IV. Simulation Examples

In this section we use the constraint mean-variance model developed in section III to undertake simulations of optimal policies and to illustrate the behavior of investment. The numerical examples are conducted under the following assumptions.

The instantaneous interest rate follows Vasicek’s model:

\[ dr_t = \alpha (b - r_t) dt + \sigma_t dw, \]

The stochastic process of riskless security, stock index fund and bond index fund are respectively given by

\[
\begin{align*}
\frac{dM_t}{M_t} &= r_t dt \\
\frac{dS_t}{S_t} &= (r_t + \theta_s) dt + \sigma_{s1} dw + \sigma_{s2} dw_r \\
\frac{dB_{T-t}(t)}{B_{T-t}(t)} &= (r_t + \theta_{T-t}(t)) dt - \sigma_{T-t}(t) dw_r \\
\frac{dB_K(t)}{B_K(t)} &= (r_t + \theta_K) dt - \sigma_K dw_r
\end{align*}
\]

where \( \theta_s \) and \( \theta_K \) are constant risk premium for stock index fund and bond index fund respectively. \( \sigma_{s1}, \sigma_{s2} \) and \( \sigma_K \) are positive constant. \( w \) is standard Brownian Motion and \( dw \) is orthogonal to \( dw_F \). The market price of risk \( \lambda \) is also assumed to be constant.\(^1\)

Instantaneous interest rate at t=0 \( r_0=0.04 \);  
Reversion strength \( \alpha=0.3 \),  
Long run interest rate level \( b=0.04 \),  
Time horizon \( T=1 \) year,  
the constant maturity \( K=1 \) year ,  
the principle guarantee rate \( \alpha=0.9 \),  
Market price of risk \( \lambda = [0.35, 0.25]^T \) \( \sigma_F=[0.04] \),  
\( \sigma_S=[0.12, 0.1] \),

\(^1\) \( B_{T-t}(t) \) is the bond price maturing at \( T \) with maturity \( T-t \). \( B_K \) is a constant-maturity bond index. In practice, the bond fund may maintain different zero coupon bonds and be managed to be a constant duration K.
As we mentioned in section III, we have to determine the Lagrange multipliers $\lambda_1$ and $\lambda_2$ at $t=0$ by solving a nonlinear system. To obtain the approximate solutions, we draw the contour plots first (figure 1). The purpose of drawing the contour plots is to roughly estimate the intervals that solutions fall in.

Figure 3. Contour Plots of Nonlinear System:

\[
\begin{align*}
(\lambda_1 - \alpha)N(j_{10}) - \lambda_2 B(0,T)N(j_{20}) + \alpha - E &= 0 \\
(\lambda_1 - \alpha)N(j_{20})B(0,T) - \lambda_2 J_0 N(j_{20} - s_{20}) + \alpha B(0,T) - 1 &= 0
\end{align*}
\]

Combining the contour plots tells us the intervals containing solutions for $\lambda_1$ and $\lambda_2$ are (1.3,1.4) and (0.2,0.3) (figure 2). Then, we apply the secant method to find the more precise solutions for $(\lambda_1, \lambda_2) = (1.36666, 0.26111)$.

Figure 4. Combined Contour Plot
Since the term structure is driven by one state variable, the bonds return are perfectly correlated and therefore one is redundant. The notation
\[ \Theta = [x_t(t), x_t(t), x_k(t), x_{t-k}(t)] \] defines the dynamic portfolio strategies for the four assets we have described in the start of this section. Because of the redundancy of \( B_k \) and \( B_{t-k}(t) \), \( \Theta = [0,0,0,1] \) is equivalent to \( \Theta = [1 - \frac{\sigma_{t-l}(t)}{\sigma_k}, 0, \frac{\sigma_{t-l}(t)}{\sigma_k}, 0] \)

\[ \text{Proof.} \quad \left(1 - \frac{\sigma_{t-l}(t)}{\sigma_k}\right) \frac{dM_t}{M_t} + \frac{\sigma_{t-l}(t)}{\sigma_k} \frac{dB_{t-l}(t)}{B_{t-l}(t)} = r_t + \frac{\sigma_{t-l}(t)}{\sigma_k} \theta_t dt + \frac{\sigma_{t-l}(t)}{\sigma_k} \sigma_k dw_t \]

Since the market price of risk for different maturities bond is the same,
\[ (r_t + \theta_t) dt + \sigma_{t-l}(t) dw_t = \frac{dB_{t-l}(t)}{B_{t-l}(t)} \]

From section B.1, the numeraire portfolio is \( h_t = [1 - h_t - h_{t-l}, h_t, 0, h_{t-l}] \)

Substitute it into the optimal strategy yields
\[ -(X_{it})[1 - h_t - h_{t-l}, h_t, 0, h_{t-l}] + [0,0,0,1 + X_{it} \] where \( X_{it} = \frac{-\lambda_i N(-d_{it})I_i}{X_t} \)
\[ = -(X_{it})[1 - h_t - h_{t-l}, h_t, 0, D] \] where \( D = -X_{it}(h_{t-l} - 1) + 1 \)
\[ = -(X_{it})[1 - h_t - h_{t-l} + D(1 - \frac{\sigma_{t-l}(t)}{\sigma_k}), h_t, D \frac{\sigma_{t-l}(t)}{\sigma_k}, 0] \]

**Simulation Results**

Figure 5 shows the DMVE portfolio sample path together with the sample path of Markowitz mean-variance efficient portfolio, based on the same price process. As the DMVE strategy takes the full advantage of information revealed in markets, stock prices and interest rate levels, it dominates the SMVE strategy.
Figure 6 shows the optimal weights on assets rely on equation (**) accompanied with the simulated dynamic asset prices evolution. In short term, as the risky assets (stock and bond index fund) tend to raise, risky assets are sold while the riskless asset (MMA) is bought and vice versa. Briefly speaking, optimal strategies are concave. The concavity of the optimal policy is related to the property of mean-variance framework or quadratic utility function, which means increasing relative risk aversion. With the increasing relative risk aversion, an raise in price of risky assets makes the investor richer, an increase in his/her risk aversion, hence decreases his holding on risky assets. In addition, the holding in money market account obviously trend high position as time goes by. The principle guarantee constraint we impose on the optimization explains the trend and the trend is also correspond to the conventional wisdom. Note that our mean-variance efficient strategies are in contrast with the OBPI (option based portfolio insurance) strategies, which state that any hedged option position whose replicating portfolio involves shifting from long risky asset position to no risky position as the risk asset price goes down implicitly involves purchase of insurance.
Figure 6. Allocation to Stock Index Fund, Bond Index Fund and Riskless Asset over Time

Figure 7 shows the optimal weights on stock index fund for different investment horizon: $T=1$, $T=3$ and $T=5$. This sensitivity analysis shows that the longer the investment horizon, the higher the initial allocation to risky asset. This means that a long-term investment investor who hardly cares the performance of short-term is able to bear the high risk in the initial time and hence leverages his portfolio.
Figure 7. Optimal Weights for Different Investment Horizon
Figure 8 shows the simulated optimal weights on stock index fund, bond index fund and riskless asset respectively for different principle guarantee rate. Not surprising, high principle guarantee rate strategies require more funds installed in riskless asset than low rate require. It is seen that from figure 6 the funds move from bond index position to riskless asset position. The little change in stock index position is due to the target return constraint. The stock index fund bearing relative high risk than other assets has higher expected return. To achieve the target return, fund managers still maintain high risky position (see also Figure 9).

Figure 8. Optimal Weights for Different Principal-Guaranteed Rate.
Figure 9. Optimal Weights for Different Target Return.