2 Proposed Model

In this section, we focus on the financial model for DC pension plan that is fully investigated in this paper. First, the financial market is introduced and the stochastic processes is employed to characterize the dynamics of the interest rate and asset prices. Then the stochastic processes are presented to describe the behavior of two background risks: labor incomes and inflation rates. Finally, the accumulated nominal fund wealth and real wealth processes are derived.

2.1 Financial Market

The financial market is assumed to be arbitrage-free, complete and continuously open over the fixed time interval $[0, T]$, where $T > 0$ denotes the retirement time of a representative shareholder. Randomness is described by two standard and independent Wiener processes $W^r(t)$ and $W^m(t)$, with $t \in [0, T]$, and defined on a complete probability space (Ω, F, P). Here, P is the real world probability, and $F = \{F(t)\}_{t\in[0,T]}$ is the filtration which represents the information structure assumed to be generated by Brownian motion and satisfying the usual conditions. $W^m(t)$ is a standard Wiener process independent of $W^r(t)$ capturing the asset risk other than the interest rate risk. The independence hypothesis on $W^r(t)$ and $W^m(t)$ implies no loss of generality since we can always shift from uncorrelated to correlated Wiener processes (and vice verse) via the Cholesky decomposition of the correlation matrix.

We suppose that the instantaneous riskless interest rate $r(t)$ follows the Vasicěk model (1977), Then, under the real world probability measure P, the process $r(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dr(t) &= a(b - r(t))dt + \sigma_r dW^r(t), \\ r(0) &= r_0, \end{aligned} \tag{1}$$

where a, b, and σ_r are strictly positive constants. Then, the interest rate presents a mean-reverting effect where the parameter b is the mean level attracting the interest rate while the strength of this attraction is measured by the parameter a.

Given the differential equation of the interest rate we can derive both its value and the value of a zero coupon bond with fixed maturity. In particular, we refer to Vasicěk (1977) for the demonstration of the following proposition.

Proposition 1 *Suppose that the interest rate $r(t)$ satisfies the stochastic differential equation (1), then:*

1. *The explicit solution of (1) is*

$$r(t) = (r_0 - b)e^{-at} + b + \sigma_r \int_0^t e^{-a(t-u)} dW^r(u), \tag{2}$$

2. *The price of a zero coupon bond with maturity $\tau > t$ is given by*

$$X^B(t,\tau,r) = e^{c(t,\tau)-\beta(t,\tau)r(t)} \tag{3}$$

where

$$\begin{aligned} \beta(t,\tau) &= \frac{1-e^{-a(\tau-t)}}{a} \\ c(t,\tau) &= -R(\infty)(\tau-t)+\beta(t,\tau)\left[R(\infty)-\frac{\sigma_r^2}{2a^2}\right]+\frac{\sigma_r^2}{4a^3}(1-e^{-2a(\tau-t)}), \end{aligned}$$

$R(\infty) = b + \frac{\sigma_r\lambda_r}{a} - \frac{\sigma_r^2}{2a^2}$ *represents the return of a zero coupon bond with maturity equal to infinity, and λ_r denotes the constant market price of risk.*

In the financial market, three kinds of assets are assumed for the pension fund manager in making the asset allocation decision. The three assets are characterized by the following processes.

1. The price process $X^0(t,r)$ of the riskless asset is given by:

$$\frac{dX^0(t,r)}{X^0(t,r)} = r(t)dt \tag{4}$$

$$X^0(0) = X_0^0$$

where the dynamics of $r(t)$, under the real probability measure P, is defined in Eq. (1). The riskless asset can be interpreted as a bank account, paying the instantaneous interest rate $r(t)$ without any default risk.

2. The dynamic process $X^S(t,r)$ of a stock price is given by:

$$\frac{dX^S(t,r)}{X^S(t,r)} = [r(t)+\sigma_{S,r}\lambda_r+\sigma_{S,m}\lambda_m]\,dt+\sigma_{S,r}dW^r(t)+\sigma_{S,m}dW^m(t), \tag{5}$$

$$X^S(0) = X_0^S,$$

where λ_r represents the risk premium of interest rate risk, and λ_m represents the risk premium of financial market risk in addition to interest rate risk; $\sigma_{S,r} > 0$ is a volatility scale factor measuring how the interest rate volatility affects the stock volatility; $\sigma_{S,m} > 0$ is a volatility scale factor measuring how the financial market risk excepted interest rate risk affects the stock volatility. Thus, the instantaneous mean of the stock index can be written as $r(t)+\sigma_{S,r}\lambda_r+\sigma_{S,m}\lambda_m$. The parameters λ_r and λ_m are assumed strictly positive so that the stock return is higher than the return of the short rate. For the sake of simplicity, in our model we introduce only one stock, which can be interpreted as a stock market index. Nevertheless, if we allow for a complete market with a finite number of stocks, no further difficulties are added to the model because the only source of difficulties is the market incompleteness.

3. A so-called rolling bond with constant time to maturity (K) whose value $X_K^B(t,r)$ follows the dynamics process:

$$\frac{dX_K^B(t,r)}{X_K^B(t,r)} = \left[r(t) + \sigma_B^K \lambda_r\right] dt - \sigma_B^K dW^r(t), \tag{6}$$

$$\sigma_B^K = \frac{1-e^{-aK}}{a}\sigma_r.$$

Actually, given the instantaneous short interest rate (2), we could assume that there exists tradable zero coupon bonds for every maturity $K \in [0,T]$. According to Proposition 1, the return of a zero coupon bond with maturity $K \in [0,T]$ is given by:

$$\frac{dX^B(t,K,r)}{X^B(t,K,r)} = (r(t) + \beta(t,K)\lambda_r)\, dt - \beta(t,K) dW^r(t),$$

where

$$\beta(t,K) = \frac{1-e^{-a(K-t)}}{a}\sigma_r.$$

Nevertheless, as pointed out in Boulier et al. (2001), it is quite unrealistic to assume the existence of infinite zero coupon bonds. Furthermore, a rolling bond seems to be very useful for fund managers and they argue that the asset allocation problem can be solved by just taking into account this bond without any loss of generality. In fact, the values $X^B(t,K,r)$ and $X_K^B(t,r)$ are linked (through the riskless asset $X^0(t)$) by the following equation:

$$\frac{dX^B(t,K,r)}{X^B(t,K,r)} = \left(1 - \frac{\beta(t,K)\sigma_r}{\sigma_B^K}\right)\frac{dX^0(t)}{X^0(t)} + \frac{\beta(t,K)\sigma_r}{\sigma_K^B}\frac{dX_K^B(t,r)}{X_K^B(t,r)}.$$

This means that the original bond can be obtained through a suitable portfolio (i.e. a linear combination) of the riskless asset and the X_K^B bond. The diffusion matrix for the considered financial market is given by:

$$\Sigma \equiv \begin{bmatrix} \sigma_{S,r} & \sigma_{S,m} \\ -\sigma_B^K & 0 \end{bmatrix},$$

and, since $\sigma_{S,m}$ and $\sigma_{S,r}$ are different from zero by hypothesis, and $\sigma_B^K \neq 0$ by construction, it follows that:

$$\det\ \Sigma = \sigma_{S,m}\sigma_B^K \neq 0.$$

Since we have as many risky assets as risk sources and the diffusion matrix is invertible, the market we consider is complete.

2.2 Labor income process

First, we formulate the dynamic evolution of the labor incomes from contributions since the employee must contribute a proportion of his labor income to the fund.

$$\frac{dL(t,r)}{L(t,r)} = \mu_L^i(t,r)dt + \sigma_{L,r}dW^r(t) + \sigma_{L,m}dW^m(t) + \sigma_L dW^L(t), \tag{7}$$

$$L(0) = L_0,$$

where $\sigma_{L,r}$ and $\sigma_{L,m}$ are the volatility factors measuring how the risk sources of interest rate and the other financial factors affect the labor incomes, while $\sigma_L \neq 0$ is a non-hedgable volatility whose risk source does not belong to the set of the financial market risk sources. This non-hedgable risk source is represented by the one-dimensional standard Brownian motion $W^L(t)$, which is assumed to be independent of $W^r(t)$ and $W^m(t)$.

After applying Itô's lemma to $logL(t)$, the explicit solution of Eq. (7) is written as:

$$L(t) = L(0)\exp\left[\begin{array}{c} \int_0^t \mu_L^i(u)du - \frac{1}{2}\left(\sigma_{L,r}^2 + \sigma_{L,m}^2 + \sigma_L^2\right)t \\ +\sigma_{L,r}W^r(t) + \sigma_{L,m}W^m(t) + \sigma_L W^L(t) \end{array}\right]. \tag{8}$$

Next, we assume that each employee contributes a constant proportion, γ, of his labor income to his personal account. Then, the defined-contribution level is characterized as follows:

$$C(t) = \gamma L(t),$$

whose evolution equation is

$$dC(t) = \gamma dL(t). \tag{9}$$

In our model, the contribution growth equals the labor income growth.

2.3 Inflation rate

In Section 2.2, we introduce a background risk, i.e., the labor income uncertainty. Now, we introduce another background risk, the inflation risk. Actually, when the portfolio problem for a pension fund is considered, the effect from the consumption price behavior needs to be incorporated due to the long duration of the investment time horizon. Hence we present the stochastic partial differential equation describing the evolution of the consumption price index (which can be interpreted as the price of the only consumption good in the economy). In particular, we assume that CPI (P) follows the diffusion process:

$$\frac{dP(t)}{P(t)} = \mu_\pi dt + \sigma_{\pi,r}dW^r(t) + \sigma_{\pi,m}dW^m(t) + \sigma_\pi dW^L(t), \tag{10}$$

$$P(0) = 1,$$

where the parameters $\sigma_{\pi,r}$ and $\sigma_{\pi,m}$ are the volatility factors measuring how the volatility of interest rate and the other financial conditions affect the price index, while $\sigma_\pi \neq 0$ is the inflation own volatility. This last parameter can be also interpreted as the non-hedgable volatility since the risk source represented by $W^L(t)$ does not belong to the set of financial market risk sources.

In particular, we call F_N the nominal fund and F the real fund. By the Fisher equation (1930), we can write (See Appendix A):

$$dF = dF_N - F_N \frac{dP}{P}. \tag{11}$$

In the above conversion equation, when we want to convert nominal fund to real fund wealth, we need to incorporate the difference which is caused by change of inflation. Noting that the difference is the form of dP/P, so the difference is related only to the increasing rate of inflation. For simplicity, in Eq. (10) we assume that the increasing rate of inflation is just a constant. Therefore P is not a state variable when we derive the optimal solution.

2.4 The fund wealth

We assume that the investment strategies of the fund manager are defined as a stochastic process $\theta(t)$ with values in R^n adapted to the natural filtration of the Brownian motion. $\theta(t)$ represents a proportion vector of assets invested in the pension fund at time t. $\theta(t)$ is predictable. Furthermore, $\theta(t) = \begin{bmatrix} 1-\theta_S-\theta_B & \theta_S & \theta_B \end{bmatrix}$ denotes the weights of the fund's money invested in the riskless asset and in the risky assets (i.e. the stock index and the rolling bond) respectively. Then the accumulated wealth process at any time $t \in [0,T]$ must satisfy:

$$\begin{aligned} dF_N &= F_N \left[(1-\theta_S-\theta_B)\frac{dX^0}{X^0} + \theta_S \frac{dX^S}{X^S} + \theta_B \frac{dX_K^B}{X_K^B} \right] + \gamma dL \\ &\quad -e_1 \max(F_N \left[(1-\theta_S-\theta_B)\frac{dX^0}{X^0} + \theta_S \frac{dX^S}{X^S} + \theta_B \frac{dX_K^B}{X_K^B} \right], 0) \\ &\quad -e_2 \min(F_N \left[(1-\theta_S-\theta_B)\frac{dX^0}{X^0} + \theta_S \frac{dX^S}{X^S} + \theta_B \frac{dX_K^B}{X_K^B} \right], 0), \end{aligned} \tag{12}$$

where γdL represents the contribution to the pension fund, and e_1 denotes the incentive fee ratio when the fund return is positive and e_2 denotes the partial floor protections when the fund return is negative. In the above equation we can see that the fund manager must charge management incentive fees or face the loss compensation, which correlates with the fund's performance. In the other words, if the fund performance is good, this indicates that the fund manager should charge higher management incentive fees. For simplicity, we assume that $e_1 = e_2 = e$. Now, we use Eq. (11) to Eq. (12), the accumulated real fund wealth process at any time $t \in [0, T]$ can be written as follows:

$$dF = F_N(1-e)\left[(1-\theta_S-\theta_B)\frac{dX^0}{X^0} + \theta_S\frac{dX^S}{X^S} + \theta_B\frac{dX_K^B}{X_K^B}\right] \tag{13}$$

$$+\gamma dL - F_N\frac{dP}{P}, \tag{14}$$

which can be also written as:

$$dF = F_N\left[(1-e)((1-\theta_S-\theta_B)\frac{dX^0}{X^0} + \theta_S\frac{dX^S}{X^S} + \theta_B\frac{dX_K^B}{X_K^B}) - \frac{dP}{P}\right] + \gamma dL. \tag{15}$$

After substituting Eq. (4), (5), (6), (7) and (10) into Eq. (15), we obtain:

$$dF = \left\{F_N\left[(1-e)\theta_G' M + (r - re - \mu_\pi)\right] + \gamma L\mu_L^i\right\}dt + \{F_N\left[\Phi' + (1-e)\theta_G'\Gamma\right] + \gamma L\Lambda'\}\,dW,$$

where

$$\begin{aligned}
\theta_G &= \begin{bmatrix} \theta_S & \theta_B \end{bmatrix}', \\
M &= \begin{bmatrix} \sigma_{S,r}\lambda_r + \sigma_{S,m}\lambda_m & \sigma_B^K\lambda_r \end{bmatrix}', \\
\Phi &= \begin{bmatrix} -\sigma_{\pi,r} & -\sigma_{\pi,m} & -\sigma_\pi \end{bmatrix}', \\
\Gamma &= \begin{bmatrix} \sigma_{S,r} & \sigma_{S,m} & 0 \\ -\sigma_B^K & 0 & 0 \end{bmatrix}, \\
\Lambda &= \begin{bmatrix} \sigma_{L,r} & \sigma_{L,m} & \sigma_L \end{bmatrix}', \\
W &= \begin{bmatrix} W^r & W^m & W^L \end{bmatrix}'.
\end{aligned}$$