portfolio problem, Monte Carlo method offers a suitable way for numerical computation. There are three steps to implement the approach. First, characterize the set of attainable wealth. Next, solve the optimization problem of maximizing terminal wealth (or utility of wealth) and find the optimal terminal wealth. Final, obtain strategies that generate the optimal terminal wealth. In short, Cox-Huang’s approach transforms the dynamic problem into static problem whose unknown is optimal terminal wealth rather than the value function in Merton’s approach.

B. Technical Background: Numeraire Portfolio and Minimum Norm Portfolio

A framework of securities market in continuous time is formulated. The following assumptions are made.
Assumption1: Stochastic structure and information flow are represented by a probability space \((\Omega, \mathcal{F}, P)\) satisfying the usual conditions.
Assumption 2: Markets are free of arbitrage, frictionless and continuous open.
Assumption 3: The set of securities contain \(N+2\) elements. The instantaneous riskless asset is \(M\) and the other \(N+1\) risky assets with price \(S_1(t), S_2(t), S_3(t), \ldots, S_N(t), B(t, T)\).

\(B(t, T)\) (or security \(N+1\)) is the zero coupon bond price at time \(t\) and yields \$1 at maturity date \(T\). The \(N+1\)-dimension vector of weights defines the portfolio policies \(\underline{x}(t)\)

\[
\underline{x}(t) = (x(t), x(t), x(t), \ldots, x(t))
\]

Here we only care the self-financing portfolios (i.e. \(\underline{x}(t)\) is self-financing) and the portfolio price evolution is

\[
\frac{dX_t}{X_t} = x_0 \frac{dM_t}{M_t} + \underline{x}^* dR_t
\]

where \(x_0 = 1 - \underline{x}^* 1_{N+1}\)

\[
dR_t = \begin{bmatrix}
    dS_1 \\
    S_1 dS_2 \\
    S_2 dS_3 \\
    S_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\[
\max_{X \in A} E[\log X_T]
\]

\[
st. X_0 = 1
\]

A is the set of all attainable policies\(^1\).

If we specify the dynamic process of securities as

\[
\frac{dM_t}{M_t} = r_t dt
\]

\[
dR_t = \mu(\cdot) dt + \Sigma(\cdot) dw_t
\]

where \(w_t\) is an \(N+1\) dimensional standard Brownian vector, \(\mu(\cdot)\) and \(r_t\) are \(F_t\)-adapted integrable process, \(\Sigma(\cdot)\) is an \((N+1) \times (N+1)\) (i.e., complete market) \(F_t\)-adapted matrix process, nonsingular a.s., and \(F_t\) is the filtration generated by \(w_t\).

Then

\[
x_t^* = h_t = (\Sigma')^{-1} \Sigma^{-1} (\mu_t - r_t 1_{N+1}) = (\Sigma')^{-1} \lambda(t),
\]

\[
H_t = \exp[\int_0^t (r_s + \frac{1}{2} \|\lambda_s\|^2) ds + \int_0^t \lambda_s' dw_s]
\]

where \(\lambda(t) = \Sigma^{-1} (\mu_t - r_t 1)\) is the global market price of risk.

The proof is presented in the Appendix A.

The numeraire portfolio has several remarkable properties.

1. It is the combination of instantaneous tangent portfolio and the risk-free asset (Long, 1990).
2. The portfolio is identified as the optimal portfolio of the logarithmic investors and is called the growth optimal portfolio. Note that the growth optimal for different horizon is identical.
3. Interpretation of \(\frac{1}{H_T}\) is also valueable. It’s also possible to identify \(\frac{1}{H_T}\) as the “pricing kernel” or “state price density” or the Riesz representation of the pricing

\(^1\) From Jensen’s inequality we know that

\[
E(\log \left( \frac{X_T}{H_T} \right)) \leq \log(E(\frac{X_T}{H_T})) = 0 \iff E(\log(X_T)) \leq E(\log(H_T))
\]
Besides, from the martingale theory, \( \frac{1}{B(0,T)H_T} \) is the Radon-Nikodym derivative defining changing the historical probability into forward probability.

## B.2 Dynamic Mean-Variance Efficient Asset Allocation without Constraint

In this section, we introduce the dynamic efficient frontier (DEF) in the standard deviation–expected return \((\sigma(X_T), E(X_T))\) space (BBP, 1998). The DEF comes from the program (P)

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \text{Var}(X_T) \\
\text{st.} & \quad E(X_T) = E, X_0 = 1, E \geq \frac{1}{B(0,T)}
\end{align*}
\]

The solution \( \mathbf{x} \) is called DMVE (dynamic mean variance efficient) strategies.

In the spirit of Cox and Huang (1989) and Merton’s formulation, (p) is equivalent to (p’)

\[
\begin{align*}
\min_{\mathbf{x}^T} & \quad \text{Var}(X_T) \\
\text{st.} & \quad E(X_T) = E, E\left(\frac{X_T}{H_T}\right) = H_0 = 1
\end{align*}
\]

The first constraint means investors seek for a target return \( E \) that is greater than risk-free return, and the second constraint describe the budget constraint under the free of arbitrage condition. Note that the initial investment is normalized to $1 without loss of generality.

We apply the Lagrangian method to deal with constrained optimization problem with equality constraints (p’).
\[
\min_{X_T, A, k_2} L \\
\text{where } L = \frac{1}{2} E(X_T^2) - k_1 (E(X_T) - E) + k_2 (E(\frac{X_T}{H_T}) - 1) \]

The first order conditions are:

\[
\frac{\partial L}{\partial X_T} = 0 \quad X_T^*(w) - k_1 + k_2 \frac{1}{H_T(w)} = 0 \quad \forall w
\]
\[
\frac{\partial L}{\partial k_1} = 0 \quad E(X_T^*) = E
\]
\[
\frac{\partial L}{\partial k_2} = 0 \quad E(\frac{X_T^*}{H_T}) = 1
\]

in addition, we also have \( E(\frac{1}{H_T}) = B(0,T) \).

With these equations, we can find the optimal wealth \( X_T^* \)

\[
X_T^*(w) = k_1 - k_2 \frac{1}{H_T(w)} \quad \text{.........................(*)}
\]

where \( k_2 = (B(0,T)E - 1)/\text{Var}(\frac{1}{H_T}) \), \( k_1 = E + k_2B(0,T) \)

Obviously, from equation (\*), any DMVE strategy yielding DMVE return \( X_T^* \) is “static” combination of \( k_1 \) unit zero coupon bonds paying off \$1 at maturity date and \( k_2 \) unit portfolios yielding payoff \( \frac{1}{H_T} \) at time T.

From (\*), we have

\[
\sigma(X_T) = |k_2| \sigma(\frac{1}{H_T}) = \frac{E(X_T)B(0,T) - 1}{\text{Var}(\frac{1}{H_T})} \sigma(\frac{1}{H_T})
\]

The equation of dynamic efficient frontier (DEF) in the \(( \sigma(X_T), E(X_T) \)) space is \( E(X_T) = \frac{1}{B(0,T)} + a\sigma(X_T) \), where \( a = \sigma(\frac{1}{H_T})/B(0,T) \)
The characteristics of DEF are as follows:

1. DEF is a straight line containing the point F (risk-free return) with slope \( a \) as shown in Figure 1.
2. DEF is generated by the static combination of two funds: the zero coupon bond and arbitrary DMVE portfolio.
3. DEF dominates SEF (static efficient frontier). By Intuition, any SMVE (static mean-variance efficient) strategy is a particular DMVE strategy. For more precise proof, since the DEF and SEF stem from the risk-free return, the only thing we concern is the slope. The slope of SEF divided by the slope of DEF is

\[
\frac{\text{slope of SEF}}{\text{slope of DEF}} = \frac{b}{a} = \text{corr}(M_T, G_T) \leq 1
\]

where \( a = \frac{1}{H_T} / B(0, T) \), \( b = \frac{[E(M_T) - 1/B(0, T)]}{\sigma(M_T)} \);

\( M_T \) is the return of standard Markowitz tangent portfolio, \( G_T \) is a DMVE return which can be generated by short selling a “minimum norm return” portfolio and investing $2 in the zero coupon bond.

Figure 1. The geometric interpretation of DEF and SEF in \((\sigma(X_T), E(X_T))\) space.

**B.3 The Minimum Norm Portfolio**

Note that reaching \( \frac{1}{H_T} \) is possible and requires an initial investment \( \frac{1}{E(H_T^2)} \) to

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2 A return \( W_T \) is SMVE if and only if it satisfies the SEF (capital market line):

\[
E(W_T) = 1/B(0, T) + b\sigma(W_T)
\]

where \( b = [E(M_T) - 1/B(0, T)]/\sigma(M_T) \)

3 The definition of minimum norm portfolio is described in the next section.


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prevent from the arbitrage opportunity. Then in our notation $H_T^m = \frac{1}{H_TE(\frac{1}{H^2})}$ is the return. A geometric interpretation of $H_T^m$ is as follows. Consider the following optimization program.

$$\min_{X_T} E(X_T^2)$$

$$st. \quad \frac{X_T}{H_T} = 1$$

The solution to the problem is

$$X_T^* = H_T^m$$

Therefore, $H_T^m$ is the return that minimizes the distance to the origin as shown in Figure 2.

It is straightforward to show that portfolio returns that minimize the expected return for a given the standard deviation are represented by the inefficient frontier (DEF-) in $(\sigma(X_T), E(X_T))$ space. Hence, any return of DEF can be obtained by short selling $1$ of its symmetrical counterpart on DEF-, using the proceeds to invest $2$ zero coupon bond.
Figure 2. The geometric interpretation of minimum norm portfolio in $(\sigma(X_T), E(X_T))$ space.

Recall that if we specify the dynamic process of securities as

$$\frac{dM_t}{M_t} = r_t dt$$
$$dR_t = \mu(.)dt + \Sigma(.)dW_t$$

The numeraire portfolio is

$$h_t = (\Sigma')^{-1} \Sigma^{-1} (\mu_t - r_t) = (\Sigma')^{-1} \lambda(t)$$

Also, we can derive the minimum norm portfolio $H^m_r$ yielding the minimum norm return $H^m_r$.

$$h^m_t = \frac{\partial \Psi(t, r_t)}{\partial r_t} (\Sigma')^{-1} (\mu_t - r_t) - h_t$$

where

$$\Psi(t, r_t) = E_t[\exp(-2(\int_t^T (r_s + \frac{1}{2} \|\lambda_s\|^2) ds + \int_t^T \lambda_s dW_s))]$$

Note that the derivation is in the Appendix B.
If we set the instantaneous interest rate as generalized Ornstein-Uhlenbeck process:

\[ dr = \mu (t, r)dt + \sigma (t, r)dw \]

Then, we have the relation between numeraire portfolio and minimum norm portfolio,

\[ h^- = -h + 2e_{N+1 \times 1} \]

the weight on riskless asset is thus \( h_1, 1_{N+1 \times 1} - 1 \).