# Chapter 2

## Optimal Asset Allocation with Extreme Returns and a VaR Constraint[*]

### 2.1 Introduction

A vast amount of empirical evidence demonstrates that asset returns generally exhibit *leptokurtic behaviors*, i.e., asset returns are negatively skewed and have fat tails (Praetz, 1972; de Vries, 1994; Campbell, Lo, & Mackinlay, 1997). Fama (1965) was the first to demonstrate that stock returns have fat tails. Many reasons account for the leptokurtic behavior of return distributions, for example, volatility changes over time, or extreme events occur. These extreme returns generate high moments differ from the high moments one would obtain with a normal distribution.

One may ask what the impact is of ignoring leptokurtic behavior in asset return models building various financial issues. Tokat, Rachev, and Schwartz (2003) used Stable non-Gaussian distributions to describe the fat tails of asset returns and demonstrated that Stable scenario modeling leads to equity asset allocations up to 26% less than those of acquired by normal scenario modeling. Hales (1997) indicated that an options pricing model with capturing fat tails in underlying asset distributions reduces pricing biases relative to the Gaussian Black-Scholes model for valuing foreign currency options. Bidarkota and McCulloch (2003) showed that accounting for fat tails in the dividends data produces an additional 13% of equilibrium equity returns in a standard consumption-based asset-pricing model.

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The comments of seminar participants at 2006 現代財務論壇學術研討會、2006 台灣財務金融學會年會暨財務金融保險不動產學術研討會、第六屆風險管理理論研討會 are greatly appreciated.
Any remaining errors are our own. 「臺大管理論叢」已接受本篇文章並預計刊登於 96 年 6 月出版之 17 卷第二期期刊.

Fat tails are not important when one is interested only in expected returns. However, when one is interested in measures that describe uncertainty or risk associated with a given asset allocation strategy, the precise shape of distribution tails matters a great deal (Jorion, 1996). In financial markets, small but non-zero probability of extreme events, regardless of how unlikely, can have a significant impact. Stulz (2002) suggested that the failure of Long-Term Capital Management（LTCM） in 1998 had an dramatic affect on risk management beliefs. Two famous financial economists, Robert Merton and Myron Scholes, were LTCM partners. Notably, LTCM was very successful in its first two years (1995 & 1996). However, it experienced two consecutive months of huge losses in 1998. The fund losses in August and September of 1998 had a probability of zero, yet they occurred. The fund exceeded the one-week VaR for five consecutive weeks. With independent weekly returns, such an event has probability of $0.0000003 (= 0.05^5)$ of occurring. Stulz proposed that four reasons account for downward bias of VaR estimates. One reason is that extreme returns, the tails of the density function, have higher probabilities than those of the normal distribution. It is likely that when one faces the mere possibility of extreme returns, he will not allocate his wealth in the same manner as one who only cares about the first and second moment.

Various alternative distributions have been developed to describe these fat tails. For example, Extreme Value Theory (EVT) addresses tail behavior of the asset returns (Embrechts, Kluppelberg, & Mikosch, 1997). This theory investigates the distribution of maximum (minimum) in large samples, thereby determining a distribution‘s tail shapes. However, information about the central characteristics of the distribution is lost. Stable Paretian distribution is an alternative model that captures fat tails and the whole return distributions (Samorodnitsky & Taqqu, 1994; Rachev & Mittnik, 2000; Tokat et al., 2003). The shortcomings of a Stable Paretian distribution are that densities and distribution functions are not known in a closed form for most Stable distributions and

Stable distributions are generally specified by their characteristic functions. Engle's (1982) autoregressive conditional heteroskedasti (ARCH) model and its various generalizations have indicated that the conditional normal or Student's t ARCH process is insufficiently fat-tailed to account for excess kurtosis in data (Nelson, 1991). Jondeau and Rockinger (2006) investigated how non-normality of returns may affect the allocation of wealth for utility-maximizing investors. A Taylor series expansion of the expected utility allows to focus on certain moments and to compute numerically the optimal portfolio allocation.

An accurate model that describes asset returns is required to measure and manage risk. One can rely on a polynomial expansion of a normal density function, such as the Gram-Charlier expansion. Jarrow and Rudd (1982) first applied the Gram-Charlier expansion to finance. This expansion addresses skewness and kurtosis. The Gram-Charlier expansion accommodates a virtually unlimited class of possible shapes and generates an approximate density function for a standardized random variable. This density function has been applied in Gallant, Hansen, and Tauchen (1990), Gallant and Nychka (1987), Gallant and Tauchen (1989), Longstaff (1995), Corrado and Su(1997a, 1997b) and Jondeau and Rockinger(2001).

This study is interested in the effect of extreme returns or fat tails on the solution of the asset allocation problem encountered by risk managers. Therefore, this study introduces Gram-Charlier expansion to approximate asset returns with negatively skewed and excess kurtosis. This study extends the analytical results obtained by Basak and Shapiro (2001) and investigates how high moments affect asset allocations when investors manage market-risk exposure using VaR-RM. The first two moments correspond to mean and variance. A negative third moment demonstrates that there are more extreme negative realizations than positive realizations. The fourth moment measures the relative concentration of values in the distribution center as compared with those at the tails.

The principal findings are as follows

(1) When risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio under the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is a bad state.

(2) The agent invests reduced funds in the stock market during intermediate states and increased amounts in the stock market during bad states when confronted with negative skewness and excess kurtosis.

(3) A risk manager cannot decrease losses in bad states, but can decrease the value of $\alpha$, the probability that a loss exceeds VaR, and the agent will suffer from reduced terminal wealth in both the good and bad states.

It is important for risk managers to precisely forecast the loss. The analytical results imply that the impact of leptokurtic asset returns is based on the shape of asset returns, and a correct measurement of leptokurtic asset returns is helpful to risk managers seeking to precisely forecast the loss.

This work is similar to that of Lucas and Klaassen (1998) and Tokat et al. (2003). Lucas and Klaassen considered a simple one-period asset allocation problem and investigated the effect of extreme returns or fat tails on optimal asset allocations using a shortfall constraint. They utilized the Student-t distribution to characterize the fat tails of asset returns and maximize expected return on the portfolio subject to a VaR constraint. They showed that the VaR of a given portfolio may be underestimated when tail behavior of asset returns is not captured accurately. Tokat et al. (2003) investigated the effects of fat-tailed returns on asset allocation decisions when an investor uses VaR and Conditional VaR as risk measures. The solution methodology comprises a multistage stochastic asset allocation problem with decision rules. The uncertainty is modeled utilizing economic scenarios with Gaussian and stable non-Gaussian distributions. Unlike Lucas and Klaassen (1998) and Tokat et al. (2003), this study focuses on solving the optimal investment problem for a VaR risk manager, and examines the effects of extreme returns or fat tails on optimal asset allocations. This study does not analyze expected utility maximization and risk management as mutually exclusive paradigms, but rather directly embeds risk management objectives into a framework that maximizes utility.

This chapter is organized as follows. In Section 2.2, the general economic framework is presented and the approach for modeling returns with a non-normal distribution is described. Section 2.3 analyzes the asset allocation problem under VaR when asset returns are non-normal. Section 2.4 presents the numerical illustrations via simplified examples. Section 2.5 presents the conclusion.

## 2.2 Economic Setting

This section formulates an economy. Financial markets are assumed arbitrage-free, continuously open between time zero and time T, and dynamically complete. Consider a frictionless securities market and the market variables are modeled on a complete probability space $(\Omega, F, P)$ where the filtration $F = (F_t), t \in [0, T]$ generated by the Brownian motion process $z(t)$. The measure $P$ is called the historical probability. The investor can transfer current wealth into future wealth by investing in some securities. The investment set consists of two well-established securities. Security $B$ is a local riskless asset and stock index $S$ is a risky asset. Section 2.2.1 shows the general assumptions of asset return in most academic financial literature. Section 2.2.2 presents the development a stock index model that allows for negatively skewed and fat-tailed. This model is employed in determining how negatively skewed and excess kurtosis affects asset allocations.

### 2.2.1 Asset returns with normal distribution

Suppose the short-term interest rate *r* is constant over time interval [0, T], and the value of a riskless bond denoted by *B* is assumed to be continuously compounded at rate *r*; that is,

$$\frac{dB(t)}{B(t)} = rdt\,, \tag{2.1}$$

or $B(t) = e^{rt}$, with $B(0) = 1$.

The stochastic process for stock index price ( with dividends reinvested) is

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz_S(t) \tag{2.2}$$

where $\mu$ is the drift rate, $\sigma$ is the stock index volatility and $z_S(t)$ is a standard Brownian motion.

The fraction of wealth invested in stock index at time t is denoted by $x_S(t)$ and the fraction of wealth invested in a riskless asset at time t is signified by $1 - x_S(t)$. We allow $x_S(t)$ to take negative as well as positive values. For any self-financing portfolio, the instantaneous change in the wealth return is then formulated as

$$\frac{dW(t)}{W(t)} = [1 - x_S(t)]\frac{dB(t)}{B(t)} + x_S(t)\frac{dS(t)}{S(t)}$$

The initial wealth of the investor is denoted by $W(0)$. The wealth *W(t)* of the investor can be expressed in the following stochastic process:

$$\frac{dW(t)}{W(t)} = [r + x_S(t)(\mu - r)]dt + x_S(t)\sigma dz_S(t) \tag{2.3}$$

$$= rdt + x_S(t)[(\mu - r)dt + \sigma dz_S(t)]$$

$$= rdt + x_S(t)\sigma dz_S^Q(t)$$

where $dz_S^Q(t) = \frac{(\mu - r)}{\sigma}dt + dz_S(t)$. Let $\kappa \equiv (\mu - r)/\sigma$, called the market price of risk. If $E[\exp(\frac{1}{2}\int_0^T \kappa^2 du)] < \infty$, then the Novikov condition holds. When markets are assumed complete, Harrison and Pliska (1981) showed that a unique state price density of the economy $\zeta(t)$ exists, as follows:

$$\zeta(t) = \exp\left\{-\int_0^t rdu - \int_0^t \kappa dz_S(u) - \frac{1}{2}\int_0^t \kappa^2 du\right\} \tag{2.4}$$

It is well known that

$$E_t^Q\left[\exp\left\{-\int_t^T rdu\right\}\right] = E_t\left[\frac{\zeta(T)}{\zeta(t)}\right]$$

where Q is the unique equivalent martingale measure defined by

$$\frac{dQ}{dP} = \exp\left\{-\int_0^T \kappa dz_S(u) - \frac{1}{2}\int_0^T \kappa^2 du\right\}.$$

The state price $\zeta(t)$ can be expressed as

$$\zeta(t) = \exp\left\{-\int_0^t rdu - \int_0^t \kappa dz_S(u) - \frac{1}{2}\int_0^t \kappa^2 du\right\}$$

$$= \exp\left\{-rt - \kappa z_S(t) - \frac{1}{2}\kappa^2 t\right\}.$$

and $\frac{\zeta(T)}{\zeta(t)} = \exp\left\{-r(T-t) - \kappa[z_S(T) - z_S(t)] - \frac{1}{2}\kappa^2(T-t)\right\}$,

$$\ln\zeta(T) = \left\{\ln\zeta(t) - (r + \frac{1}{2}\kappa^2)(T-t) - \kappa[z_S(T-t)]\right\}.$$

Let $m = \ln(\zeta(t)) - (r + \frac{1}{2}\kappa^2)(T-t)$, $v = \kappa\sqrt{T-t}$ and

$Z = \frac{\ln\zeta(T) - m}{v} = \frac{-\kappa z_S(T-t)}{v}$. When risky asset returns follow a normal distribution, then $\frac{-\kappa z_S(T-t)}{v}$ also follows a standard normal distribution. $m$ and $v$ are the conditional mean and standard deviation of $\ln\zeta(T)$ implied by the risk-neutral pricing measure Q. The density function of $Z$, the standardized value of $\ln\zeta(T)$, is given by

$$f(z) = \phi(z) \tag{2.5}$$

where $\phi(\cdot)$ denotes the standard normal density function.

### 2.2.2 Asset returns with fat-tailed distribution

Distribution with high tail probability compared to a normal distribution with the same mean and variance are called fat-tailed. Skewness and kurtosis measure the shape of a probability distribution. Skewness measures the degree of asymmetry, with symmetry implying zero skewness, and negative skewness indicates a relatively long left tail compared to the right. Tails of a distribution means that the regions are located far from center. Kurtosis is the extent to which probability in concentrated at the center. Because kurtosis is especially sensitive to tail-weight, high kurtosis is almost synonymous with fat tails. A fat-tailed distribution is more prone to extreme values, which are sometimes called outliers.

In this section, a riskless bond is presented and model of the stock index is developed that allows for negatively skewed and fat tails. In other words, stock index return skewness is typically negative and kurtosis for stock index return is positive. The Gram-Charlier expansion accommodates a virtually unlimited class of possible shapes, and yields an approximate density function for a standardized random variable. This study considers a Gram-Charlier expansion of the density function of the state price. The state price density process (or pricing kernel) has a critical role in deriving optimal trading strategies. Longstaff (1995), Corrado and Su (1997a, 1997b) and Jondeau and Rockinger (2001) applied a Gram-Charlier expansion for the underlying asset to develop risk-neutral density and derive an option pricing model.

Let Z be the standardized value of $\ln\zeta(T)$

$$Z = \frac{\ln\zeta(T) - m}{v}$$

where $\ln\zeta(T) = \left\{ \ln\zeta(t) - (r + \frac{1}{2}\kappa^2)(T-t) - \kappa[z_S(T-t)] \right\}$. Let $m = \ln(\zeta(t)) - (r + \frac{1}{2}\kappa^2)(T-t)$, $v = \kappa\sqrt{T-t}$ and $Z = \frac{\ln\zeta(T) - m}{v} = \frac{-\kappa z_S(T-t)}{v}$. When the stock index is developed that allows for negatively skewed and fat tails, then stock index return skewness is typically negative and kurtosis is positive. $\frac{\kappa z_S(T-t)}{v}$ follows Gram-Charlier expansion of the density function. $\frac{\kappa z_S(T-t)}{v}$ has negative skewness and positive kurtosis. Since skewness denotes the third moment and kurtosis denotes the fourth moment, skewness for $\frac{-\kappa z_S(T-t)}{v}$ is positive and kurtosis for $\frac{-\kappa z_S(T-t)}{v}$ is still positive.

Furthermore, let $f$(Z; $\beta$, $\delta$) denote the risk-neutral density for Z implied by Q. For the lognormal density function, the Gram-Charlier expansion is given by

$$f(z;\beta,\delta) = (1 + \beta(z^3 - 3z) + \delta(z^4 - 6z^2 + 3))\phi(z) \qquad (2.6)$$

where $\beta = \frac{\varsigma}{6}$ , $\delta = \frac{\chi}{24}$ and $\varsigma$, $\chi$ denote skewness and kurtosis of the density function, $\ln\zeta(T)$, respectively. Assuming the existence of one risky asset only, $\varsigma$ and $\chi$ also denote the skewness and kurtosis of the return on this risky asset. Under a normal specification, the skewness and kurtosis coefficients $\varsigma = 0$ and $\chi = 0$ respectively; by substituting them into *f*(Z; $\beta$, $\delta$) generate the special case of a standard normal density $\phi(\cdot)$ .When the stock index is developed that allows for negatively skewed and fat tails, then the state price skewness and kurtosis are positive (Jondeau and Rockinger, 2001).

### 2.3 Portfolio Optimization under VaR-RM

This section considers the dynamic investment problem of a VaR risk manager. Value-at-Risk describes the loss that can occur over a given period and at a given confidence level, due to market risk exposure (Jorion, 1996). The VaR at $\alpha$ percent is the loss that has a $\alpha$ percent probability of being exceeded. The definition can be presented as

$$P(W(0) - W(T) \le VaR(\alpha)) = 1 - \alpha \,, \quad 0 \le \alpha \le 1 .$$

Our objective is to embed VaR risk management objectives into a framework that maximizes utility. We assume that an additional constraint is imposed on the agent's optimization problem and requiring the $VaR(\alpha)$ to be maintained below the prespecified level,

$$VaR(\alpha) \le W(0) - \underline{W} \,,$$

where the floor $\underline{W}$ is specified exogenously. Following the above equations, the VaR constraint can be expressed as

$$P(W(T) \geq \underline{W}) \geq 1-\alpha$$

The investor's utility function is assumed to belong to the class of constant relative risk aversion (CRRA) utility functions.

$$U(W(T)) = \frac{W(T)^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0 \tag{2.7}$$

where the function $U(\cdot)$ is assumed twice continuously differentiable, strictly increasing, strictly concave and to satisfy $\lim_{x\to 0} U'(x) = \infty$ and $\lim_{x\to\infty} U'(x) = 0$.

The manager's objective is to maximize expected utility and maintain the VaR of terminal wealth at a prespecified level. The optimal strategy that applies to the manager can be formulated as

$$\max_{W(T)} E[\frac{W(T)^{1-\gamma}}{1-\gamma}] \tag{2.8}$$

*subject to*

$$E[\zeta(T)W(T)] = \zeta(0)W(0),$$

$$P(W(T) \geq \underline{W}) \geq 1-\alpha.$$

The wealth process is assumed to be a self-financing process. In this case, the portfolio problem can be solved by the martingale solution technique proposed by Cox and Huang (1989), (1991). Following Cox and Huang (1989), the investor's problem is solved in two steps. First, the investor's optimal terminal wealth is determined. Then the dynamic strategy that replicates it is derived. Basak and

Shapiro (2001) showed that the solution to this program is

$$W(T)^{VaR} = \begin{cases} (y\zeta(\mathrm{T}))^{-1/\gamma}, & if \quad \zeta(T) \leq \underline{\zeta} \\ \underline{W}, & if \quad \underline{\zeta} \leq \zeta(T) \leq \overline{\zeta} \\ (y\zeta(\mathrm{T}))^{-1/\gamma}, & if \quad \overline{\zeta} \leq \zeta(T) \end{cases} \tag{2.9}$$

where $\underline{W} \equiv (y\underline{\zeta})^{-1/\gamma}$, $\overline{\zeta}$ is such that $P(\zeta(T) > \overline{\zeta}) \equiv \alpha$ and $\underline{\underline{W}} \equiv (y\overline{\zeta})^{-1/\gamma}$. Lagrange multiplier $y \geq 0$ solves $E(\zeta(T)W(T;y)) = \zeta(0)W(0)$ (see Appendix A).

The optimal wealth at each date t can be computed according to

$$\zeta(t)W(t)^{VaR} = E\left[\zeta(T)W(T)^{VaR}\right] \tag{2.10}$$

where E is the expected condition on the information available at time t. Using the martingale approach is in principle straightforward for solving the optimal wealth process described in the Appendix B.

To perform a detailed examination of the optimal behavior under the VaR-RM strategy, this study specialize the setting to Gram-Charlier expansion of the density function of state prices with a constant interest rate and market price of risk. Let $\beta$ and $\delta$ denote the skewness and kurtosis of the density function, $\ln\zeta(T)$, respectively. Then

**<u>Proposition 2.1</u>**:

The time t optimal wealth is given by

$$W^{VaR}(t)=\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[1+G_1\left(v'\right)\right]$$

$$-\left\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[N\left(-d_1(\underline{\zeta})\right)+Q\left(-d_1\left(\underline{\zeta}\right),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\underline{\zeta})\right)+Q\left(-d_2\left(\underline{\zeta}\right),v\right)\right]\right\}$$

$$+\left\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[N\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1\left(\overline{\zeta}\right),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\overline{\zeta})\right)+Q\left(-d_2\left(\overline{\zeta}\right),v\right)\right]\right\}$$

(2.11)

where $G_1(a)=\beta a^3+\delta a^4$

$$G_2(a,b)=\beta\left(a^2-3ab+3b^2-1\right)+\delta\left(-a^3+4a^2b-6ab^2+3a-4b+4b^3\right)$$

$$Q(a,b)=N(a)G_1(b)+\phi(a)G_2(a,b)$$

$$\Gamma(t)\equiv\left(\frac{1-\gamma}{\gamma}\right)\left(r+\frac{\|\kappa\|^2}{2}\right)\left(T-t\right)+\left(\frac{1-\gamma}{\gamma}\right)^2\frac{\|\kappa\|^2}{2}\left(T-t\right)$$

$$d_2(x)\equiv\frac{\ln\left(\frac{x}{\zeta(t)}\right)+\left(r-\frac{\|\kappa\|^2}{2}\right)\left(T-t\right)}{\|\kappa\|\sqrt{T-t}}$$

$$d_1(x)\equiv d_2(x)+\frac{1}{\gamma}\|\kappa\|\sqrt{T-t}$$

$$v'=\frac{\gamma-1}{\gamma}v.$$

Equation (2.11) represents time t optimal wealth that has three components: optimal wealth of a non-risk manager; cost of a put option on the benchmark-wealth, wealth of a non-risk manager, with strike price $\underline{W}$; and, the form of shorting a portfolio of a binary option. Consequently, the second and third terms attempt to keep time-T wealth at $\underline{W}$ in intermediate states. Notably, G and Q functions in equation (2,11) exhibit the impact of skewed and fat tails. The solutions when asset returns are normally distributed are special cases of the solutions presented when $\beta = 0$ and $\delta = 0$.

**Proposition 2.2**:

The fraction of wealth invested in stock index is

$$x_S^{VaR}(t) =$$

$$\frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\left(\frac{1}{\gamma}\right)\Big\{W(t) - \underline{W}e^{-r(T-t)}\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right) + Q\left(-d_2(\underline{\zeta}), v\right) - \mathrm{N}\left(-d_2(\bar{\zeta})\right) - Q\left(-d_2(\bar{\zeta}), v\right)\right]$$

$$- \frac{(\underline{\underline{W}} - \underline{W})e^{-r(T-t)}}{\| \kappa \| \sqrt{T-t}}\left[\phi\left(-d_2(\bar{\zeta})\right)\left(1 + \varphi\left(\alpha(\bar{\zeta})\right)\right)\right]\Big\}, \tag{2.12}$$

Let $x_S^{VaR}(t) = q^{VaR}(t) x_S^B(t)$, where $x_S^B(t)$ is the benchmark value, the fraction of wealth invested in stock index of a non-risk manager, and $q^{VaR}(t)$ is the exposure to risky asset relative to the benchmark, and

$$x_S^B(t) = \frac{1}{\gamma}(\sigma(t)^{\mathrm{T}})^{-1}\kappa, \tag{2.13}$$

then

$$q^{VaR}(t) = 1 - \frac{\underline{W}e^{-r(T-t)}\left\{\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right) - \mathrm{N}\left(-d_2(\bar{\zeta})\right)\right] + \left[Q\left(-d_2(\underline{\zeta}), v\right) - Q\left(-d_2(\bar{\zeta}), v\right)\right]\right\}}{W^{VaR}(t)}$$

$$+ \frac{\gamma e^{-r(T-t)}\phi\left(d_2(\bar{\zeta})\right)(\underline{W} - \underline{\underline{W}})\left[1 + \varphi\left(\alpha(\bar{\zeta})\right)\right]}{W^{VaR}(t)\|\kappa\|\sqrt{T-t}} \tag{2.14}$$

where $\varphi\left(\alpha(x)\right) = \beta\left[\alpha^3(x) - 3\alpha(x)\right] + \delta\left[\alpha^4(x) - 6\alpha^2(x) + 3\right]$

$$\alpha(x) = \frac{\ln(x) - m}{v} = d_1(x) + v' = d_2(x) + v$$

The G, Q and $\varphi$ functions of Eq. (2.12) exhibit the impact of skewed and fat tails (see Appendix C). When $\beta = 0$ and $\delta$=0, the G, Q and $\varphi$ functions all equal 0, and the results in Eqs. (2.12) and (2.13) are the same as those obtained by Basak and Shapiro (2001), indicating that the solutions when asset returns are normally distributed are a special case of solutions presented when $\beta = 0$ and $\delta$=0.

## 2.4 Numerical Illustrations

This section provides some numerical examples for analyzing how deviations from normality affect the end-of-horizon wealth and risk exposure of the VaR risk manager described in the previous section. The principal parameters for the financial market presented by Basak and Shapiro (2001) are applied. The parameters describing skewness and kurtosis by Corrado and Su (1997a, 1997b) are employed for numerical illustrations. Figures 2.1-2.5 show the impact of skewness and kurtosis on optimal terminal wealth and risk exposure of a VaR risk manager. Moreover, Figs. 2.1-2.4 plot the results obtained by Basak and Shapiro (2001) when $\beta = 0$ and $\delta$=0.

Figure 2.1 shows how negative skewness affects optimal terminal wealth. The agent behaves similarly to a benchmark agent, a non-risk manager, in both good states (low $\zeta(T)$) and bad states (high $\zeta(T)$), and the manager insures against in intermediate states. Optimal terminal wealth is larger than $\underline{W}$ in good states, and optimal terminal wealth equals $\underline{W}$ in intermediate states. The probability in bad states equals $\alpha$. $\alpha$ is set to $0.01$; in other words, the probability of optimal terminal wealth being less than $\underline{W}$ is 0.01. However, optimal terminal wealth is far less than $\underline{W}$ in bad states and is less than $\underline{\underline{W}}$ in bad states. The impact of negative skewness can be ignored in both good and intermediated states. The agent will experience a great loss in bad states when confronted with negative skewness. Figure 2.2 presents the effect caused by excess kurtosis. The impact of excess kurtosis is identical to that of negative skewness.

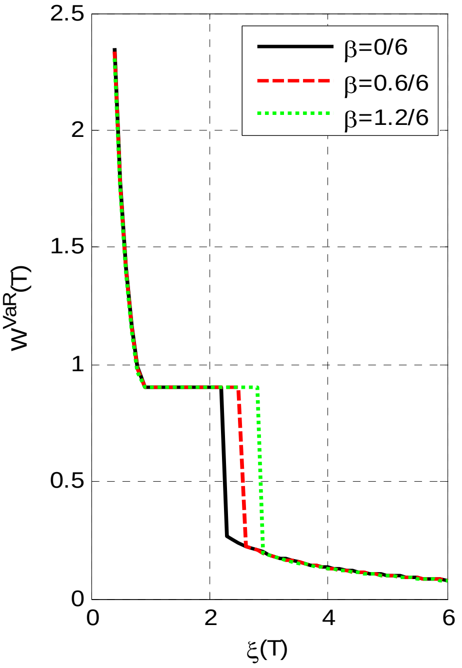

**Figure 2.1 Optimal terminal wealth of a VaR agent: Impact of negative skewness**

Figure 2.1 presents the effect of negative skewness on optimal terminal wealth, $W(T)$, for a risk manager with $\delta = 0$, $\beta = 0$ (solid line), $\beta = 0.1$ (dashed line) and $\beta = 0.2$ (dotted line). The parameters utilized are $\gamma = 0.8$, $\alpha = 0.01$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$. Then,

$\beta$=0, $\delta$=0: $\underline{\zeta}$=0.8625, $\overline{\zeta}$=2.2267, $\underline{\underline{W}} = 0.2750$;

$\beta$=0.1, $\delta$=0: $\underline{\zeta}$=0.8577, $\overline{\zeta}$=2.5867, $\underline{\underline{W}} = 0.2264$;

$\beta$=0.2, $\delta$=0: $\underline{\zeta}$=0.8502, $\overline{\zeta}$=2.8334, $\underline{\underline{W}} = 0.1999$.

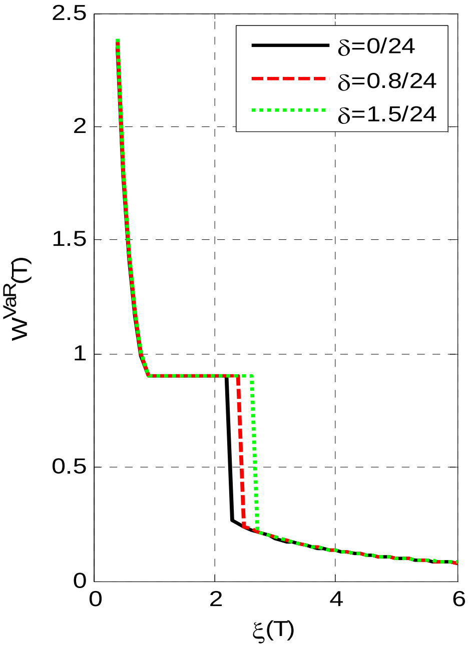

**Figure 2.2 Optimal terminal wealth of a VaR agent: Impact of excess kurtosis**

Figure 2.2 plots the effect of excess kurtosis on optimal terminal wealth, $W(T)$, for a risk manager with $\beta = 0$, $\delta = 0$ (solid line), $\delta = 0.8/24$ (dashed line) and $\delta = 1.5/24$ (dotted line). The parameters used are $\gamma = 0.8$, $\alpha = 0.01$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$. Then,

$\beta$=0, $\delta$=0: $\underline{\zeta}$=0.8625, $\overline{\zeta}$=2.2267, $\underline{\underline{W}} = 0.2750$;

$\beta$=0, $\delta$=0.8/24: $\underline{\zeta}$=0.8679, $\overline{\zeta}$=2.4344, $\underline{\underline{W}} = 0.2479$;

$\beta$=0, $\delta$=1.5/24: $\underline{\zeta}$=0.8718, $\overline{\zeta}$=2.6446, $\underline{\underline{W}} = 0.2248$.

Figure 2.3 illustrates how negative skewness and excess kurtosis affect optimal terminal wealth. The agent will have a substantial loss in bad states when confronted with negative skewness and excess kurtosis. When risky asset returns follow a normal distribution ($\beta = 0, \delta = 0$, solid line), the probability of optimal terminal wealth being below $\underline{\underline{W_1}}$ is 0.05 and $\underline{\underline{W_1}}$ is 0.4256. When risky asset return has negative skewness and excess kurtosis ($\beta = 0.2, \delta = 1.5/24$, dashed line), the probability of optimal terminal wealth being less than $\underline{\underline{W_2}}$ is 0.05 and $\underline{\underline{W_2}}$ is 0.3301. Moreover, when risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio based on the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is bad. However, the VaR manager may mistakenly believe that the probability of achieving optimal terminal wealth of below 0.3301 is 0.0138. In fact, that probability is 0.05.

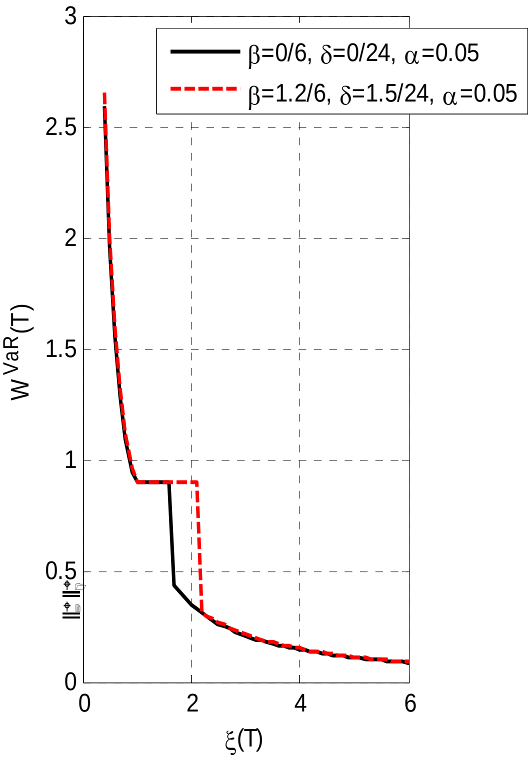

**Figure 2.3 Optimal terminal wealth of a VaR agent: Impact of negative skewness and excess kurtosis**

Figure 2.3 plots the effects of negative skewness and excess kurtosis on optimal terminal wealth, $W(T)$, for a risk manager with $\beta=0\,,\delta=0$ (solid line), $\beta=0.2\,,\delta=1.5/24$ (dashed line). The parameters used are $\gamma=0.8$, $\alpha=0.05$, $W_0=1.0$, $\underline{W}=0.9$, $r=0.05$, $\|\kappa\|=0.4$, $T=1$, $\zeta(0)=1$. Then,

$\beta=0\,,\delta=0$ : $\underline{\zeta}$ =0.9313, $\overline{\zeta}$ =1.6954, $\underline{\underline{W_1}}=0.4256$;

$\beta=0.2\,,\delta=1.5/24$: $\underline{\zeta}$ =0.9498, $\overline{\zeta}$ =2.1188, $\underline{\underline{W_2}}=0.3301$·

Figure 2.4 presents the influence of negative skewness and excess kurtosis on risk exposure. The risk manager invests a higher fraction of his wealth in the bond in the good states than a non-risk manager. As $\zeta(t)$ increases, the VaR agent begins to increase his equity exposure and invests a higher proportion of his wealth in the stock index. When $\zeta(t)$ becomes sufficiently high to deter the agent from taking further risks, the agent returns to his benchmark strategy. Asset allocation attempts to replicate a portfolio of binary options and insuring the intermediate states region. When a risk manager confronts negative skewness and excess kurtosis, the agent invests reduced funds in the stock market in intermediate states and increases their investment in the stock market in bad states.

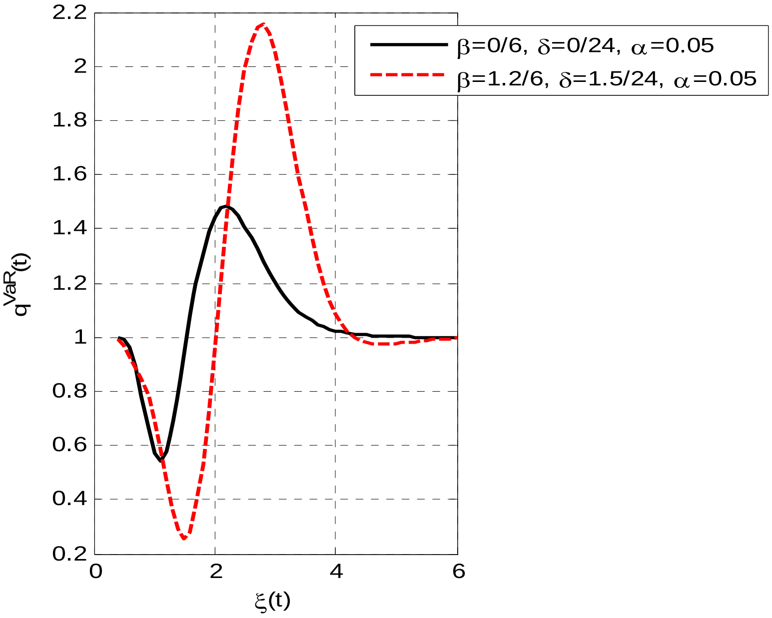

**Figure 2.4 Risk exposure of a VaR agent: Impact of negative skewness and excess kurtosis**
Figure 2.4 presents the effect of negative skewness and excess kurtosis on exposure to risky assets relative to the benchmark, $q(t)$, for a risk manager with $\beta = 0, \delta = 0$ and $\alpha = 0.05$ (solid line), $\beta = 0.2$, $\delta = 1.5/24$ and $\alpha = 0.05$ (dashed line). The parameters used are $\gamma = 0.8$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $t = 0.5$, $\zeta(0) = 1$.

Importantly, a risk manager must forecast the loss precisely and to estimate the probability that the end-of-horizon state is bad. When risky asset returns have negative skewness and excess kurtosis, the agent will experience an unexpected and large loss when the end-of-horizon state is a bad state (Fig. 2.3). A risk manager cannot decrease the loss in a bad state, but can decrease the value of $\alpha$, the probability that a loss exceeds VaR, and moreover, the agent will simultaneously have reduced terminal wealth in both the good and bad states. Figure 2.5 illustrates this result.

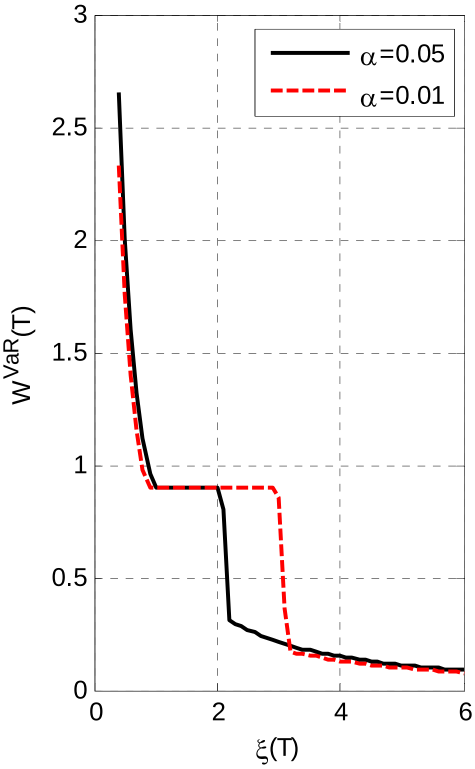

**Figure 2.5 Optimal terminal wealth of a VaR agent: Impact of $\alpha$**
Figure 2.5 shows the effect of $\alpha$ on optimal terminal wealth, $W(T)$, for a risk manager with $\beta = 0.2$, $\delta = 1.5/24$ and $\alpha = 0.05$ (solid line), $\beta = 0.2$, $\delta = 1.5/24$ and $\alpha = 0.01$ (dashed line). Parameters used are $\gamma = 0.8$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$.

Lagrange multiplier, *y,* means that approximately *y* additional unit of expected utility can be increased if one additional dollar in initial wealth invests. Table 2.1 illustrates Lagrange multipliers, *y,* under several α, $\gamma$ and $\underline{W}$ values (i.e., α = 0(portfolio insurance), 0.01 and 0.05； $\gamma$=0.6, 0.8 and 1.2； $\underline{W}$=0.9, 0.95 and 0.99). It shows that *y* increases when we decrease the $\gamma$ values and α values or increases the floor $\underline{W}$. When risk aversion is low, that is for lower $\gamma$, VaR risk management can increase more expected utility. When α approaches 0, that is portfolio insurance, more expected utility can be increased. When VaR agent increases the floor $\underline{W}$， more expected utility also can be increased.

**Table 2.1 Summary of Lagrange multipliers**

| | | **$\underline{W}$=0.9** | | | **$\underline{W}$=0.95** | | | **$\underline{W}$=0.99** | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\gamma$ | | | $\gamma$ | | | $\gamma$ | | |
| | | 0.6 | 0.8 | 1.2 | 0.6 | 0.8 | 1.2 | 0.6 | 0.8 | 1.2 |
| α | 0 | 1.602 | 1.308 | 1.057 | 1.762 | 1.421 | 1.113 | 1.964 | 1.574 | 1.193 |
| | 0.01 | 1.532 | 1.261 | 1.038 | 1.660 | 1.349 | 1.082 | 1.804 | 1.456 | 1.143 |
| | 0.05 | 1.380 | 1.168 | 1.004 | 1.454 | 1.217 | 1.030 | 1.523 | 1.269 | 1.062 |

## 2.5 Discussions

Lucas and Klaassen (1999) considered a simple one-period asset allocation problem and investigated the effect of extreme returns or fat tails on optimal asset allocations using a shortfall constraint. They utilized the Student-t distribution to characterize the fat tails of asset returns and maximize expected return on the portfolio subject to a VaR constraint. Their analysis revealed that the degree of the shortfall probability plays a crucial role in determining the effects of the choice between a fat-tailed and a normal distribution. If the shortfall probability is moderately large, say, 5%, then the assumption of fat tails results in more aggressive asset allocations. If the shortfall probability is small, say, 1%, then the use of leptokurtic distribution leads to more prudent asset allocations. Their results imply that the VaR of a given portfolio may be underestimated when tail behavior of asset returns is not captured accurately. They suggested that a good characterization of the distribution of asset returns is needs in a financial decision context involving downside risk.

Tokat, Rachev and Schwartz (2003) analyzed the effects of the distributional assumptions on optimal asset allocation. A multistage stochastic asset allocation problem with decision rule has been set up. The optimal asset allocations found under normal and stable scenarios are compared. The analysis suggested that the normal scenarios may greatly underestimate risks depending on the utility function of the decision maker. If the agent has very low or very high risk averse, then the Gaussian and stable non-Gaussian scenarios result in similar allocation. When the risk aversion of the agent is between these two extreme cases, the two distributional assumptions result in very different allocation. Since stable economic scenarios model extreme events more realistically, they suggested more conservative asset allocations.

This study provided solutions to the intertemporal investment strategy by an agent in a complete market setting in which asset returns are negatively skewed and have excess kurtosis. Therefore, the Gram-Charlier expansion was utilized to approximate asset returns with negatively skewed and excess kurtosis. How do extreme returns or fat tails affect optimal asset allocation and end-of-horizon wealth? First, when risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio under the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is a bad state. Second, the agent invests reduced funds in the stock market during intermediate states and increased amounts in the stock market during bad states when confronted with negative skewness and excess kurtosis. Third, a risk manager cannot decrease losses in bad states, but can decrease the value of $\alpha$, the probability that a loss exceeds VaR, and the agent will suffer from reduced terminal wealth in both the good and bad states.

The impact of skewness and kurtosis is large and should not be neglected. Analytical results suggest that the total impact of leptokurtic asset returns depends on the shape of asset returns and correctly measuring leptokurtic asset returns is very important for optimal asset choice made by risk managers.

The major conclusions of Lucas and Klaassen *(*1999), Tokat et al.( 2003) and this study is shown in table 2.2.

**Table 2.2 The major conclusions of Lucas and Klaassen (1999), Tokat et al.( 2003) and this study**

| Authors | Method | Conclusions |
|---|---|---|
| Tokat, Rachev and Schwartz ( 2003) | multistage stochastic asset allocation;<br>Stable non-Gaussian distributions;<br>non-risk management | 1. The risk aversion of the agent plays a crucial role in determining the effects of the choice between a fat-tailed and a normal distribution.<br>2. They suggested more conservative asset allocations. |
| Lucas and Klaassen (1999) | one-period;<br>Student-t distribution;<br>VaR constraint | 1. The degree of the shortfall probability plays a crucial role in determining the effects of the choice between a fat-tailed and a normal distribution.<br>2. The VaR of a given portfolio may be underestimated when tail behavior of asset returns is not captured accurately. |
| This paper | Gram-Charlier expansion;<br>VaR constraint | 1. The VaR manager will experience an unexpected and great loss when the end-of-horizon state is a bad state when risky asset return has negative skewness and excess kurtosis.<br>2. The agent invests reduced funds in the stock market during intermediate states and increased amounts in the stock market during bad states. |