Chapter 4

One Factor Double Mixture Distribution Copula Models for Pricing CDOs

4.1. The Introduction of Closed Skew Normal Distribution

As we mentioned before, Domínguez-Molina et al. (2004) (References [11]) expanded the closed skew normal distribution (CSN), which has the properties similar to the normal distribution, and additional parameters to bring the skewness and kurtosis. Azzalini's (1996) skew-normal distribution is also a special case of it. Different from that, the CSN distribution has the convolution property under sums of independent random vectors and the convolution for the joint distribution of independent random vectors. Now we introduce this distribution as follows:

Assume $y = \mu + E_1$ where $E_1 \sim N\left(0, \sigma_1^2\right)$

$z = -\upsilon + DE_1 + E_2$ where $E_2 \sim N\left(0, \sigma_2^2\right)$

Then $\begin{pmatrix} Y \\ Z \end{pmatrix} \sim BN\left(\begin{pmatrix} \mu \\ -\upsilon \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & D\sigma_1^2 \\ D\sigma_1^2 & D^2\sigma_1^2 + \sigma_2^2 \end{pmatrix}\right)$

Thus, we can prove that:

$$f\left(y \mid z > 0\right) = \sigma_1^{-1}\phi_1\left(\frac{y - \mu}{\sigma_1}\right)\Phi_1\left(\frac{-\upsilon + D\left(y - \mu\right)}{\sigma_2}\right) \Big/ \Phi_1\left(\frac{-\upsilon}{\sqrt{D^2\sigma_1^2 + \sigma_2^2}}\right)$$

$$= \frac{\phi_1\left(y; \mu, \sigma_1^2\right)\Phi_1\left(D\left(y - \mu\right); \upsilon, \sigma_2^2\right)}{\Phi_1\left(0; \upsilon, D^2\sigma_1^2 + \sigma_2^2\right)}$$

where $\phi_1\left(\bullet\right)$ and $\Phi_1\left(\bullet\right)$ denote the density function and the distribution function of the standard normal distribution. This equation is the p.d.f. of univariate CSN distribution. We can denote it as $Y \sim CSN_{1,1}\left(\mu, \sigma_1^2, D, \upsilon, \sigma_2^2\right)$.

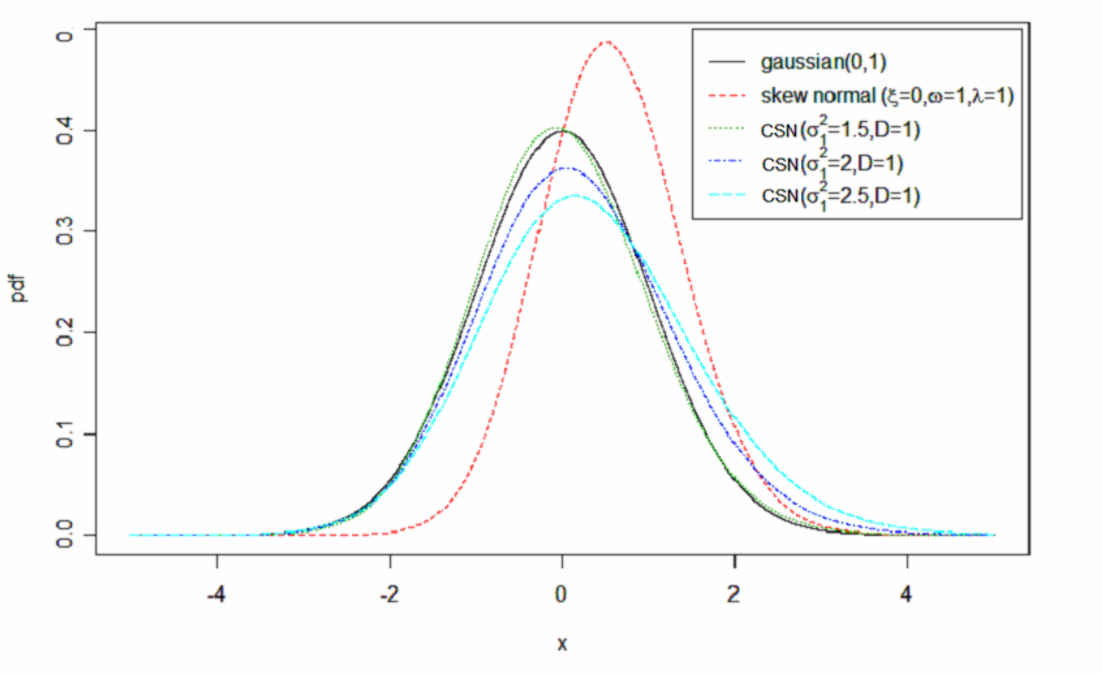

Figure 7: Closed skew normal distribution with different scale parameters $\sigma_1^2 = 1.5,\ 2,$ and 2.5 in comparison to the normal distribution and skew normal with parameters $\xi = 0,\ \omega = 1,$ and $\lambda = 1$. From this plot we can find that the more σ_1^2 increases, the distribution is getting flatter.

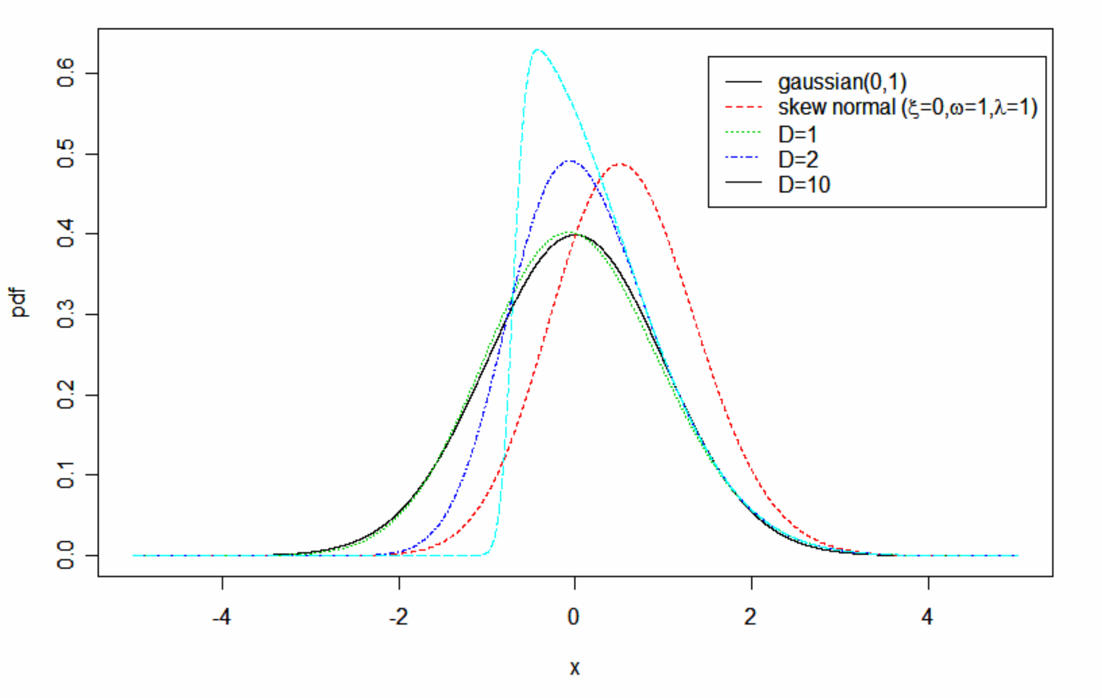

Figure 8: Closed skew normal distribution with different shape parameters $D = 1,\ 2,$ and 10 in comparison to the normal distribution and skew normal with parameters $\xi = 0,\ \omega = 1,$ and $\lambda = 1$. The more D increases, the distribution is getting sharper.

Generally speaking, the density of the CSN distribution is:

$$f_{p,q}\left(y;\mu,\Sigma,D,\upsilon,\Delta\right)=\frac{\phi_p\left(y;\mu,\Sigma\right)\Phi_q\left(D\left(y-\mu\right);\upsilon,\Delta\right)}{\Phi_q\left(0;\upsilon,\Delta+D\Sigma D'\right)},\ y\in\Re^p$$

where $p\geq 1,\ q\geq 1$, $\mu\in\Re^p,\ \Sigma\left(p\times p\right)>0,\ D\left(q\times p\right),\ \upsilon\in\Re^q,\ \Delta\left(q\times q\right)>0$ and $\phi_p\left(\cdot\ ;\eta,\Psi\right),\ \Phi_p\left(\cdot\ ;\eta,\Psi\right)$ are the p.d.f and c.d.f of a *p*-dimensional normal distribution with mean $\eta\in\Re^p$ and Ψ is a $p\times p$ covariance matrix .

We denote a *p*-dimensional random vector which is CSN distribution with parameters $\mu,\ \Sigma,\ D,\ \upsilon,\ \Delta$ as $Y\sim CSN_{p,q}\left(\mu,\Sigma,D,\upsilon,\Delta\right)$.

The difference between CSN distribution and skew normal distribution are: parameters υ and Δ, which allow the possibility of closure properties for the conditional and marginal densities (these were presented in Domínguez-Molina et al. (2004). References [12]); the inclusion of $\Phi_q\left(\cdot\right)$, for $q\geq 1$, which allows the possibility of closure properties for the sum and the joint distribution of independent CSN random vectors. The parameters $\mu,\ \Sigma$ and D have the same meaning as in the other skew normal families, which are location, scale and skewness parameters.

Lemma 4.1.1

If $Y\sim CSN_{1,1}\left(\mu,\sigma_1^2,D,\upsilon,\sigma_2^2\right),\ Y\in\Re^1$*, then the moment generating function of* Y *is:*

$$M_Y\left(t\right)=\frac{\Phi_1\left(D\sigma_1^2t;\upsilon,D^2\sigma_1^2+\sigma_2^2\right)}{\Phi_1\left(0;\upsilon,D^2\sigma_1^2+\sigma_2^2\right)}e^{\mu t+\frac{\sigma^2t^2}{2}},\qquad t\in\Re^1$$

From the Lemma 4.1.1 we can derive the first and second derivatives of Y:

$$M'_Y(t)\big|_{t=0}=\mu+\frac{\phi_1\left(\frac{-\upsilon}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}\right)}{\Phi_1\left(\frac{-\upsilon}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}\right)}\times\frac{D\sigma_1^2}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}\qquad(4.1.1)$$

$$M''_Y(t)\big|_{t=0}=\mu^2+\sigma_1^2+\frac{\phi_1\left(\frac{-\upsilon}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}\right)\frac{D\sigma_1^2}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}}{\Phi_1\left(\frac{-\upsilon}{\sqrt{D^2\sigma_1^2+\sigma_2^2}}\right)}\left[2\mu+\frac{D\sigma_1^2\upsilon}{D^2\sigma_1^2+\sigma_2^2}\right]\qquad(4.1.2)$$

In order to reduce parameters, we set $\upsilon=0$, therefore we can get the mean and variance of Y:

$$E(Y) = \mu + \frac{2D\sigma_1^2}{\sqrt{2\pi}\sqrt{D^2\sigma_1^2 + \sigma_2^2}}$$

$$Var(Y) = \sigma_1^2 - \frac{4D^2\sigma_1^4}{(2\pi)(D^2\sigma_1^2 + \sigma_2^2)} = \frac{D^2\sigma_1^4(\pi - 2) + \pi\sigma_1^2\sigma_2^2}{\pi(D^2\sigma_1^2 + \sigma_2^2)} > 0$$

Theorem 4.1.1

The CSN distribution is closed under translation, i.e. for an arbitrary constant $b \in R$ and $Y \sim CSN_{1,1}(\mu, \sigma_1^2, D, \upsilon, \sigma_2^2)$, we can prove that

$$Y + b \sim CSN_{1,1}(\mu + b, \sigma_1^2, D, \upsilon, \sigma_2^2)$$

and if c is a real number. Then

$$cY \sim CSN_{1,1}(c\mu, c^2\sigma_1^2, Dc^{-1}, \upsilon, \sigma_2^2)$$

we can see the proof in the Domínguez-Molina et al. (2004) (References [11]).

Corollary 4.1.1

If Y_1, Y_2 are independent distributed random variables, Y_1, $Y_2 \in \Re^1$ and $Y_i \sim CSN_{1,1}(\mu_i, \sigma_{i1}^2, D_i, \upsilon_i, \sigma_{i2}^2)$, $i = 1, 2$, then

$$a_1Y_1 + a_2Y_2 \sim CSN_{1,2}(\mu^\dagger, \Sigma^\dagger, D^\dagger, \upsilon^\dagger, \Delta^\dagger)$$

where $\mu^\dagger = a_1\mu_1 + a_2\mu_2$, $\Sigma^\dagger = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$

$$D^\dagger = \begin{bmatrix} \dfrac{D_1a_1\sigma_{11}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} \\ \dfrac{D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} \end{bmatrix},\ \upsilon = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \end{bmatrix},\ \Delta^\dagger = \begin{bmatrix} \sigma_{12}^2 + \dfrac{D_1^2a_2^2\sigma_{11}^2\sigma_{21}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} & -\dfrac{D_1a_1\sigma_{11}^2 \times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} \\ -\dfrac{D_1a_1\sigma_{11}^2 \times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} & \sigma_{22}^2 + \dfrac{D_2^2a_1^2\sigma_{11}^2\sigma_{21}^2}{a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2} \end{bmatrix}$$

Proof:

$$M_{Y_i}(t) = \frac{\Phi_1(D_i\sigma_{i1}^2t; \upsilon_i, D_i^2\sigma_{i1}^2 + \sigma_{i2}^2)}{\Phi_1(0; \upsilon_i, D_i^2\sigma_{i1}^2 + \sigma_{i2}^2)} e^{\mu_it + \frac{\sigma_{i1}^2t^2}{2}}$$

By Theorem 4.1.1, we write the moment generating function as follows:

$$M_{a_1Y_1 + a_2Y_2}(t) = M_{Y_1}(a_1t) \times M_{Y_2}(a_2t)$$

$$= \frac{\Phi_1(D_1a_1\sigma_{11}^2t; \upsilon_1, D_1^2\sigma_{11}^2 + \sigma_{12}^2)\Phi_1(D_2a_2\sigma_{21}^2t; \upsilon_2, D_2^2\sigma_{21}^2 + \sigma_{22}^2)}{\Phi_1(0; \upsilon_1, D_1^2\sigma_{11}^2 + \sigma_{12}^2)\Phi_1(0; \upsilon_2, D_2^2\sigma_{21}^2 + \sigma_{22}^2)} e^{(a_1\mu_1 + a_2\mu_2)t + \frac{(a_1^2\sigma_{11}^2 + a_2^2\sigma_{21}^2)t^2}{2}}$$

$$
=\frac{\Phi_2\left(\begin{bmatrix} D_1 a_1 \sigma_{11}^2 t \\ D_2 a_2 \sigma_{21}^2 t\end{bmatrix};\begin{bmatrix}\upsilon_1\\ \upsilon_2\end{bmatrix},\begin{bmatrix} D_1^2\sigma_{11}^2+\sigma_{12}^2 & 0 \\ 0 & D_2^2\sigma_{21}^2+\sigma_{22}^2\end{bmatrix}\right)}{\Phi_2\left(\begin{bmatrix}0\\0\end{bmatrix};\begin{bmatrix}\upsilon_1\\ \upsilon_2\end{bmatrix},\begin{bmatrix} D_1^2\sigma_{11}^2+\sigma_{12}^2 & 0 \\ 0 & D_2^2\sigma_{21}^2+\sigma_{22}^2\end{bmatrix}\right)} e^{(a_1\mu_1+a_2\mu_2)t+\frac{\left(a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2\right)t^2}{2}}
$$

$$
=\frac{\Phi_2\left(D^\dagger\Sigma^\dagger t;\upsilon,\Delta^\dagger+D^\dagger\Sigma^\dagger D^{\dagger\prime}\right)}{\Phi_2\left(0;\upsilon,\Delta^\dagger+D^\dagger\Sigma^\dagger D^{\dagger\prime}\right)} e^{t\mu^\dagger+\frac{1}{2}t'\Sigma^\dagger t}
$$

$$
\therefore\ \mu^\dagger = a_1\mu_1+a_2\mu_2
$$

$$
\Sigma^\dagger = a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2
$$

$$
D^\dagger=\begin{bmatrix}\dfrac{D_1a_1\sigma_{11}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\\ \dfrac{D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\end{bmatrix}
$$

$$
\Delta^\dagger=\begin{bmatrix} D_1^2\sigma_{11}^2+\sigma_{12}^2 & 0 \\ 0 & D_2^2\sigma_{21}^2+\sigma_{22}^2\end{bmatrix}-D^\dagger\Sigma^\dagger D^{\dagger\prime}
$$

$$
=\begin{bmatrix} D_1^2\sigma_{11}^2+\sigma_{12}^2 & 0 \\ 0 & D_2^2\sigma_{21}^2+\sigma_{22}^2\end{bmatrix}
$$

$$
-\begin{bmatrix}\dfrac{D_1a_1\sigma_{11}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\\ \dfrac{D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\end{bmatrix} a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2\begin{bmatrix}\dfrac{D_1a_1\sigma_{11}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2} & \dfrac{D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\end{bmatrix}
$$

$$
=\begin{bmatrix} D_1^2\sigma_{11}^2+\sigma_{12}^2 & 0 \\ 0 & D_2^2\sigma_{21}^2+\sigma_{22}^2\end{bmatrix}-\begin{bmatrix}\dfrac{D_1^2a_1^2\sigma_{11}^4}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2} & \dfrac{D_1a_1\sigma_{11}^2\times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\\ \dfrac{D_1a_1\sigma_{11}^2\times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2} & \dfrac{D_2^2a_2^2\sigma_{21}^4}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\end{bmatrix}
$$

$$
=\begin{bmatrix}\sigma_{12}^2+\dfrac{D_1^2a_2^2\sigma_{11}^2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2} & -\dfrac{D_1a_1\sigma_{11}^2\times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\\ -\dfrac{D_1a_1\sigma_{11}^2\times D_2a_2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2} & \sigma_{22}^2+\dfrac{D_2^2a_1^2\sigma_{11}^2\sigma_{21}^2}{a_1^2\sigma_{11}^2+a_2^2\sigma_{21}^2}\end{bmatrix}
$$

Corollary 4.1.2

$Y\sim CSN_{p,q}\left(\mu,\Sigma,D,\upsilon,\Delta\right)$ if and only if, $a'Y\sim CSN_{1,q}\left(\mu_a,\Sigma_a,D_a,\upsilon_a,\Delta_a\right)$ for every $a\neq 0$, p-vector in $\Re^p$, where $\mu_a,\Sigma_a,D_a,\upsilon_a,\Delta_a$ are presented as follows:

$$
\mu_a=a'\mu,\ \Sigma_a=a'\Sigma a,\ D_a=D\Sigma a\Sigma_a^{-1},
$$

$$
\upsilon_a=\upsilon,\ \Delta_a=\Delta+D\Sigma aa'\Sigma D'\Sigma_a^{-1}
$$

The proof is derived in Domínguez-Molina et al. (2004) (References [11]). We can use corollary 4.1.2 derive the marginal density function of the multivariate CSN distribution.

4.2. One Factor Double CSN Copula Model

The steps for deriving the pricing formulas for the LHP assumptions under the one factor double CSN copula model are similar to those were proved in one factor double NIG copula model. From Corollary 4.1.1 we know that as long as the factors M and X_i are CSN distributed, the A_i also follows the CSN distribution.

We first assume $M \sim CSN_{1,1}\left(\mu_1,\sigma_{11}^2,D_1,0,\sigma_{12}^2\right)$, $X_i \sim CSN_{1,1}\left(\mu_2,\sigma_{21}^2,D_2,0,\sigma_{22}^2\right)$. By Theorem 4.1.1 and Corollary 4.1.1 we will get:

$$\sqrt{\rho_i}M \sim CSN_{1,1}\left(\sqrt{\rho_i}\mu_1,\rho_i\sigma_{11}^2,\frac{D_1}{\sqrt{\rho_i}},0,\sigma_{12}^2\right) \tag{4.2.1}$$

$$\sqrt{1-\rho_i}X_i \sim CSN_{1,1}\left(\sqrt{1-\rho_i}\mu_2,(1-\rho_i)\sigma_{21}^2,\frac{D_2}{\sqrt{1-\rho_i}},0,\sigma_{22}^2\right) \tag{4.2.2}$$

$$A_i \sim CSN_{1,2}\left(\mu_{A_i},\Sigma_{A_i},D_{A_i},\upsilon_{A_i},\Delta_{A_i}\right) \tag{4.2.3}$$

where

$$\mu_{A_i} = \sqrt{\rho_i}\mu_1 + \sqrt{1-\rho_i}\mu_2,\ \Sigma_{A_i} = \rho_i\sigma_{11}^2 + \left(1-\rho_i\right)\sigma_{21}^2$$

$$D_{A_i} = \begin{bmatrix} \dfrac{D_1\sqrt{\rho_i}\sigma_{11}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} \\ \dfrac{D_2\sqrt{1-\rho_i}\sigma_{21}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} \end{bmatrix},\ \upsilon_{A_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Delta_{A_i} = \begin{bmatrix} \sigma_{12}^2 + \dfrac{D_1^2\left(1-\rho_i\right)\sigma_{11}^2\sigma_{21}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} & -\dfrac{D_1\sqrt{\rho_i}\sigma_{11}^2\times D_2\sqrt{1-\rho_i}\sigma_{21}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} \\ -\dfrac{D_1\sqrt{\rho_i}\sigma_{11}^2\times D_2\sqrt{1-\rho_i}\sigma_{21}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} & \sigma_{22}^2 + \dfrac{D_2^2\rho_i\sigma_{11}^2\sigma_{21}^2}{\rho_i\sigma_{11}^2+\left(1-\rho_i\right)\sigma_{21}^2} \end{bmatrix}$$

For simplicity we assume the factor M and X_i have the same distribution. Further more, we restrict the parameters in order to standardize the distribution of M and X_i. i.e. M and X_i has zero mean and unit variance. Therefore the mean and variance of $A_i(t)$ also equal to 0 and 1. Besides, considering a homogeneous portfolio of *m* credit instruments, the standardized asset return up to time t of the i-th issuer in the portfolio,

$A_i(t)$, is assumed to be of the form:

$$A_i(t) = \sqrt{\rho}M + \sqrt{1-\rho}X_i \quad 1 \le i \le n, \ -1 \le a \le 1$$

So we can see the distribution of factors and $A_i(t)$ below:

$$M,\ X_i \xrightarrow{iid} CSN_{1,1}\left(\pm\sqrt{\sigma_1^2 - 1}, \sigma_1^2, \pm\frac{\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}}, 0, \sigma_2^2 \right) \quad (4.2.4)$$

where $1 < \sigma_1^2 < \pi/(\pi-2)$. Under our assumption, the density function for M and X_i are independent of σ_2, that is, no matter what the value of σ_2 is, the density of M would not change. We denote these factors as $M \sim CSN_{1,1}(\mu_M, \Sigma_M, D_M, \upsilon_M, \Delta_M)$ and $X_i \sim CSN_{1,1}(\mu_X, \Sigma_X, D_X, \upsilon_X, \Delta_X)$.

Substitute the parameters of M and X_i in Equation 4.2.4 into Equation 4.2.1 ~ 4.2.3, we get the following distributions :

$$\sqrt{\rho}M \sim CSN_{1,1}\left(\pm\sqrt{\rho}\sqrt{\sigma_1^2-1}, \rho\sigma_1^2, \pm\frac{\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sqrt{\rho}\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}}, 0, \sigma_2^2 \right)$$

$$\sqrt{1-\rho}X_i \sim CSN_{1,1}\left(\pm\sqrt{1-\rho}\sqrt{\sigma_1^2-1}, (1-\rho)\sigma_1^2, \pm\frac{\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sqrt{1-\rho}\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}}, 0, \sigma_2^2 \right)$$

$$A_i \sim CSN_{1,2}(\mu_A, \Sigma_A, D_A, \upsilon_A, \Delta_A)$$

where

$$\mu_A = \sqrt{\rho}\mu + \sqrt{1-\rho}\mu = \pm\left(\sqrt{\rho}+\sqrt{1-\rho}\right)\sqrt{\sigma_1^2-1},\ \Sigma_A = \sigma_1^2$$

$$D_A = \begin{cases} \begin{bmatrix} \dfrac{\sqrt{\rho}\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}} \\ \dfrac{\sqrt{1-\rho}\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}} \end{bmatrix}, & \text{if } D \text{ is positive} \\ \begin{bmatrix} -\dfrac{\sqrt{\rho}\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}} \\ -\dfrac{\sqrt{1-\rho}\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}} \end{bmatrix}, & \text{if } D \text{ is negative} \end{cases}$$

$$\upsilon_A = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \Delta_A = \begin{bmatrix} \sigma_2^2 + \dfrac{\pi\left(\sigma_1^2 - 1\right)\sigma_2^2 (1-\rho)}{\pi - (\pi - 2)\sigma_1^2} & -\dfrac{\pi\left(\sigma_1^2 - 1\right)\sigma_2^2 \sqrt{\rho}\sqrt{1-\rho}}{\pi - (\pi - 2)\sigma_1^2} \\ -\dfrac{\pi\left(\sigma_1^2 - 1\right)\sigma_2^2 \sqrt{\rho}\sqrt{1-\rho}}{\pi - (\pi - 2)\sigma_1^2} & \sigma_2^2 + \dfrac{\pi\left(\sigma_1^2 - 1\right)\sigma_2^2 \rho}{\pi - (\pi - 2)\sigma_1^2} \end{bmatrix}$$

From the distribution of asset returns we find that it will be four possible combinations for μ_A and D_A, so we plot these densities of A_i's in Figure 9. If making both of the signs of μ_A and D_A are all positive or negative, the distribution of A_i's are skewed to the right, otherwise they are skewed to the left. After we substitute them into the loss function and use numerical method to derive the fair tranche spreads, we find those distributions that have the same skewness will receive the same results. When we have opposite sign of μ_A and D_A, we can get the minimum absolute error spreads over all tranches. The pricing results are presented in Table 7. In the next chapter we set the μ_A is positive and D_A is negative.

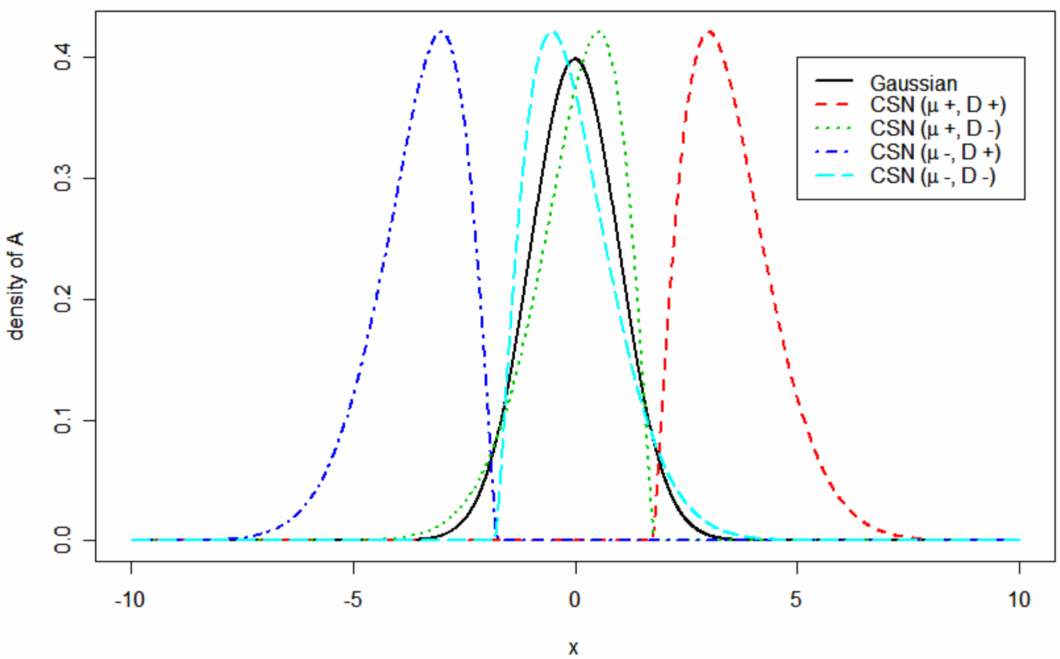

Figure 9: The densities for A_i s, all of them have the parameters $\rho = 0.2,\ \sigma_1^2 = 2.75$. These four combinations have different type of distributions.

Under the CSN copula model the asset returns $(A_1,\ldots,A_N)$ are mapped to the default times $(\tau_1,\ldots,\tau_N)$. $F_{p,q(\mu,\Sigma,D,\upsilon,\Delta)}(\cdot)$ means the c.d.f. of closed skew normal with dimension $p \times q$ and parameter $(\mu,\Sigma,D,\upsilon,\Delta)$. Besides, make the LHP assumption, the probability of the issuer i to default before time t is:

$$P\left(t_i \le t\right) = P\left(A_i\left(t\right) \le C\left(t\right)\right) = F_{1,2(\mu_A,\Sigma_A,D_A,\upsilon_A,\Delta_A)}\left(C\left(t\right)\right)$$

Hence the threshold $C(t)$ can be written as:

$$C\left(t\right) = F^{-1}_{1,2(\mu_A,\Sigma_A,D_A,\upsilon_A,\Delta_A)}\left(p\left(t\right)\right)$$

where $p\left(t\right)$ is the default probability of each of the companies, and assume t is exponentially distributed.

Conditional on the common factor $M\left(t\right)$, the probability of the *i*-th issuer's default at time *t* is:

$$\begin{aligned}
P\left(t_i \le t \mid M\right) &= P\left(A_i(t) \le C(t) \mid M\right) \\
&= P\left(X_i(t) \le \frac{C(t) - \sqrt{\rho}M(t)}{\sqrt{1-\rho}} \mid M\right) \\
&= F_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}\left(\frac{C(t) - \sqrt{\rho}M(t)}{\sqrt{1-\rho}}\right)
\end{aligned} \tag{4.2.5}$$

Assume the reference portfolio has zero recovery rates and contains N instruments. Then the probability of k issuers default is binomial distributed as below:

$$k\left|M \sim Binomial\left(N, F_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}\left(\frac{C(t) - \sqrt{\rho}M(t)}{\sqrt{1-\rho}}\right)\right)\right.$$

The analytic steps for CSN copula factor model are similar to one factor double NIG copula model. Next consider an infinitely large homogeneous portfolio with the asset returns following a one factor double CSN copula model. Then we can derive the portfolio loss distribution:

$$G_\infty\left(t,x\right) = 1 - F_{1,1(\mu_M,\Sigma_M,D_M,\upsilon_M,\Delta_M)}\left(\frac{C(t) - \sqrt{1-\rho}F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(x)}{\sqrt{\rho}}\right) \tag{4.2.6}$$

with $x \in \left[0,\ 1\right]$

Then we can compute the tranche expected loss in the one factor double CSN copula model. We use the Equation 3.1.1 which are introduced in Section 3.1 and rewrite it as:

$$EL_{(K_L,K_U)}(t)=\frac{1}{K_U-K_L}\int_{K_L}^{K_U}(x-K_L)\,dG_\infty(t,x)+\left(1-G_\infty(t,K_U)\right) \tag{4.2.7}$$

Because the integral $\int_{K_L}^{K_U}(x-K_L)\,dG_\infty(t,x)$ has no analytical solution and has to be computed numerically, the inverse distribution function of the CSN distribution is quite computationally intensive. Computing this integral numerically involves the evaluation of the inverse distribution function numerous times. However, it is very easy to avoid this by eliminating it by means of the variable change. Let

$$\gamma=F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(x)$$

In order to compute the integral, we need the density function of the portfolio loss, which is the derivation of Equation 4.2.6:

$$g_\infty(t,x)=\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{f_{1,1(\mu_M,\Sigma_M,D_M,\upsilon_M,\Delta_M)}\left(\dfrac{C(t)-\sqrt{1-\rho}\,F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(x)}{\sqrt{\rho}}\right)}{f_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}\left(F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(x)\right)} \tag{4.2.8}$$

Therefore we can derive the first part of the tranche expected loss mentioned above.

$$\begin{aligned}&\int_{K_L}^{K_U}(x-K_L)\,g_\infty(t,x)\,dx\\&=\int_{F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(K_L)}^{F^{-1}_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(K_U)}\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\left(F_{1,1(\mu_X,\Sigma_X,D_X,\upsilon_X,\Delta_X)}(\gamma)-K_L\right)\\&\quad\times f_{1,1(\mu_M,\Sigma_M,D_M,\upsilon_M,\Delta_M)}\left(\frac{C(t)-\sqrt{1-\rho}\,\gamma}{\sqrt{\rho}}\right)d\gamma\end{aligned} \tag{4.2.9}$$

Because we have not find the R package for calculating CSN distribution yet, we have to compute the cumulative probability and the inverse mapping numerically. This is time consuming. However, we receive a more improved result than one factor double NIG copula model in senior tranches. (The numerical results are presented in Table 3 and 4). In order to get an advanced model to make each of the model generated tranche spreads similar to the market quote; in the next section we introduce the one factor double mixture distribution of NIG and CSN copula model.

4.3. One Factor Double Mixture Distribution of NIG and CSN Distribution Copula Model

In Kalemanove et al. (2007) we already know that the one factor double NIG copula model have excellent pricing result in junior tranches, and use the one factor double CSN copula model we introduced in previous section, we obtain a better fitting price in senior tranches. Therefore we consider a mixture distribution of NIG and CSN distributions, both of them have zero mean and unit variance. The criterion we have presented in Section 3.5 and 4.2. In Equation 4.3.1 we express a mixture distribution W as:

$$W = \begin{cases} U \text{ with probability } p \\ V \text{ with probability } 1-p \end{cases}, \quad 0 \le p \le 1 \tag{4.3.1}$$

where p is the proportion of the NIG and $1-p$ is the proportion of the CSN component distribution in the mixture distribution W. Hence we can write the pdf of the W is:

$$\begin{aligned} f(w) &= p \cdot f_{NIG}\left(w; \alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right) + (1-p) \cdot f_{CSN1,1}\left(w; \sqrt{\sigma_1^2 - 1}, \sigma_1^2, -\frac{\sqrt{\pi(\sigma_1^2-1)}\sigma_2}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}}, 0, \sigma_2^2\right) \\ &= p \cdot \frac{\frac{\gamma^3}{\alpha}\exp\left(\frac{\gamma^4}{\alpha^2} + \beta\left(w + \frac{\beta\gamma^2}{\alpha^2}\right)\right)}{\pi\sqrt{\left(\frac{\gamma^3}{\alpha^2}\right)^2 + \left(w + \frac{\beta\gamma^2}{\alpha^2}\right)^2}} K_1\left(\alpha\sqrt{\left(\frac{\gamma^3}{\alpha^2}\right)^2 + \left(w + \frac{\beta\gamma^2}{\alpha^2}\right)^2}\right) \\ &\quad + (1-p) \cdot 2\phi_1\left(\frac{w - \sqrt{\sigma_1^2 - 1}}{\sigma_1}\right)\left\{1 - \Phi_1\left[\frac{\sqrt{\pi(\sigma_1^2-1)}}{\sigma_1\sqrt{\pi-(\pi-2)\sigma_1^2}}\left(w - \sqrt{\sigma_1^2 - 1}\right)\right]\right\} \end{aligned} \tag{4.3.2}$$

Use the mixture function of NIG and CSN distributions in Equation 4.3.2 as the densities of M and X_i's. Whereas the distribution of M and X_i's are mixture, the distribution of A_i's are not stable under convolution. We need to compute the distribution function of A_i's numerically. Hence we can derive the loss distribution with procedures similar to the double NIG copula model. We refer it as a one factor double mixture distribution of NIG and CSN distribution copula model. To reduce the number of parameters, we set $\beta = 0, p = 0.5$, so we only have to estimate α, σ_1^2, and the correlation ρ. These results are presented in the next chapter.

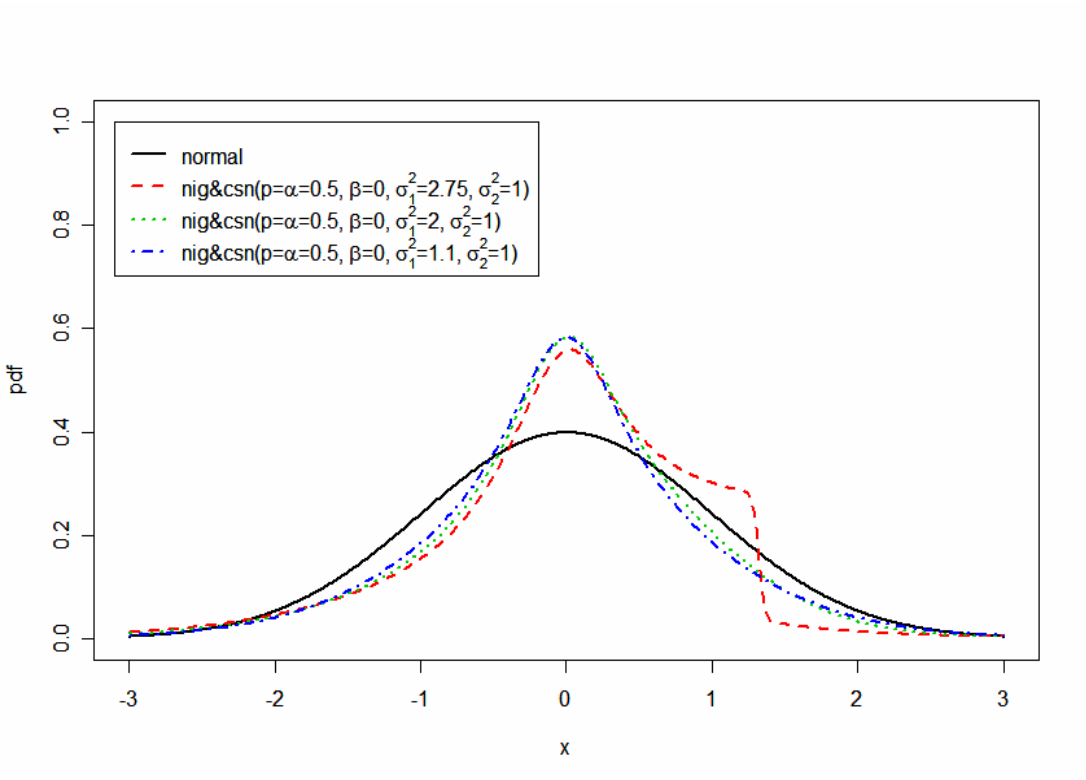

Figure 10: The density function for three different levels of σ_1^2, although the skewness parameter of NIG distribution is zero, the distribution of these combinations also slightly skew to the left because of different σ_1^2's.