3 Asset allocation problem

The goal of the fund manager is to choose a portfolio strategy in order to maximize the expected value of the terminal utility function $K(F(T))$.

3.1 The stochastic optimal control

The stochastic optimal control problem is written as follows:

$$\underset{\theta}{Max} E_0 \left[K(F(T)) \mid F(0) = F_0, z(0) = z_0 \right],$$

$$\begin{aligned} d\begin{bmatrix} z \\ F \end{bmatrix} &= \begin{bmatrix} \mu_z \\ F_N \left[(1-e)\theta_G' M + (r - re - \mu_\pi) \right] + \gamma L \mu_L^i \end{bmatrix} dt \\ &+ \begin{bmatrix} \Omega' \\ F_N \left[\Phi' + (1-e)\theta_G' \Gamma \right] + \gamma L \Lambda' \end{bmatrix} dW, \end{aligned}$$

where

$$\begin{aligned} z &= \begin{bmatrix} r & L \end{bmatrix}', \\ \mu_z &= \begin{bmatrix} a(b-r) & L\mu_L^i \end{bmatrix}', \\ \Omega' &= \begin{bmatrix} \sigma_r & 0 & 0 \\ L\sigma_{L,r} & L\sigma_{L,m} & L\sigma_L \end{bmatrix}. \end{aligned}$$

The scale variables F and z represent the two state variables, while the elements of θ_G represent the two control variables.

Let $J(t; F_0, z_0)$ denote the value function of our optimal control problem, then it follows that:

$$J(t; F_0, z_0) = E_t \left[K(F(T)) \mid F(0) = F_0, z(0) = z_0 \right].$$

And the Hamiltonian equation is that:

$$\begin{aligned} H &= \mu_z' J_z + J_F \left[F_N((1-e)\theta_G' M + r - re - \mu_\pi) + \gamma L \mu_L^i \right] \\ &+ \frac{1}{2} tr(\Omega' \Omega J_{zz}) + (F_N \Phi' + F_N (1-e)\theta_G' \Gamma + \gamma L \Lambda') \Omega J_{zF} \\ &+ \frac{1}{2} J_{FF} [F_N^2 (1-e)^2 \theta_G' \Gamma' \Gamma \theta_G + F_N^2 \Phi' \Phi + \gamma^2 L^2 \Lambda' \Lambda \\ &+ 2F_N^2 (1-e)\theta_G' \Gamma' \Phi + 2\gamma L F_N (1-e)\theta_G' \Gamma' \Lambda + 2\gamma L F_N \Lambda' \Phi)], \end{aligned}$$

where we denote

$$J_z \equiv \frac{\partial J}{\partial z}, J_F \equiv \frac{\partial J}{\partial F}, J_{zz} \equiv \frac{\partial^2 J}{\partial z^2}, J_{FF} \equiv \frac{\partial^2 J}{\partial F^2}, \text{ and } J_{zF} \equiv J_{Fz} \equiv \frac{\partial^2 J}{\partial z \partial F}.$$

The first order condition gives us the following linear system of two equations and two unknowns:

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \theta_G} = J_F F_N(1-e)M + F_N(1-e)\Gamma\Omega J_{zF} \\ &\quad + J_{FF}[F_N^2(1-e)^2\Gamma\Gamma'\theta_G + F_N^2(1-e)\Gamma\Phi + \gamma L F_N(1-e)\Gamma\Lambda]. \end{aligned}$$

From the above equation, we obtain the optimal weights θ_G^*.

$$\begin{aligned} \theta_G^* &= -\frac{J_F}{J_{FF}}\frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}M - \frac{1}{J_{FF}}\frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega J_{zF} \qquad (1) \\ &\quad -\frac{1}{1-e}(\Gamma\Gamma')^{-1}\Gamma\Phi - \frac{1}{F_N(1-e)}\gamma L(\Gamma\Gamma')^{-1}\Gamma\Lambda. \end{aligned}$$

In order to illustrate the optimal behavior, we adopt the results of Markus and William (2004) and rewrite Eq.(1) as follows:

$$\begin{aligned} \theta_G^{*\prime} &= -A\frac{J_F}{F_N(1-e)\cdot J_{FF}}\theta_M{}' - B\frac{1}{F_N(1-e)\cdot J_{FF}}J_{zF}'\theta_Y{}' \\ &\quad -C\frac{1}{(1-e)}\theta_P{}' - D\frac{\gamma L}{F_N(1-e)}\theta_L{}', \end{aligned}$$

where

$$\begin{aligned} \theta_M &= \frac{M'(\Gamma\Gamma')^{-1}}{I'M'(\Gamma\Gamma')^{-1}}, \quad \theta_Y = \frac{\Omega\Gamma'(\Gamma\Gamma')^{-1}}{I'\Omega\Gamma'(\Gamma\Gamma')^{-1}}, \quad \theta_P = \frac{\Phi\Gamma'(\Gamma\Gamma')^{-1}}{I'\Phi\Gamma'(\Gamma\Gamma')^{-1}}, \quad \theta_L = \frac{\Lambda\Gamma'(\Gamma\Gamma')^{-1}}{I'\Lambda\Gamma'(\Gamma\Gamma')^{-1}} \\ A &= I'M'(\Gamma\Gamma')^{-1}, \quad B = I'\Omega\Gamma'(\Gamma\Gamma')^{-1}, \quad C = I'\Phi\Gamma'(\Gamma\Gamma')^{-1}, \quad D = I'\Lambda\Gamma'(\Gamma\Gamma')^{-1}. \end{aligned}$$

The vectors $\theta_M{}'$, $\theta_P{}'$ and $\theta_L{}'$ are two dimensional with elements that sum to 1, and $\theta_Y{}'$ is of dimension 3×2 with row elements that sum to 1; A, B, C and D are real constants. The optimal portfolio consists of five single portfolios: the market portfolio $\theta_M{}'$, the hedge portfolio for the state variables $\theta_Y{}'$, the hedge portfolio for the inflation risk $\theta_P{}'$, the hedge portfolio for the salary uncertainty $\theta_L{}'$ and the riskless asset. Thus we can state the following results.

1. The first term is a market portfolio, and its investment weight is equal to $-A\frac{J_F}{F_N(1-e)\cdot J_{FF}}$. We should note that this is a speculative component proportional to both the portfolio Sharpe ratio and the inverse of the Arrow-Pratt risk aversion index. In other words, this portfolio's investment weight will be influenced by the fund manager's risk aversion index.

2. The second term is a state variables (i.e., the interest rate and labor income uncertainties) hedge portfolio. This component provides a detailed mutual fund in the capital market to hedge the uncertainties.

3. The third and forth components are interesting. For the background risks (salary uncertainty and inflation risk) there exist no perfect hedging instruments in the financial markets. But the third and forth portfolios show how the background risks can be hedged in the capital market, and these components are preference-free components depending only on the diffusion terms of assets and background variables.

According to our five separation mutual funds, a pension fund manager who plans to hedge the market risk, the interest rate risk, the inflation rate risk and the labor income uncertainty should invest the wealth in the following five funds.

1. The market portfolio θ_M' with level $-A\frac{J_F}{F_N(1-e)\cdot J_{FF}}$.
2. The state variables hedge portfolio θ_Y' with level $-B\frac{1}{F_N(1-e)\cdot J_{FF}}J_{zF}'$.
3. The inflation hedge portfolio θ_P' with level $-C\frac{1}{(1-e)}$.
4. The salary uncertainty hedge portfolio θ_L' with level $-D\frac{\gamma L}{F_N(1-e)}$.
5. The riskless asset with level $1+A\frac{J_F}{F_N(1-e)\cdot J_{FF}}+B\frac{1}{F_N(1-e)\cdot J_{FF}}J_{zF}'+C\frac{1}{(1-e)}+D\frac{\gamma L}{F_N(1-e)}$.

Note that when we take the incentive fees or loss compensations into account, we find that the weights in risky assets increase. In other words, the levels that are invested in the market portfolio, the state variables hedge portfolio, the inflation hedge portfolio and the salary uncertainty hedge portfolio increase; while the level which is invested in the riskless asset decreases. This seems rational and reasonable. Since the fund manager charges the management incentive fee from the pension account and the management fee ratio is positively correlated with the pension fund's preference (i.e. the fund's return), it is necessary to pay extra money in the hedging components.

3.2 An exact solution

In the financial literature since Merton (1969, 1971), the condition of separability in wealth by product represents a common assumption in the attempt to explicitly solve the optimal portfolio problem. Accordingly following the previous works in Battocchio and Menoncin (2004), our value function is assumed to be given by the product of two terms: an increasing and concave function of the wealth F, and an exponential function depending on time and on the interest rates. Then, the value function J can be written as follows:

$$J(t;F,z) = U(F)e^{h(z,t)}. \tag{2}$$

First, we substitute the optimal asset allocation θ_G^* into the Hamiltonian, obtaining:

$$\begin{aligned} H^* \; = \; & \mu_z' J_z + J_F\left[F_N(r-re-\mu_\pi)+\gamma L\mu_L^i-(A'+B')\Gamma'(\Gamma\Gamma')^{-1}M\right]+\frac{1}{2}tr(\Omega'\Omega J_{zz}) \quad (3) \\ & -\frac{1}{2}\frac{(J_F)^2}{J_{FF}}M'(\Gamma\Gamma')^{-1}M-\frac{J_F}{J_{FF}}M'(\Gamma\Gamma')^{-1}\Gamma\Omega J_{zF}+(A'+B')(I-\Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega J_{zF} \\ & -\frac{1}{2}\frac{1}{J_{FF}}J_{zF}\Omega'\Gamma'(\Gamma\Gamma')^{-1}\Gamma\Omega J_{zF}+\frac{1}{2}J_{FF}(A'+B')(I-\Gamma'(\Gamma\Gamma')^{-1}\Gamma)(A+B), \end{aligned}$$

where we denote $A = F_N\Phi$, $B = \gamma L\Lambda$, and that I is the identity matrix.

Second, substituting (2) into the HJB equation (3), we obtain:

$$
\begin{aligned}
J(t; F, z)h_t + H^* &= 0, \\
h(T, z(T)) &= 0,
\end{aligned}
$$

and after dividing by J, we can write the HJB equation in the following way:

$$
\begin{aligned}
0 \quad = \quad & h_t + \mu_z' h_z + \frac{U_F}{U}\left[F_N(r - re - \mu_\pi) + \gamma L\mu_L^i - (A' + B')\Gamma'(\Gamma\Gamma')^{-1}M\right] \\
& + \frac{1}{2}tr(\Omega'\Omega(h_{zz} + h_z h_z')) - \frac{1}{2}\frac{(U_F)^2}{U_{FF}U}M'(\Gamma\Gamma')^{-1}M - \frac{(U_F)^2}{U_{FF}U}M'(\Gamma\Gamma')^{-1}\Gamma\Omega h_z \\
& + \frac{U_F}{U}(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega h_z \\
& - \frac{1}{2}\frac{(U_F)^2}{U_{FF}U}h_z\Omega'\Gamma'(\Gamma\Gamma')^{-1}\Gamma\Omega h_z \\
& + \frac{1}{2}\frac{U_{FF}}{U}(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)(A + B).
\end{aligned}
$$

We assume an exponential utility function of the form:

$$
U(F) = \beta_1 e^{\beta_2 F},
$$

for which we have:

$$
\begin{aligned}
\frac{U_F}{U} &= \beta_2, \\
\frac{(U_F)^2}{U_{FF}U} &= 1.
\end{aligned}
$$

Therefore, the HJB equation can be written as follows:

$$
\begin{aligned}
0 \quad = \quad & h_t + [\mu_z' - M'(\Gamma\Gamma')^{-1}\Gamma\Omega + \beta_2(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega]h_z + \frac{1}{2}tr(\Omega'\Omega h_{zz}) \\
& - \frac{1}{2}h_z\Omega'\Gamma'(\Gamma\Gamma')^{-1}\Gamma\Omega h_z + \beta_2\left[F_N(r - re - \mu_\pi) + \gamma L\mu_L^i - (A' + B')\Gamma'(\Gamma\Gamma')^{-1}M\right] \\
& - \frac{1}{2}M'(\Gamma\Gamma')^{-1}M + \frac{1}{2}\beta_2^2(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)(A + B).
\end{aligned}
$$

This kind of partial differential equation can be solved using the $Feynman-Kac$ theorem, and so we can find the functional form of $h(z; t)$, which is given by:

$$
h(z; t) = E_t\left[\int_t^T g(\widetilde{z}(s), s)ds\right],
$$

where

$$
\begin{aligned}
d\widetilde{z}(s) &= [\mu_z' - M'(\Gamma\Gamma')^{-1}\Gamma\Omega + \beta_2(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega]'ds + \Omega(\widetilde{z}(s), s)'dW, \\
\widetilde{z}(s) &= z(s), \\
g(\widetilde{z}(t), t) &= \left[F_N(r - re - \mu_\pi) + \gamma L\mu_L^i - (A' + B')\Gamma'(\Gamma\Gamma')^{-1}M\right] - \frac{1}{2}\frac{1}{\beta_2}M'(\Gamma\Gamma')^{-1}M \\
& \quad + \frac{1}{2}\beta_2(A' + B')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)(A + B).
\end{aligned}
$$

Finally, the optimal portfolio is written as follows:

$$\begin{aligned}\theta^*_G &= -\frac{1}{\beta_2}\frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}M - \frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega\cdot\int_t^T \frac{\partial}{\partial z}E_t\left[g(\widetilde{z}(s),s)\right]ds \\ &\quad -\frac{1}{1-e}(\Gamma\Gamma')^{-1}\Gamma\Phi - \frac{1}{F_N(1-e)}\gamma L(\Gamma\Gamma')^{-1}\Gamma\Lambda.\end{aligned}$$

3.3 Second component of the optimal portfolio

Now we are interested in the second component $\theta^*_{G(2)}$ of the optimal portfolio, which is the state variables hedge portfolio. Since the quadratic term $M'(\Gamma\Gamma')^{-1}M$ does not depend on the state variables, this term is deleted. We rearrange the $\theta^*_{G(2)}$ as the following equation.

$$\theta^*_{G(2)} = \frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega\cdot\int_t^T \frac{\partial}{\partial z}E_t\left[Q_1+Q_2\right]ds, \tag{4}$$

where

$$\begin{aligned}Q_1 &= [F_N(r-re-\mu_\pi)] - F_N\Phi\Gamma'(\Gamma\Gamma')^{-1}M + \frac{1}{2}\beta_2F_N^2\sigma_\pi^2, \\ Q_2 &= \gamma L\mu_L^i - \gamma L\Lambda\Gamma'(\Gamma\Gamma')^{-1}M + \frac{1}{2}\beta_2[-2\gamma LF_N\sigma_L\sigma_\pi + \gamma^2L^2\sigma_L^2].\end{aligned}$$

Accordingly, the derivative of the expected value in Eq. (4) can be written as follows:

$$\begin{bmatrix}\frac{\partial}{\partial r(t)}E_t[Q_1+Q_2] \\ \frac{\partial}{\partial L(t)}E_t[Q_1+Q_2]\end{bmatrix} = \begin{bmatrix}(1-e)F_N(t) \\ \gamma\mu_L^i - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi F_N(t) + \beta_2\gamma^2\sigma_L^2E_t[\widetilde{L}]\end{bmatrix}.$$

The only term we have to compute is the expected value of the modified process of labor incomes, that is to say the modified real contribution.

Now we carry out the necessary computation for the modified process of L. In particular, we need to compute the matrix product

$$[-M'(\Gamma\Gamma')^{-1}\Gamma\Omega + \beta_2(F_N\Phi' + \gamma L\Lambda')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega]'.$$

For simplicity, we assume that $\Gamma'(\Gamma\Gamma')^{-1}\Gamma = I$. Notice that the original problem is in Appendix B. According to what has already been presented in the previous sections, we can write:

$$[-M'(\Gamma\Gamma')^{-1}\Gamma\Omega]' = \begin{bmatrix} w_1 \\ Lw_2\end{bmatrix},$$

where w_1 and w_2 are given by

$$\begin{aligned}w_1 &\equiv \sigma_r\cdot\lambda_r, \\ w_2 &\equiv -\frac{2\sigma_{L,m}\sigma_{S,r}\lambda_r}{\sigma_{S,m}} - \sigma_{L,m}\lambda_m + \sigma_{L,r}\lambda_r.\end{aligned}$$

Thus, the modified differential of the state variables $\widetilde{z}_s$ can be written as

$$\begin{bmatrix} d\widetilde{r} \\ \frac{d\widetilde{L}}{\widetilde{L}} \end{bmatrix} = \begin{bmatrix} a(b-\widetilde{r})-w_1 \\ \mu_L^i - w_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_r & 0 & 0 \\ \sigma_{L,r} & \sigma_{L,m} & \sigma_L \end{bmatrix} \begin{bmatrix} dW^r \\ dW^m \\ dW^L \end{bmatrix}.$$

In particular, for $s \geq t$, the solution of the interest rate process is

$$\widetilde{r}(s) = \widetilde{r}(t)e^{a(t-s)} + \frac{ab-w_1}{a}(1-e^{a(t-s)}) + \sigma_r e^{-as}\int_t^s e^{a\tau} dW^r(\tau).$$

The solution of the labor incomes process is

$$\begin{aligned} \widetilde{L}(s) \quad &= \quad \widetilde{L}(t)\exp[(\mu_L^i - w_2 - \frac{1}{2}\sigma_{L,r}^2 - \frac{1}{2}\sigma_{L,m}^2 - \frac{1}{2}\sigma_L^2)(s-t) \\ &\quad +\sigma_{L,r}(W^r(s)-W^r(t)) + \sigma_{L,m}(W^m(s)-W^m(t)) + \sigma_L(W^L(s)-W^L(t))]. \end{aligned}$$

Then, according to the boundary equation ($\widetilde{z}(s) = z(s)$) we can obtain the expected value:

$$E_t[\widetilde{L}(s)] = L(t)e^{R(s-t)},$$

where

$$\begin{aligned} R(s-t) \quad &= \quad (\mu_L^i - w_2 - \frac{1}{2}\sigma_{L,r}^2 - \frac{1}{2}\sigma_{L,m}^2 - \frac{1}{2}\sigma_L^2)(s-t) \\ &\quad +\sigma_{L,r}(W^r(s)-W^r(t)) + \sigma_{L,m}(W^m(s)-W^m(t)) \\ &\quad +\sigma_L(W^L(s)-W^L(t)). \end{aligned}$$

Thus, the integral defining the second component of the optimal portfolio is given by

$$\int_t^T \frac{\partial}{\partial z} E_t\left[Q_1 + Q_2\right] ds = \begin{bmatrix} (1-e)F_N(t) \\ \gamma\mu_L^i - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi F_N(t) + \beta_2\gamma^2\sigma_L^2 L(t)e^{R(s-t)} \end{bmatrix}.$$

Finally, we rewrite the second optimal portfolio component as

$$\theta_{G(2)}^* = \frac{1}{F_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega \cdot \begin{bmatrix} (1-e)F_N(t) \\ \gamma\mu_L^i - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi F_N(t) + \beta_2\gamma^2\sigma_L^2 L(t)e^{R(s-t)} \end{bmatrix}.$$