Chapter 3

One Factor Double NIG Copula Model for Pricing CDOs

For deriving the one factor double NIG copula model, first we introduce how to calculate the loss distribution and the risk neutral premium of the CDOs, this is in the Section 3.1. Second, for continuing this chapter, we provide brief instructions for copula method and one factor copula model in Section 3.2 and 3.3. Third, in the Section 3.4 the main properties of the NIG distribution are recalled. Finally, the last section provides the one factor double NIG copula model under large homogeneous portfolio (LHP) assumption. That is, assuming it is possible to approximate the real reference portfolio with a portfolio which consists of a large number of equally weighted instruments. These instruments have the same term structure of default probability, recovery rates, and correlations to the common factor.

3.1. The Loss Distribution and Fair CDO Premium

We can use the method of pricing cash CDOs to price the synthetic CDO. Provided the synthetic CDO has the properties of the cash CDO (such as investing the collateral pool to the risk natural assets and the CDS spread payment dates are similar to the premium payment date of the CDO tranches), we can regard it as cash CDO and price it with the same model.

We consider a synthetic CDO with a reference portfolio which consist of credit default swaps only. As long as no credit events occur, the tranche holders (protection seller) would receive premium payments by the CDO issuer (protection buyer). If the default losses of the reference credit portfolio dose not exceed the principal of tranches, the tranche holders still can receive the spread payment on the outstanding notional. Otherwise, they get nothing but compensate for the losses.

The investors are responsible to the losses from K_L to K_U of the reference portfolio. Assume the cumulative loss on the reference portfolio at time t_i is $L_{portfolio}\left(t_i\right)$, then we can write the percentage cumulative loss is

$$L_{(K_L,K_U)} = \frac{\left(\min\left(L_{portfolio}(t), K_U\right) - K_L\right)^+}{K_U - K_L}$$

Given a continuous portfolio loss distribution function $F(x;t)$, the expected percentage of tranche loss can be calculated as:

$$\begin{aligned}
&EL_{(K_L,K_U)}(t) \\
&= \frac{1}{K_U - K_L} \int_{K_L}^{1} \left(\min(x, K_U) - K_L\right) dF(x;t) \\
&= \frac{1}{K_U - K_L} \left(\int_{K_L}^{K_U} (x - K_L) dF(x;t) + \int_{K_U}^{1} (K_U - K_L) dF(x;t) \right) \\
&= \frac{1}{K_U - K_L} \left(\int_{K_L}^{1} (x - K_L) dF(x;t) - \int_{K_U}^{1} (x - K_L) dF(x;t) + \int_{K_U}^{1} (K_U - K_L) dF(x;t) \right) \\
&= \frac{1}{K_U - K_L} \left(\int_{K_L}^{1} (x - K_L) dF(x;t) - \int_{K_U}^{1} (x - K_U - K_L + K_L) dF(x;t) \right) \\
&= \frac{1}{K_U - K_L} \left(\int_{K_L}^{1} (x - K_L) dF(x;t) - \int_{K_U}^{1} (x - K_U) dF(x;t) \right) \qquad (3.1.1)
\end{aligned}$$

Set $t_1 < \ldots < t_n = T$ are the regular spread payment dates (usually quarterly). T denotes the maturity for the synthetic CDO. s is the spread and t_0 is the valuation date (with $t_0 < t_1$).

With above condition, we can compute the premium leg of the tranche, which is the present value of all expected spread payments except the equity tranches:

$$\text{Premium leg} = \sum_{i=1}^{n} \Delta t_i \cdot s \left[1 - EL_{(K_L,K_U)}(t_i)\right] B(t_0, t_i) \qquad (3.1.2)$$

where $\Delta t_i = t_i - t_{i-1}$. The Δt_i's are all equal to a quarter of a year, and $B(t_0, t_i)$ is a discount factor.

The expected percentage loss decreases by time. For the equity tranche, the protection buyer will pay an up-fount fee to the protection seller initially and spread payment quarterly. Further, the premium payments are made in arrears. By the same mean, the premium is paid on the outstanding notional $1 - EL_{(K_L,K_U)}(t_m)$ at time t_m. Once at time t_m the credit event occurs, the protection payment will be made immediately, and the amount is equal to the tranche loss during the previous payment period. So we can calculate the expected value of the discounted default payments.

$$\begin{aligned}
\text{Protection leg} &= \int_{t_0}^{t_n} B(t_0, s)\, dEL_{(K_L,K_U)}(s) \\
&\approx \sum_{i=1}^{n} \left[EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1}) \right] B(t_0, t_i) \qquad (3.1.3)
\end{aligned}$$

The fair price s for the synthetic CDO should make the premium leg and protection leg equal. Thus, from Equation 3.1.2 and 3.1.3 we can get the fair credit spread:

$$s = \frac{\sum_{i=1}^{n}\left[EL_{(K_L,K_U)}(t_i) - EL_{(K_L,K_U)}(t_{i-1})\right]B(t_0,t_i)}{\sum_{i=1}^{n}\Delta t_i\left[1 - EL_{(K_L,K_U)}(t_i)\right]B(t_0,t_i)} \tag{3.1.4}$$

From this equation, as long as finding the expected loss of a tranche, we can easily calculate fair prices for synthetic CDOs. Unfortunately, the roughest thing is to compute the loss function on the reference portfolio, which is needed to calculate the expected tranche loss. So the following section we introduce the NIG distribution for deriving the portfolio loss function and the expected tranche loss.

3.2. Copula Method

As we told before, to price the synthetic CDOs, we have to derive the loss function. The main step of derivation of loss function is to fit the multivariate joint distribution function of reference portfolio. However, if the portfolio is consisted of a large number of instruments, it is hard to derive or compute the joint distribution. Recently, the copula method was applied extensively in financial field. It was first introduced by Abe Sklar in 1959. (see Nelsen R.B. (2005)). Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions and these one-dimensional margins are uniform. Following we introduce some theorems used in this article.

Definition 3.2.1: the definition of copula function

An n-dimensional copula (or n-copula) is a function C with the following properties:

1. $C:[0,1]^n \rightarrow [0,1]$
2. C *is grounded*[6] *and n-increasing*[7].
3. *For every* $u_i,\ \forall i = 1,\ldots,n$ in I *:* $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$

---

[6] $C(\underset{\sim}{u}) = 0$ if at least one coordinate of $\underset{\sim}{u}$ is 0.

[7] The C-volume of every B is larger or equal to zero for all n-boxes B whose vertices lie in Dom C. The n-boxes $B = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$, $\underset{\sim}{a} \le \underset{\sim}{b}$, where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in I^n. See Roger B. Nelsen (2006).

Definition 3.2.2: n-dimensional distribution

An n-dimensional distribution function is a function F *with domain* $\Re^n$ *such that*

1. F *is n-increasing.*
2. $F(\underset{\sim}{t}) = 0$ *for all* $\underset{\sim}{t}$ *in* $\Re^n$ *such that* $t_k = -\infty$ *for at least one* k*, and* $F(\infty, \infty, ..., \infty) = 1$.

Thus F is grounded, and because the Domain of F is $\Re^n$, the one-dimensional margins of F are the functions F_k, $\forall k = 1,...,n$ given by Domain $F_k = \Re$, where $F_k(x) = F(\infty,...,\infty, x, \infty,...,\infty)$ for all x in $\Re$.

From Definition 3.2.1 and 3.2.2, we can link the n-dimensional distribution and n-dimensional copula. If $F_1, F_2, ..., F_n$ are univariate cumulative density functions, and $C(F_1(x_1), F_2(x_2), ..., F_n(x_n))$ is a multivariate cumulative density function with marginal distribution functions $F_1, F_2, ..., F_n$. Therefore we know that the copula function is a function of joint distribution functions. The Sklar's theorem is the statement of this structure.

Theorem 3.2.1: Sklar's theorem in n-dimensions

Let F *be an n-dimensional distribution function with margins* $F_1, F_2, ..., F_n$. *Then there exists an n-copula* C *such that for all* $\underset{\sim}{x}$ *in* $\overline{R}^n$,

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{3.2.1}$$

If $F_1, F_2, ..., F_n$ are all continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran}\, F_1 \times \text{Ran}\, F_2 \times ... \times \text{Ran}\, F_n$. Conversely, if C is and n-copula and $F_1, F_2, ..., F_n$ are distribution functions, then the function F is defined by Equation 3.2.1 is an n-dimensional distribution function with margins $F_1, F_2, ..., F_n$.

Theorem 3.2.2: *Let* $F, C, F_1, F_2, ..., F_n$ *be as in Theorem 3.2.1, and let* $F_1^{-1}, F_2^{-1}, ..., F_n^{-1}$ *be quasi-inverse of* $F_1, F_2, ..., F_n$*, respectively. Then for any* $\underset{\sim}{u}$ *in* I^n

$$C(u_1, u_2, ..., u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_n^{-1}(u_n)) \tag{3.2.2}$$

when $F_1, F_2, ..., F_n$ are continuous, the above result holds for copulas as well and provides a method of constructing copulas form joint distribution functions. Afterward we will introduce how to use the copula method to describe the correlation between the asset returns in the reference portfolio and the CDO pricing procedure.

3.3. One Factor Copula Model

The reference portfolio assets consist of N financial instruments. Assume N is large. We consider making a large homogeneous portfolio (LHP) assumption.

Assume the default times (or the survival time) are exponentially distributed with parameter λ. We use the average portfolio CDS spread and the fixed recovery rate to estimate it. Then the default probability of company *i* before time *t* is $p_i(t) = 1 - e^{-\lambda t}$.

Now we introduce the one factor model. $A_i(t)$, the asset return up to time *t* of the *i*-th issuer in the reference portfolio depending on the common factor $M(t)$, can be written as:

$$A_i(t) = \sqrt{\rho_i} M(t) + \sqrt{1-\rho_i} X_i(t) \tag{3.3.1}$$

where $M(t)$ and $X_i(t),\ i = 1,...,N$ are identically independent distributed random variables, assume they are both normally distributed. Since $A_i(t)$ is a linear function of these two factors. $A_i(t)$ follows a standard normal distribution as well. Because of the equation, conditional on the common factor $M(t)$, these asset returns are independent.

To price a CDO, we have to know the joint distribution of $A = (A_1,...,A_N)$, which is a multivariate normally distributed vector. Hence A has a joint distribution with a $N \times N$ correlation matrix Σ. The joint distribution of A can be presented as follows:

$$\begin{aligned} F\left(C_1(t),...,C_N(t)\right) &= \Pr\left(A_1(t) \le C_1(t),...,A_N(t) \le C_N(t)\right) \\ &= \Phi_N\left(C_1(t),...,C_N(t);\Sigma\right) \end{aligned} \tag{3.3.2}$$

where $F(\cdot)$ is the joint distribution function of A with margins $F_1,...,F_N$.

Set the vector $\mathrm{T} = (\tau_1,...,\tau_N)$ contains the default times for N companies. Assume the vector T has the Gaussian copula structure, which is expressed as $C_\Sigma^{Gaussian}$. By using the Gaussian copula and Sklar's theorem, we can write the joint distribution function of T as follows:

$$\begin{aligned} &P(\tau_1,...,\tau_N) \\ &= C_\Sigma^{Gaussian}\left(p_1(\tau_1),...,p_N(\tau_N)\right) \\ &= \Phi_N\left(\Phi_1^{-1}\left(p_1(\tau_1)\right),...,\Phi_1^{-1}\left(p_N(\tau_N)\right);\Sigma\right) \\ &= \Phi_N\left(C_1(\tau_1),...,C_N(\tau_N);\Sigma\right) \end{aligned} \tag{3.3.3}$$

where $\Phi_k(\cdot\,;\Sigma)$ is a multivariate normal distribution function with dimension *k* and covariance matrix Σ. $U_i,\ i=1,...,N$ is a uniform random variable. $P(\cdot)$ is the joint distribution function of T and $C_i(t)$ is the threshold of issue *i* at time *t*. Then we combine Equation 3.3.2 and 3.3.3 as follow:

$$F\left(C_1(t),...,C_N(t)\right)=\Pr\left(\tau_1\le t,...,\tau_N\le t\right)$$

In other words, the i-th issue defaults at time t if and only if its asset return is less than a threshold $C_i(t)$, and the default times $(\tau_1,...,\tau_N)$ and the asset returns $(A_1,...,A_N)$ are one-to-one mapping. Thus, we can write it with this form:

$$p_i(t)=P\left[A_i(t)\le C_i(t)\right]$$

So the threshold $C_i(t)$ can be computed as:

$$C_i(t)=\Phi_1^{-1}\left(p_i(t)\right) \qquad (3.3.4)$$

Let's sum up above concepts, before defining the critical value as done in Equation 3.3.4, we firstly adopt a one factor structure and secondly assume a Gaussian copula for $(A_1,...,A_N)$. Such a model is called "one factor double Gaussian copula model". Of course we can substitute the Gaussian distribution with other distributions. In the last section, we would recall the one factor model with the NIG distribution.

3.4. Main Properties of the NIG Distribution

The NIG distribution is a mixture of normal and IG distributions. A non-negative random variable Y has an Inverse Gaussian distribution with parameters α and β, both are positive. Its density function is of the form:

$$f_{IG}(y;\alpha,\beta)=\begin{cases}\dfrac{\alpha}{\sqrt{2\pi\beta}}\,y^{-\frac{3}{2}}e^{-\frac{(\alpha-\beta y)^2}{2\beta y}} & \text{if } y>0\\ 0 & \text{if } y\le 0\end{cases}$$

We can write it as $Y\sim IG(\alpha,\beta)$.

A random variable X follows a normal inverse Gaussian distribution with parameters $\alpha,\ \beta,\ \mu$ and δ if:

$$X|Y = y \sim N(\mu + \beta y, y)$$
$$Y \sim IG(\delta\gamma, \gamma^2) \text{ with } \gamma = \sqrt{\alpha^2 - \beta^2}$$

with parameters satisfying $0 \le |\beta| \le \alpha$ and $\delta > 0$. We write the random variable $X \sim NIG(\alpha, \beta, \mu, \delta)$ and denote the density and distribution functions by $f_{NIG}(x; \alpha, \beta, \mu, \delta)$ and $F_{NIG}(x; \alpha, \beta, \mu, \delta)$. Then the density of a random variable $X \sim NIG(\alpha, \beta, \mu, \delta)$ is:

$$f_{NIG}(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) \tag{3.4.1}$$

where $K_1 = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\omega(t + t^{-1})\right)dt$ is the modified Bessel function of the third kind.

The density function depends on these four parameters:

1. $\alpha > 0$ determines the shape (steepness).
2. β with $0 \le |\beta| \le \alpha$ determines the skewness.
3. μ determines the location.
4. $\delta > 0$ is a scaling parameter.

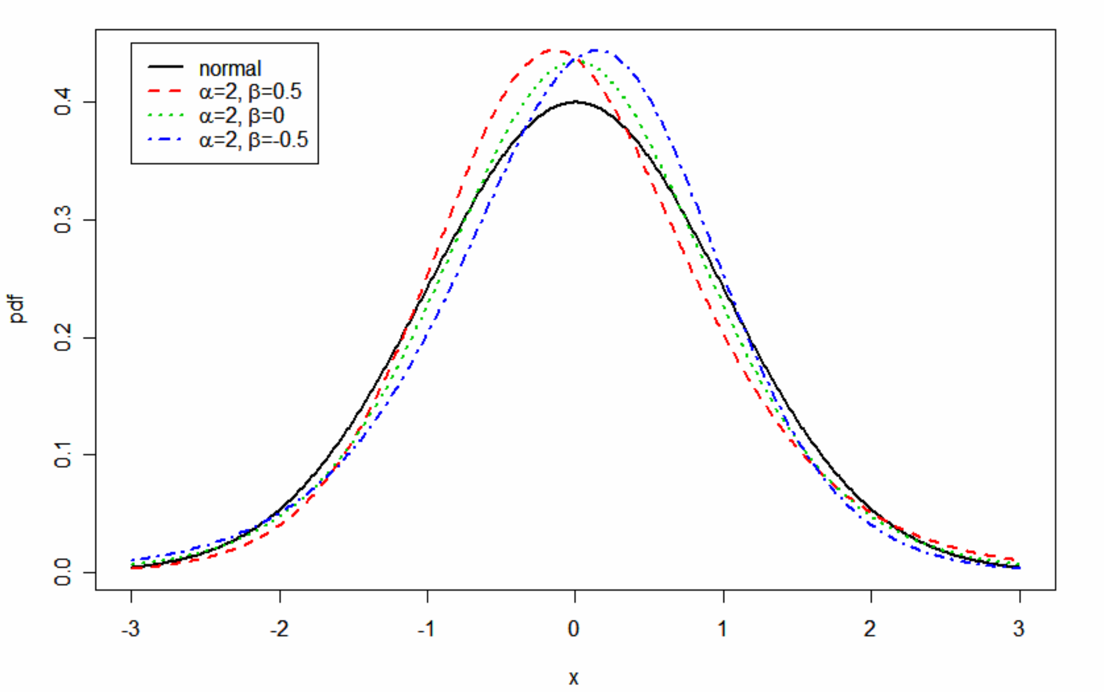

Figure 5: We plot Normal inverse Gaussian distribution with shape and skewness parameters $\alpha = 2,\ \beta = 0.5$, $\alpha = 2,\ \beta = -0.5$ and $\alpha = 2,\ \beta = 0$ in comparison to the normal distribution, where the distribution of Normal inverse Gaussian distribution with zero skewness parameter is symmetric. If the skewness parameter is positive, the distribution is skew to the right; if the skewness parameter is negative, the density is skew to the left.

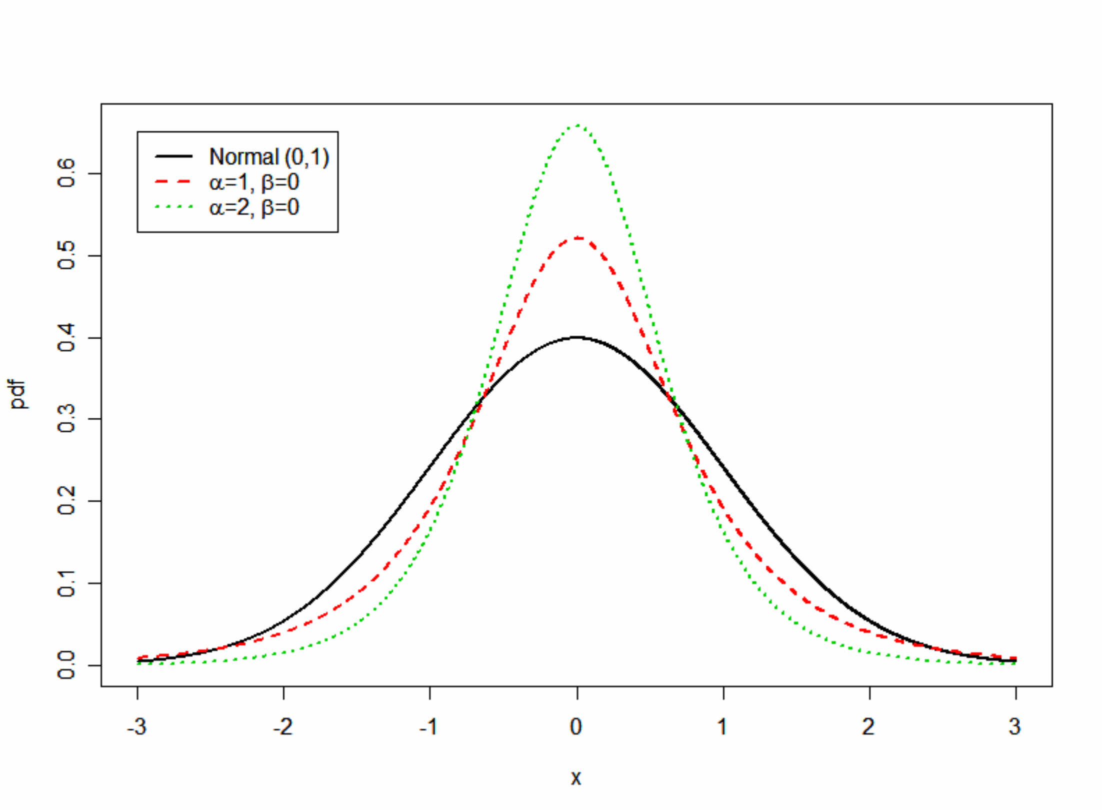

Figure 6: This figure shows that when the shape parameter increases the steepness of the density increases monotonically.

Given the moment generating function of $NIG(\alpha,\beta,\mu,\delta)$:

$$M_{NIG}(x;\alpha,\beta,\mu,\delta)=\frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2-(\beta+t)^2}\right)} \tag{3.4.2}$$

We can simply prove the mean and variance of $X \sim NIG(\alpha,\beta,\mu,\delta)$ are:

$$E(X)=\mu+\delta\frac{\beta}{\gamma} \qquad \text{and} \qquad V(X)=\delta\frac{\alpha^2}{\gamma^3}$$

The important properties of the NIG distribution are the scaling and closure property:

$$X \sim NIG(\alpha,\beta,\mu_1,\delta_1),\ Y \sim NIG(\alpha,\beta,\mu_2,\delta_2)$$
$$\Rightarrow cX \sim NIG\left(\frac{\alpha}{c},\frac{\beta}{c},c\mu_1,c\delta_1\right)$$
$$\Rightarrow X+Y \sim NIG(\alpha,\beta,\mu_1+\mu_2,\delta_1+\delta_2)$$

The calculation of density, cumulative probability value, quantile function and generation of Normal Inverse Gaussian distribution belongs to the package fAsianOptions of R. It can be found at: http://cran.csie.ntu.edu.tw/

3.5. LHP Approximation in the One Factor Double NIG Copula Method

Recall the one factor model we introduced before:

$$A_i(t) = \sqrt{\rho_i}M(t) + \sqrt{1-\rho_i}X_i(t)$$

In order to standardize the distribution of both factors and the asset returns with zero mean and a unit variance, assume $M(t)$ and $X_i(t)$, $i = 1,\ldots,N$ are independent NIG variables with following parameters:

$$M(t) \sim NIG\left(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right)$$

$$X_i(t) \sim NIG\left(\sqrt{\frac{1-\rho}{\rho}}\alpha, \sqrt{\frac{1-\rho}{\rho}}\beta, -\sqrt{\frac{1-\rho}{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \sqrt{\frac{1-\rho}{\rho}}\frac{\gamma^3}{\alpha^2}\right)$$

For non-zero β we get a skewed distribution. Then by the property of scaling and the closure under convolution, the distributions of asset returns are:

$$A_i(t) \sim NIG\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right)$$

To simplify notation we denote

$$F_{NIG}\left(s\alpha, s\beta, -s\frac{\beta\gamma^2}{\alpha^2}, s\frac{\gamma^3}{\alpha^2}\right) \text{ with } F_{NIG(s)}(x)$$

Like the Equation 3.3.5, because of the stability under convolution the default thresholds $C_i(t)$ are simple and relatively fast to be computed:

$$C_i(t) = F^{-1}_{NIG\left(\frac{1}{\sqrt{\rho}}\right)}\left(p_i(t)\right)$$

Because of the LHP assumption, all obligors of the reference portfolio have the same threshold $C(t)$, and the principals and recovery amounts of all obligors in the portfolio are the same. In addition the correlations between assets are all the same. i.e. $\rho_i = \rho$

The following steps are correspondent with those for one factor double Gaussian copula model (see O'Kane and Schlögl (2001)). We calculate the default probability conditional on the common factor. That is, given the common factor $M(t)$, find the probability of asset for issue i falls below the threshold $C(t)$.

$$p_i(t|M) = F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{C(t)-\sqrt{\rho}M(t)}{\sqrt{1-\rho}}\right)$$

For simplicity, we assume the recovery rate is zero. Then the probability of k issuers default is binomial distributed with parameter $\left(N, F_{NIG\left(\sqrt{1-\rho}/\sqrt{\rho}\right)}\left(\left(C(t)-\sqrt{\rho}M(t)\right)/\sqrt{1-\rho}\right)\right)$. To obtain the unconditional percentage portfolio loss, we have to integrate over the common factor $M(t)$.

$$P\left(L(t)=\frac{k}{N}\right)$$
$$=\int_{-\infty}^{\infty}\binom{N}{k}F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{C(t)-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)^k\left(1-F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{C(t)-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)\right)^{N-k}dF_{NIG(1)}(m)$$

We can calculate the cumulative probability of the percentage portfolio loss:

$$G_N(t,x)$$
$$=\sum_{k=0}^{[Nx]}P\left(L(t)=\frac{k}{N}\right)$$
$$=\sum_{k=0}^{[Nx]}\int_{-\infty}^{\infty}\binom{N}{k}F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{C(t)-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)^k\left(1-F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{C(t)-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)\right)^{N-k}dF_{NIG(1)}(m)$$

(3.5.1)

The calculation of the loss distribution in Equation 3.5.1 is computationally intensive for large N. So we can use the large portfolio limit approximation which was proposed by Vasicek (2002) to derive the loss distribution, which is given by:

$$G_\infty(t,x)=1-F_{NIG(1)}\left(\frac{C(t)-\sqrt{1-\rho}F^{-1}_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(x)}{\sqrt{\rho}}\right)$$

Form Kalemanova et al. (2007) we learned how to compute the expected loss by numerical methods and eliminating variables. From the Equation 3.1.1, we calculate the expected loss in the tranche with thresholds $\left(K_L, K_U\right)$:

$$
\begin{aligned}
& EL_{(K_L,K_U)}(t) \\
& = \frac{\sqrt{1-\rho}}{\sqrt{\rho}\left(K_U - K_L\right)} \int_{F^{-1}_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(K_L)}^{F^{-1}_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(K_U)} \left(F_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(\gamma) - K_L \right) f_{NIG(1)}\left(\frac{C(t) - \sqrt{1-\rho}\gamma}{\sqrt{\rho}} \right) d\gamma \\
& \quad + F_{NIG(1)}\left(\frac{C(t) - \sqrt{1-\rho}F^{-1}_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(K_U)}{\sqrt{\rho}} \right)
\end{aligned}
\tag{3.5.2}
$$

where

$$\gamma = F^{-1}_{NIG\left(\sqrt{\frac{1-\rho}{\rho}}\right)}(x)$$

If we assume the asset returns have the same recovery rate *R*, which are not zero. The expected percentage loss of the tranche taking losses form K_L to K_U is given by:

$$EL^R_{(K_L,K_U)}(t) = EL_{\left(\frac{K_L}{1-R},\frac{K_U}{1-R}\right)}(t)$$

The most important advantage of the one factor double NIG copula model is that the threshold $C(t)$ is simple and fast to be computed due to the closure property of the NIG distribution. However, this model still not fit the market price precisely, especially in senior tranches. (We can see the results in Table 3 and 4). Therefore in the next chapter, we present the one factor double CSN copula model and one factor mixture distribution copula model to get a better fit.