Chapter 2

Literature Review

According to the Securities Industry and Financial Markets Association[3], since 1998 the CDO market has experienced an average annual growth rate of 150%. Aggregate global CDO annual issuance totaled $ 157 billion in 2004, $ 272 billion in 2005, $ 552 billion in 2006 and $ 503 billion in 2007[4]. Research firm Celent estimates the size of the CDO global market to close to $2 trillion by the end of 2006[5]. Because of the rising for the issuance of CDOs, how to price CDOs becomes an important issue. Now we introduce several models for pricing CDOs.

2.1. Binomial Expansion Technique (BET)

The BET methodology is based on the concept of Diversity Score (DS) and is an application of the binomial formula from probability theory to a simplified version of the portfolio. Replace the pool of correlated heterogeneous assets into a pool of DS uncorrelated homogeneous assets each with the same notional amount $\tilde{N} = \sum_i N_i / m$, m is the Diversity Score, which makes the first two moments of the loss distribution match those of the original pool. Then we compute the weighted average life (WAL), weighted average coupon (WAC), weighted average recovery rate (WAR) and weighted average rating factor (WARF), hence the tranche spread can be easily and fast calculated by using the binomial distribution, without imputing a large amount of collateral data. However Garcia et al. (2005) conclude that in BET we assumed the correlation was hidden in the DS methodology, it might not be enough to catch all the complexities of the instrument. Then confirm that BET will give lower losses than Gaussian copula under the high correlation assumption. i.e. the BET method would underestimate the expect loss with the high correlation assumption.

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[3] The Securities Industry and Financial Markets Association (SIFMA) is an industry trade group representing securities firms, banks, and asset management companies in the U.S., Europe, and Asia. SIFMA was formed on November 1, 2006, from the merger of The Bond Market Association and the Securities Industry Association.

[4] Resource: http://www.sifma.org/research/pdf/SIFMA_CDOIssuanceData2008.pdf

[5] Resource: Celent http://www.celent.com/PressReleases/20051031/CDOMarket.htm

2.2. Copula Model

From Garcia et al. (2005), it reveals that an important problem of CDO pricing is the default correlation. Li (2000) introduced using the Gaussian copula to capture the "time-until-default". Li (2000) applied the Sklar's theorem to obtain the joint distribution of the survival times as follows:

$$
\begin{aligned}
F(t_1, t_2, \ldots, t_n) &= C\left(F_1(t_1), F_2(t_2), \ldots, F_n(t_n)\right) \\
&= \Phi_n\left(\Phi_1^{-1}\left(F_1(t_1)\right), \Phi_1^{-1}\left(F_2(t_2)\right), \ldots, \Phi_1^{-1}\left(F_n(t_n)\right); \Sigma\right)
\end{aligned}
$$

where the distribution function of the survival time t_i is $F_i(t)$, and $\Phi_n(\cdot\,;\Sigma)$ is a multivariate normal distribution function with dimension n and covariance matrix Σ. Then set asset returns $A_1, A_2, \ldots, A_n$ equal to $\Phi_1^{-1}\left(F_1(t_1)\right), \Phi_1^{-1}\left(F_2(t_2)\right), \ldots,$ $\Phi_1^{-1}\left(F_n(t_n)\right)$, hence can use Monte Carlo simulation to simulate these default times. It simplifies the default correlation between survival times as the correlation matrix of asset returns. After that we can calculate the loss distribution of every tranche. From the equality of premium leg and protection leg we can get the fair spread. We also can use other copulas to price CDOs. The copula model can reveal the default correlations between the asset portfolio and with flexibility, whereas Monte Carlo simulation technique have to be used if the default correlation structure is completely general, it's hard to work when there is a large portfolio size.

2.3. One Factor Copula Model

Because of the disadvantage of copula model we mentioned above, one factor copula model provides another concept that the credit events of different names in reference portfolio are independent conditional on a common factor (the market factor). This method can fast and simply compute the loss functions for different time horizons and avoid time consuming of the Monte Carlo simulation.

The thought of one factor Gaussian copula model was first introduced by O'Kane and Schlögl (2001). This model has become the market standard due to its simplicity. But the essential problem is, under the Large Homogeneous Portfolio assumption the calculation for correlations implied by market prices should be equal among all tranches of the same CDO. Instead, we observe an implied correlation skew. The main explanation for this phenomenon is the lack of tail dependence of the Gaussian copula. Many researchers sought other distributions to bring more tail dependence. Andersen and Sidenius (2004) extended the one factor Gaussian copula with random recovery and random factor loadings. Different form the one factor double Gaussian copula model, the recovery rates are also affected by common factor and idiosyncratic factor, and the factor loadings depend on the common factors. It has been confirmed that use random factor loadings can produce a heavy upper tail in portfolio loss distributions, thus it is differ markedly from the distribution of the one factor double Gaussian copula model (i.e. with lower probability of zero loss and a fat upper tail). It is capable of producing correlation skews similar to those observed in the market.

After that, many authors proposed other different approaches (i.e. use other copulas with more tail dependence), like the Student t-copula in O'kane and Schloegl (2003) and the double t-copula in Hull and White (2004). Burtschell et al. (2005) compared Gaussian factor copula model with several copula models, such as the stochastic correlation extension to Gaussian copula, Student t-copula, double t-copula, Clayton copula and Marshall-Olkin copula. It showed that the results of Student t-copula and Clayton copula models were similar to the Gaussian factor copula model. The Marshall-Olkin copula leads to a dramatic fattening of the tail of the loss distributions. The results for double t-copula and stochastic correlation copula were closer to market quotes than others. The random factor loading introduced in Andersen and Sidenius (2004) leads to a correlation smile to stochastic correlation copula.

Dezhong et al. (2006) extended the Gaussian factor copula and double t-copula to one factor double mixture distribution of t and Gaussian distribution copula model and one factor double t distribution with factional degrees of freedom copula model. In Hull and White (2004), the degrees of freedom of the t distribution are integers. Compared with the double t-copula model, the two models introduced in Dezhong et al. (2006) were more heavy-tailed and do indeed fit the CDX NA IG 5-year index tranches market data better over time.

Unfortunately, one factor double mixture distribution of t and Gaussian distribution copula model and one factor double t distribution with factional degrees of freedom copula model also have the same problem as the double t-copula, that is, the instability of the Student t-distribution under convolution. We have to use a numerical root search procedure to calculate the default thresholds, and this is time consuming.

Without above problems, Kalemanova et al. (2007) presented a modification of the LHP model replacing the Student t-distribution with the Normal Inverse Gaussian (NIG) distribution. The NIG distribution still provides more tail dependence than one factor double Gaussian copula model, and in addition, it is stable under convolution under certain conditions. The one factor double NIG copula model is simple and speeds up computation of the default thresholds. Besides, the model fit for the double NIG copula model is similar to the double t-copula model and gives more flexibility in the modeling of the dependence structure of a reference portfolio. However, similar to the double t-copula, the double NIG copula model just fit the second tranche well, but extremely overprice the most two senior tranches.

Lüscher (2005) extends the one factor double NIG copula model with stochastic factor loadings and compared the pricing result of one factor double Gaussian copula model, one factor double Gaussian copula model with stochastic factor loadings, one factor double NIG copula model and one factor double NIG copula model with stochastic factor loadings. Because the one factor double NIG copula model with stochastic factor loadings can generate lower probability for zero loss and fatter upper tail for loss distribution than other models, the pricing of CDO tranches are more precise than other three models in this paper.

2.4. Normal Inverse Gaussian Distribution

To understand the normal inverse Gaussian distribution, we have to know about the multivariate normal inverse Gaussian (MNIG) distribution. The multivariate normal distribution does not allow for fat tails or skewness to the marginal distributions. Barndorff-Nielsen (1997) presented the MNIG distribution that accommodates heavy tails and skewness at the same time, the multivariate normal inverse Gaussian distribution is obtained as mean-variance mixture of the multivariate normal distribution. The MNIG distribution arises form a multivariate normal density mixed by the Inverse Gaussian (IG) distribution.

2.5. Closed Skew Normal Distribution

Azzalini (1985) first presents the formal version of the skew normal (SN) distribution, which has the properties of normal families and the shape parameter as well. The multivariate skew normal distribution reference was made by Azzalini and Dalla-Valle (1996). The multivariate skew normal (MSN) distribution is a generalization of the normal distribution to model in a natural way, skewness feature in the distribution. This family also has properties similar to the normal distribution. Then Arellano-Valle et al. (2004) derived the skew-generalized normal (SGN) distribution, which contains Azzalini's (1985) skew-normal distribution as a special case. This distribution has an important property, the absolute value of a skew-generalized normal random variable is half-normal. This implies that all the moments of the skew-generalized normal distribution are finite, and its even moments coincide with those of the standard normal distribution. In this article, the SGN model with flexibility and increased skewness and kurtosis ranges were able to captures features of a data that other models missed.

However, Domínguez-Molina et al. (2004) (References [11]) bring up the multivariate skew normal distribution families disobey the following two properties:

- The closure for the joint distribution of independent members of the multivariate skew normal family.
- The closure under linear combinations other than those given by nonsingular matrices.

Thus Domínguez-Molina et al. (2004) (References [11]) expanded the closed skew normal distribution (CSN), which has the convolution property under sums of independent random vectors and the convolution for the joint distribution of independent CSN distributions. The family also has properties similar to the normal distribution. Azzalini's (1985) skew-normal distribution is a special case of it, too. In addition, just as the SN distribution, the CSN distribution also has extra parameters to control the shape.

In order to compare CSN and mixture models, we review the main properties of NIG distributions and the one factor double NIG copula model in Chapter 3, and before that, we first introduce the general approach for pricing CDOs.