III. Dynamic Mean-Variance Efficient Asset Allocation under Principal-Guaranteed Constraint

Unfortunately, MV portfolio strategies yielding possible negative wealth become over-leveraged in some cases. In order to overcome the undesirable features, we impose another constraint on the MV investors program.

The mean-variance optimization problem is rewritten as

$$\min_{x_T} \ Var(X_T)$$

$$st. \ E(X_T) = E, \ X_0 = 1, \ E\left(\frac{X_T}{H_T}\right) = H_0 = 1, \ X_T \geq \alpha, \ \alpha \geq 0$$

We apply the Lagrangian method to solve the constrained optimization problem with inequality constraints.

The Karush-Kuhn-Tucker necessary conditions are

$$\frac{\partial L}{\partial X_T} = 0, \ X_T^* - \lambda_1 + \lambda_2 \frac{1}{H_T} - \lambda_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0, \ E(X_T^*) = E$$

$$\frac{\partial L}{\partial \lambda_2} = 0, \ E\left(\frac{X_T^*}{H_T}\right) = 1$$

$$\lambda_3 \geq 0$$

$$\frac{\partial L}{\partial \lambda_3} \leq 0, \ \alpha - X_T \leq 0$$

$$\lambda_3 \frac{\partial L}{\partial \lambda_3} = 0, \ \lambda_3 (\alpha - X_T) = 0$$

Consequently,
\[ \lambda_3 > 0 \ , \ X^*_T = \alpha \]

\[ \lambda_3 = 0 \ , \ X^*_T = \lambda_1 - \lambda_2 \frac{1}{H_T} = \lambda_2 (\lambda' - \frac{1}{H_T}) \quad \text{where} \quad \lambda' = \frac{\lambda_2}{\lambda_2} \]

\[ E(X^*_T) = E \ , \ E\left(\frac{X^*_T}{H_T}\right) = 1 \]

If \( \lambda_3 \) is binding, \( X^*_T = \lambda_2 (\lambda' - \frac{1}{H_T}) \). In the other case, the optimal terminal wealth is \( X^*_T = \alpha \). Hence, we identify the efficient strategy as a duplication of the
principal-guaranteed put option on the minimum norm portfolio

\[ X^*_T = \lambda_2 (\lambda' - I_T)1_A - \alpha 1_A + \alpha \]

where, \( I_T = \frac{1}{H_T} \), \( I_A = 1 \), \( I_T < \lambda' - \frac{\alpha}{\lambda_2} \)
\( I_A = 0 \), o.w.

If we specify the dynamic evolution of assets, we can achieve the optimal weights that generate the optimal attainable wealth

\[ \frac{dM_t}{M_t} = r_t dt \]

\[ dR_t = \mu(.) dt + \Sigma(.) dw_t \]

\[ (dr_t = a(b - r_t) dt + \sigma r_t dw_t) \]

Before progressing, one thing tedious we left in optimization problem needs to be completed is to find out the Lagrangian multipliers. The multiplier is determined through the budget constraints.

\[ E(X^*_T) = E \ , \ E\left(\frac{X^*_T}{H_T}\right) = 1 \]

This implies a nonlinear system
\[
\begin{align*}
\lambda_1 E(I_{1_A}) - \lambda_2 E(I_{T1_A}) - \alpha E(I_{1_A}) + \alpha &= E \\
\lambda_1 E(I_{T1_A}) - \lambda_2 E(I_{T^21_A}) - \alpha E(I_{T1_A}) + \alpha E(I_{T}) &= 1
\end{align*}
\]

Equivalently,

\[
\begin{align*}
(\lambda_1 - \alpha)N(j_{20}) - \lambda_2 B(0,T)N(j_{20}) &= E - \alpha \\
(\lambda_1 - \alpha)N(j_{20})B(0,T) - \lambda_2 I_0 N(j_{20} - s_{20}) &= 1 - \alpha B(0,T)
\end{align*}
\]

where

\[
\begin{align*}
\hat{j}_{1t} &= \frac{\ln \hat{\lambda}_1 - \alpha}{\hat{\lambda}_2} + \frac{Y(T-t) + \frac{1}{2} \int_t^T |\lambda|^2 ds}{s_{1t}} \\
n, \ Y(T-t) &= b(T-t) + (r_t - b) \frac{1 - e^{-a(T-t)}}{a} \\
s_{1t} &= V^2(T-t) + |\gamma_t|^2 (T-t) + 2 \frac{\sigma_r \dot{\lambda}}{a} (T-t - \frac{1 - e^{-a(T-t)}}{a}) \\
V^2(T-t) &= \frac{\sigma_{\dot{\lambda}}}{a^2} [T-t - 2 \frac{1 - e^{-a(T-t)}}{a} + \frac{1 - e^{-2a(T-t)}}{2a}] \\
\hat{j}_{2t} &= \frac{\lambda_1 - \alpha}{I_0} + \frac{1}{2} s_{2t} \\
s_{2t}^2 &= \int_t^T |\gamma_t|^2 ds \quad , \quad \gamma_t = \dot{\lambda}(t) - \sigma_{\dot{\lambda}} \\
I_0 &= \exp(-2Y(T) + |\lambda|^2 T + 2V^2(T) + 4 \frac{\sigma_r \dot{\lambda}}{a} (T - \frac{1 - e^{-a(T)}}{a}))
\end{align*}
\]

\(N(.)\) is the cumulative normal distribution function. Actually, to solve a nonlinear system is more difficult than to solve a linear system. In a nonlinear system, the numerical method may not work that well as in linear system. Since the solution is probably more than one, we will apply the contour plot to pick the reasonable approximate solution at the start. And then we implement the secant method to find the more precise solution.

Recall that the optimal terminal wealth is

\[
X_T^* = \lambda_2 (\dot{\lambda} - I_T) 1_A - \alpha 1_A + \alpha
\]
At time $t$, according to the martingale asset pricing theory, the constrained optimal portfolio value is therefore

$$X_t = B(t,T)E_t^{QT}[\lambda^*_2(\lambda^* - I_t)1_A] + \alpha B(t,T)[1 - E_t^{QT}(1_A)]$$

$$= \lambda^*_2(\text{Euro put}) + \alpha B(t,T)(1 - E_t^{QT}(1_A))$$

In spite of the absence of such an option in current market, from the option pricing theory, the European put option can be replicated by the dynamic hedging policies.

$$X_t = \lambda^*_2[-N(-d_{1t})I_t + \lambda^* N(-d_{2t})B(t,T)] + (1 - N(j_{2t}))\alpha B(t,T)]$$

$$= -\lambda^*_2 N(-d_{1t})I_t + [\alpha - \alpha N(j_{2t}) + \lambda^*_2 N(-d_{2t})]B(t,T)]$$

$$\ln \frac{I_t}{\lambda^*_2 B(t,T)} + \frac{1}{2} s_{2t}^2$$

with $d_{1t} = \frac{\ln \frac{I_t}{\lambda^*_2 B(t,T)}}{s_{2t}}, \quad d_{2t} = d_{1t} - s_{2t}$

The holding value at time $t$,

$$-\lambda^*_2 N(-d_{1t})I_t \quad \text{dollars in minimum norm portfolio}$$

$$[\alpha - \alpha N(j_{2t}) + \lambda^*_2 N(-d_{2t})]B(t,T) \quad \text{dollars in zero coupon bond}$$

Therefore, the dynamic optimal weights at time $t$

$$\frac{-\lambda^*_2 N(-d_{1t})I_t}{X_t} \quad \text{to minimum norm portfolio}$$

$$\frac{[\alpha - \alpha N(j_{2t}) + \lambda^*_2 N(-d_{2t})]B(t,T)}{X_t} \quad \text{to zero coupon bond}$$

In the other form,

$$\frac{-\lambda^*_2 N(-d_{1t})I_t}{X_t}(-h + 2e_{N+1}) + \frac{[\alpha - \alpha N(j_{2t}) + \lambda^*_2 N(-d_{2t})]B(t,T)}{X_t}e_{N+1} \quad \text{to risky asset}$$

$$\frac{-\lambda^*_2 N(-d_{1t})I_t}{X_t}[-h, 1_{N+1} - 1] \quad \text{to riskless asset}$$
With the help of the optimal strategies, we can undertake straightforward the simulation in next section.