1 Introduction

In this thesis, all meromorphic functions are defined in the whole complex plane.

In 1929, Nevanlinna [4, 5] proved that if two non-constant meromorphic functions f and g share five distinct values, then they are identical, and if f and g share four distinct values, then f is a Möbius transformation of g. Therefore, it is natural to ask what happens if f and g share three distinct values? Obviously, we can not expect that f and g have some particular relation. However, if we impose some other conditions, for example, multiplicities, order, deficient values and some others on f and g, then we can get some further relations between f and g. In this thesis, we will use the theory of value distribution to study some well-known results in these aspects, especially, the results proved by H. X. Yi [10, 11, 12] and H. Ueda [8, 9].

In the next section, we review some basic theories of value distribution. Two meromorphic functions sharing three values and their basic properties are studied in section 3. In section 4, we study the relation between multiplicities and uniqueness for two meromorphic functions that share three values. Finally, we study the relation between deficient values and uniqueness for two meromorphic functions that share three values in section 5.

2 Basic Theory of Value Distribution

In this section, we will review some basic theories in value distribution which can be found in [1, 2, 14].

First of all, we define the positive logarithmic function.

Definition 2.1 *For $x \geq 0$, define*

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad 0 \leq x < 1. \end{cases}$$

It is obvious that

$$\log x = \log^+ x - \log^+ \frac{1}{x}$$

hold for all positive numbers x.

Definition 2.2 *Let $f(z)$ be a meromorphic function. For $r > 0$, we define*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

which is the average of the positive logarithm of $|\, f(z)\, |$ on the circle $|\, z\, | = r$.

Definition 2.3 *Let $f(z)$ be a meromorphic function. For $r > 0$, we define*

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ denotes the number of poles of $f(z)$ in the disc $|\, z\, | \leq t$, multiple poles are counted according to their multiplicities. $n(0, f)$ denotes the multiplicity of poles of $f(z)$ at the origin. $N(r, f)$ is called the counting function of poles of $f(z)$.

Definition 2.4 *Let $f(z)$ be a meromorphic function. For $r > 0$, we define*

$$T(r, f) = m(r, f) + N(r, f).$$

$T(r, f)$ is called the characteristic function of $f(z)$.

Definition 2.5

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta} - a)|} d\theta.$$

Definition 2.6

$$N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n\left(t, \frac{1}{f-a}\right) - n\left(0, \frac{1}{f-a}\right)}{t} dt + n\left(0, \frac{1}{f-a}\right) \log r,$$

where $n\left(t, \frac{1}{f-a}\right)$ *denotes the number of zeros of* $f(z)-a$ *in the disc* $|\, z\,| \le t$ *counting multiplicities and* $n\left(0, \frac{1}{f-a}\right)$ *denotes the multiplicity of zeros of* $f(z) - a$ *at the origin.*

Definition 2.7

$$T\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right).$$

From the Poisson-Jensen's formula, we have

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1).$$

Furthermore, the characteristic functions of $f(z)$ and $\frac{1}{f(z)-a}$ are related as follows.

Theorem 2.8 (The first fundamental theorem) *Suppose that* $f(z)$ *is meromorphic in* $|\, z\,| < R$ $(\le \infty)$ *and* a *is any complex number. Then, for* $0 < r < R$, *we have*

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + \log|\, c_\lambda\,| + \varepsilon(a, r),$$

where c_λ *is the first non-zero coefficient of the Laurent expansion of* $\frac{1}{f(z)-a}$ *at the origin, and*

$$|\, \varepsilon(a, r)\,| \le \log^+ |\, a\,| + \log 2.$$

Now, we can state the most remarkable result in the theory of value distribution, namely, the second fundamental theorem. First, we recall the definition of a meromorphic function.

Definition 2.9 *Let $f(z)$ be a meromorphic function. The order of $f(z)$ is defined to be*

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r},$$

and the lower order of $f(z)$ is defined to be

$$\mu = \underline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r}.$$

Theorem 2.10 (The second fundamental theorem) *Suppose $f(z)$ is a non-constant meromorphic function and $a_1, a_2, a_3, \cdots, a_q$ are q (≥ 3) distinct values in the extended complex plane. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) - N_1(r) + S(r,f),$$

where $N_1(r) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right)$, and $S(r,f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}\right) + O(1)$. Moreover, $S(r,f)$ satisfies

(i) $S(r,f) = O(\log r)$ *if $f(z)$ is of finite order,*

(ii) $S(r,f) = O(\log T(r,f) + \log r)$ *if $f(z)$ is of infinite order.*

For $0 < r < R$, let $\overline{n}\left(0, \frac{1}{f-a}\right)$ be the number of distinct zeros of $f(z) - a$ in $|\,z\,| \leq r$, and any of them be counted only once. Let

$$\overline{n}\left(0, \frac{1}{f-a}\right) = \begin{cases} 0 & \text{if } f(0) \neq a, \\ 1 & \text{if } f(0) = a. \end{cases}$$

and

$$\overline{N}\left(r,\frac{1}{f-a}\right)=\int_0^r \frac{\overline{n}\left(t,\frac{1}{f-a}\right)-\overline{n}\left(0,\frac{1}{f-a}\right)}{t}dt+\overline{n}\left(0,\frac{1}{f-a}\right)\log r,$$

which is called the reduced counting function of $f(z)-a$.

Theorem 2.11 (Another form of the second fundamental theorem) *Suppose $f(z)$ is a non-constant meromorphic function and $a_1, a_2, a_3, \cdots, a_q$ are q (≥ 3) distinct values in the extended complex plane. Then*

$$(q-2)T(r,f)<\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f),$$

where $S(r,f)$ is given as in theorem 2.10.

Definition 2.12 *Let $f(z)$ be a non-constant meromorphic function and a be any complex number. The deficiency of a with respect to $f(z)$ is defined by*

$$\delta(a,f)=\varliminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\varlimsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

It is obvious that $0\leq\delta(a,f)\leq 1$.

Definition 2.13 *If $\delta(a,f)>0$, then the complex number a is called a deficient value of $f(z)$. The deficient value is also called exceptional value in the sense of Nevanlinna.*

3 Functions Sharing Three Common Values

In this section, we will study the basic properties of two meromorphic functions which share three values CM.

First, we introduce some symbols and definitions.

Let $f(z)$ and $g(z)$ be meromorphic functions and $a \in \mathbb{C}_\infty$. If every zero of $f(z) - a$ is also a zero of $g(z) - a$ (ignoring multiplicity), then we write

$$f = a \Rightarrow g = a \text{ or } g = a \Leftarrow f = a.$$

If each zero z_0 of $f(z) - a$ with multiplicity $\nu_f(z_0)$ is also a zero of $g(z) - a$ with multiplicity $\nu_g(z_0) \geq \nu_f(z_0)$, then we write

$$f = a \rightarrow g = a \text{ or } g = a \leftarrow f = a.$$

Hence $f = a \Leftrightarrow g = a$ means that $f - a$ and $g - a$ have the same zeros (ignoring multiplicity), $f = \infty \Leftrightarrow g = \infty$ means that $f(z)$ and $g(z)$ have the same poles (ignoring multiplicity), $f = a \rightleftharpoons g = a$ means that $f(z) - a$ and $g(z) - a$ have the same zeros (counting multiplicity), and $f = \infty \rightleftharpoons g = \infty$ means that $f(z)$ and $g(z)$ have the same poles (counting multiplicity).

Definition 3.1 *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions and a be a complex number.*

(i) *If $f = a \rightleftharpoons g = a$, it is said that $f(z)$ and $g(z)$ share a CM;*

(ii) *If $f = a \Leftrightarrow g = a$, it is said that $f(z)$ and $g(z)$ share a IM.*

Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing three distinct values a_1, a_2, a_3 CM, without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, otherwise, we can consider the following two functions

$$F(z) = \frac{f(z) - a_1}{f(z) - a_3} \cdot \frac{a_2 - a_3}{a_2 - a_1} \quad \text{and} \quad G(z) = \frac{g(z) - a_1}{g(z) - a_3} \cdot \frac{a_2 - a_3}{a_2 - a_1},$$

which share 0, 1, ∞ CM.

Theorem 3.2 [5]**.** *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions. If $f(z)$ and $g(z)$ share distinct values a_1, a_2 and a_3 IM, then*

$$T(r, f) < 3T(r, g) + S(r, f),$$

$$T(r, g) < 3T(r, f) + S(r, g).$$

Theorem 3.3 [5]**.** *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing 0, 1, ∞ CM. If $f(z) \not\equiv g(z)$ then*

(i) *there exist two entire functions $\beta(z)$ and $\gamma(z)$ satisfying $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$ such that*

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1} \quad \textit{and} \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1}. \tag{3.1}$$

(ii)

$$T(r, g) = O(T(r, f)), \quad (r \to \infty, r \notin E),$$

$$T(r, e^{\beta}) = O(T(r, f)), \quad (r \to \infty, r \notin E),$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \to \infty, r \notin E).$$

Proof. Since $f(z)$ and $g(z)$ share 0, 1, ∞ CM, then there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$\frac{f(z)}{g(z)} = e^{\alpha(z)} \quad \text{and} \quad \frac{f(z) - 1}{g(z) - 1} = e^{\beta(z)}. \tag{3.2}$$

$f(z) \not\equiv g(z)$ implies that $e^{\alpha(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv 1$, $e^{\beta(z)-\alpha(z)} \not\equiv 1$. And so from (3.2), we get

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\beta(z)-\alpha(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\beta(z)+\alpha(z)} - 1},$$

which gives (3.1) with $\gamma(z) = \beta(z) - \alpha(z)$. By Theorem 3.2, we have

$$T(r, g) < 3T(r, f) + S(r, f). \tag{3.3}$$

This together with (3.2) implies

$$\begin{aligned} T(r, e^{\alpha}) &\leq T(r, f) + T(r, g) + O(1) \\ &< 4T(r, f) + S(r, f) \end{aligned}$$

and

$$\begin{aligned} T(r, e^{\beta}) &\leq T(r, f) + T(r, g) + O(1) \\ &< 4T(r, f) + S(r, f). \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} T(r, e^{r}) &= T(r, e^{\beta-\alpha}) \\ &\leq T(r, e^{\beta}) + T(r, e^{\alpha}) + O(1) \\ &< 8T(r, f) + S(r, f). \end{aligned} \tag{3.5}$$

By (3.3), (3.4) and (3.5), we complete the proof of Theorem 3.3 ❑

Definition 3.4 *Let* $\mathcal{A} = \{f | f$ *is a non-constant meromorphic function satisfy* $\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r, f)\}$. *Members in* $\mathcal{A}$ *are called functions of class* $\mathcal{A}$.

It is clear that all functions in $\mathcal{A}$ *are transcendental meromorphic functions.*

Theorem 3.5 [3]. *Suppose* $f(z)$, $g(z)$, $h(z) \in \mathcal{A}$ *share 1 IM. Then at least two of them are the same.*

Theorem 3.6 [6]. *Let* $g_j(z)$ $(j = 1, 2, \cdots, p)$ *be transcendental entire functions and* a_j $(j = 1, 2, \cdots, p)$ *be non-zero constants. If* $\sum_{j=1}^{p} a_j g_j(z) = 1$, *then* $\sum_{j=1}^{p} \delta(0, g_j) \leq p - 1$.

Theorem 3.7 [6]. *Let* $f_j(z)$ $(j = 1, 2, 3)$ *be meromorphic functions and* $f_1(z)$ *be nonconstant. If*

$$\sum_{j=1}^{3} f_j(z) \equiv 1$$

and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r) \quad (r \in I),$$

where $\lambda < 1$, $T(r) = \max_{1\le j\le 3}\{T(r, f_j)\}$ *and* $I \subset (0, \infty)$ *is of infinite linear measure, then* $f_2(z) \equiv 1$ *or* $f_3(z) \equiv 1$.

Definition 3.8 *Let* $f(z)$ *be a meromorphic function and* a *be any finite value. If* $f(z) - a$ *has no zeros, then* a *is called a Picard exceptional value of* $f(z)$.

Theorem 3.9 [3]. *There are at most two distinct non-constant meromorphic functions sharing three distinct values CM.*

Proof. Suppose Theorem 3.9 is not true. Without loss of generality, we assume that the three shared values are 0, 1, ∞, then there exist three non-constant meromorphic functions $f(z)$, $g(z)$ and $h(z)$ sharing 0, 1, ∞, and $f(z) \not\equiv g(z)$, $f(z) \not\equiv h(z)$, $g(z) \not\equiv h(z)$.

If two of 0, 1, ∞, say 0, ∞, are the Picard exceptional values of f, then by Theorem 3.5, we see that at least two of $f(z)$, $g(z)$, $h(z)$ are identically equal to each other, which contradicts the assumption. Hence at least two of 0, 1, ∞, say 0, ∞, are not the Picard exceptional values of $f(z)$. Hence $f(z)$ has poles and zeros.

Since $f(z)$ and $g(z)$ share 0, 1, ∞ CM, by Theorem 3.3, we have

$$f(z) = \frac{e^{\beta_1(z)} - 1}{e^{\gamma_1(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-\beta_1(z)} - 1}{e^{-\gamma_1(z)} - 1}, \tag{3.6}$$

where $\beta_1(z)$ and $\gamma_1(z)$ are entire functions, and $e^{\beta_1(z)} \not\equiv$ constant, $e^{\gamma_1(z)} \not\equiv$ constant, $e^{\beta_1(z)} \not\equiv e^{\gamma_1(z)}$.

Similarly, from the assumption that $f(z)$ and $h(z)$ share $0, 1, \infty$ CM, we obtain

$$f = \frac{e^{\beta_2(z)} - 1}{e^{\gamma_2(z)} - 1} \quad \text{and} \quad h = \frac{e^{-\beta_2(z)} - 1}{e^{-\gamma_2(z)} - 1}, \tag{3.7}$$

where $\beta_2(z)$ and $\gamma_2(z)$ are entire functions, and $e^{\beta_2(z)} \not\equiv$ constant, $e^{\gamma_2(z)} \not\equiv$ constant, $e^{\beta_2(z)} \not\equiv e^{\gamma_2(z)}$.

Equations (3.6) and (3.7) give

$$f(z)=\frac{e^{\beta_1(z)}-1}{e^{\gamma_1(z)}-1}=\frac{e^{\beta_2(z)}-1}{e^{\gamma_2(z)}-1}. \tag{3.8}$$

If $e^{\beta_1(z)}\equiv e^{\beta_2(z)}$, then from (3.8), we get $e^{\gamma_1(z)}\equiv e^{\gamma_2(z)}$, which together with (3.6) and (3.7) gives that $g(z)\equiv h(z)$. This contradicts the assumption, and so $e^{\beta_1(z)}\not\equiv e^{\beta_2(z)}$. If $e^{\beta_2(z)-\beta_1(z)}\equiv c$, where $c\ (\neq 0,1)$ is a constant, then (3.8) shows that the zeros of $f(z)$ must be the zeros of $e^{\beta_1(z)}-1$ and the zeros of $e^{\beta_2(z)}-1=ce^{\beta_1(z)}-1$. This is impossible because $e^{\beta_1(z)}-1$ and $ce^{\beta_1(z)}-1$ have no common zeros, and hence $e^{\beta_2(z)-\beta_1(z)}\not\equiv$ constant.

Similarly, if $e^{\gamma_1(z)}\equiv e^{\gamma_2(z)}$, then from (3.8), we get $e^{\beta_1(z)}\equiv e^{\beta_2(z)}$, which together with (3.6) and (3.7) gives that $g(z)\equiv h(z)$. This contradicts the assumption, and so $e^{\gamma_1(z)}\not\equiv e^{\gamma_2(z)}$. If $e^{\gamma_2(z)-\gamma_1(z)}\equiv c$, where $c\ (\neq 0,1)$ is a constant, then (3.8) shows that the poles of $f(z)$ must be the zeros of $e^{\gamma_1(z)}-1$ and the zeros of $e^{\gamma_2(z)}-1=ce^{\gamma_1(z)}-1$. This is impossible because $e^{\gamma_1(z)}-1$ and $ce^{\gamma_1(z)}-1$ have no common zeros, and hence $e^{\gamma_2(z)-\gamma_1(z)}\not\equiv$ constant. (3.8) yields

$$e^{\gamma_2(z)}-e^{\beta_2(z)+\gamma_1(z)-\beta_1(z)}+e^{\beta_2(z)-\beta_1(z)}+e^{\gamma_1(z)-\beta_1(z)}-e^{\gamma_2(z)-\beta_1(z)}=1. \tag{3.9}$$

Applying Theorem 3.6 to (3.9) means that at least one of $e^{\beta_2(z)+\gamma_1(z)-\beta_1(z)}$, $e^{\gamma_1(z)-\beta_1(z)}$, $e^{\gamma_2(z)-\beta_1(z)}$ is a constant. We distinguish three cases below.

Case 1. Suppose $e^{\gamma_1(z)-\beta_1(z)}\equiv k_1$, where $k_1\ (\neq 0,1)$ is a constant. Then $e^{\gamma_1(z)}=k_1e^{\beta_1(z)}$. Substituting this into (3.9) gives

$$e^{\gamma_2(z)}-k_1e^{\beta_2(z)}+e^{\beta_2(z)-\beta_1(z)}-e^{\gamma_2(z)-\beta_1(z)}=1-k_1. \tag{3.10}$$

Again, applying Theorem 3.6 to (3.10), we know that $e^{\gamma_2(z)-\beta_1(z)}$ is a constant. Let $e^{\gamma_2(z)-\beta_1(z)}\equiv c_1$, then $e^{\gamma_2(z)-\gamma_1(z)}=\frac{c_1}{k_1}$, a contradiction. Hence $e^{\gamma_1(z)-\beta_1(z)}\not\equiv$ constant.

Case 2. Suppose $e^{\gamma_2(z)-\beta_1(z)}\equiv k_2$, where $k_2\ (\neq 0)$ is a constant. Then $e^{\gamma_2(z)}=k_2e^{\beta_1(z)}$. Substituting this into (3.9) yields

$$k_2e^{\beta_1(z)}-e^{\beta_2(z)+\gamma_1(z)-\beta_1(z)}+e^{\beta_2(z)-\beta_1(z)}+e^{\gamma_1(z)-\beta_1(z)}=1+k_2. \tag{3.11}$$

If $1+k_2=0$, it follows from (3.11) that

$$-e^{2\beta_1(z)-\beta_2(z)-\gamma_1(z)}+e^{-\gamma_1(z)}+e^{-\beta_2(z)}=1. \tag{3.12}$$

Applying Theorem 3.7 to (3.12) gives $e^{2\beta_1(z)-\beta_2(z)-\gamma_1(z)}=-1$ and $e^{-\gamma_1(z)}=-e^{-\beta_2(z)}$. And so we deduce that $e^{2(\gamma_1(z)-\beta_1(z))}=1$, a contradiction.

If $1+k_2\neq 0$, then applying Theorem 3.6 to (3.11), we see that $e^{\beta_2(z)+\gamma_1(z)-\beta_1(z)}=c_2\ (\neq 0)$ is a constant. Substituting $e^{\gamma_1(z)}=c_2e^{\beta_1(z)-\beta_2(z)}$ into (3.11) yields

$$k_2e^{\beta_1(z)}+e^{\beta_2(z)-\beta_1(z)}+c_2e^{-\beta_2(z)}=1+k_2+c_2.$$

Again using Theorem 3.6, we get $1+k_2+c_2=0$, which follows that

$$-k_2e^{2\beta_1(z)-\beta_2(z)}-c_2e^{\beta_1(z)-2\beta_2(z)}=1.$$

From this and Theorem 3.6, we see that both $e^{2\beta_1(z)-\beta_2(z)}$ and $e^{\beta_1(z)-2\beta_2(z)}$ are constants, and hence $e^{\beta_1(z)}$ and $e^{\beta_2(z)}$ are constants. This contradicts the assumption. So $e^{\gamma_2(z)-\beta_1(z)}$ is not a constant.

Case 3. Suppose $e^{\beta_2(z)+\gamma_1(z)-\beta_1(z)}\equiv k_3$, where $k_3\ (\neq 0)$ is a constant. Then $e^{\beta_2(z)-\beta_1(z)}=k_3e^{-\gamma_1(z)}$. Substituting this into (3.9) gives

$$e^{\gamma_2(z)}+k_3e^{-\gamma_1(z)}+e^{\gamma_1(z)-\beta_1(z)}-e^{\gamma_2(z)-\beta_1(z)}=1+k_3.$$

From this and by Theorem 3.6, we get $k_3=-1$. And thus the above equation becomes

$$e^{\beta_1(z)}-e^{\beta_1(z)-\gamma_1(z)-\gamma_2(z)}+e^{\gamma_1(z)-\gamma_2(z)}=1. \tag{3.13}$$

Now applying Theorem 3.7 to (3.13) means $-e^{\beta_1(z)-\gamma_1(z)-\gamma_2(z)}\equiv 1$ and $e^{\beta_1(z)}\equiv -e^{\gamma_1(z)-\gamma_2(z)}$. From this, we obtain $e^{2\gamma_2(z)}\equiv -1$, this is a contradiction. The proof of Theorem 3.9 is completed. ❑

Theorem 3.10 [3]. *Let $f(z)$, $g(z)$, $h(z)$ and $k(z)$ be non-constant meromorphic functions and a_j $(j=1,2,3)$ be three distinct values in the extended complex plane. If $f(z)$, $g(z)$, $h(z)$, $k(z)$ share a_1, a_2 CM and share a_3 IM, then at least two of $f(z)$, $g(z)$, $h(z)$, $k(z)$ are the same.*

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = \infty$, $a_3 = 1$. Suppose that Theorem 3.10 is not true, namely $f(z)$, $g(z)$, $h(z)$, $k(z)$ are different from one another. By Theorem 3.2, we obtain $S(r,f) = S(r,g) = S(r,h) = S(r,k) := S(r)$. For the sake of convenience, we write

$$\overline{N}(r,0) = \overline{N}\left(r,\frac{1}{f}\right), \quad \overline{N}(r,\infty) = \overline{N}(r,f), \quad \overline{N}(r,1) = \overline{N}\left(r,\frac{1}{f-1}\right).$$

If two of $\overline{N}(r,0)$, $\overline{N}(r,\infty)$, $\overline{N}(r,1)$ are $S(r)$, then Theorem 3.5 shows that at least two of $f(z)$, $g(z)$, $h(z)$ are the same. This contradicts the assumption. Hence at least two of $\overline{N}(r,0)$, $\overline{N}(r,\infty)$, $\overline{N}(r,1)$ are not equal to $S(r)$. Without loss of generality, we may assume that

$$\overline{N}(r,0) \neq S(r), \quad \overline{N}(r,1) \neq S(r). \tag{3.14}$$

Since $f(z)$, $g(z)$, $h(z)$, $k(z)$ share $0, \infty$ CM, we have

$$\frac{f(z)}{g(z)} = e^{\alpha}(z), \quad \frac{f(z)}{h(z)} = e^{\beta}(z) \quad \text{and} \quad \frac{f(z)}{k(z)} = e^{\gamma}(z), \tag{3.15}$$

where $\alpha(z)$, $\beta(z)$, $\gamma(z)$ are entire functions. If $\alpha(z) \equiv c$ is a constant, then (3.14) and (3.15) imply that $e^{\alpha}(z) \equiv 1$, and thus $f(z) \equiv g(z)$. This contradicts the assumption, and so $\alpha(z)$ is not a constant. Similarly, $\beta(z)$, $\gamma(z)$, $\alpha(z) - \beta(z)$, $\alpha(z) - \gamma(z)$, $\beta(z) - \gamma(z)$ are not constants. Let

$$\frac{f(z)-1}{g(z)-1} = A, \quad \frac{f(z)-1}{h(z)-1} = B \quad \text{and} \quad \frac{f(z)-1}{k(z)-1} = C, \tag{3.16}$$

where A, B, C are meromorphic functions. From (3.14) and (3.16), we see that none of A, B, C, $\frac{A}{B}$, $\frac{A}{C}$, $\frac{B}{C}$ are constants. Solving $g(z)$ from (3.15) and (3.16) gives

$$g(z) = \frac{A-1}{A-e^{\alpha}(z)},$$
$$g(z) = \frac{\frac{B}{A}-1}{\frac{B}{A}-e^{\beta(z)-\alpha(z)}}e^{\beta(z)-\alpha(z)},$$
$$g(z) = \frac{\frac{C}{A}-1}{\frac{C}{A}-e^{\gamma(z)-\alpha(z)}}e^{\gamma(z)-\alpha(z)}.$$

The above three equations imply

$$e^{\alpha(z)-\beta(z)} = \frac{e^{\alpha(z)}(B-A)+(A-AB)}{B-AB}, \tag{3.17}$$

$$e^{\alpha(z)-\gamma(z)} = \frac{e^{\alpha(z)}(C-A)+(A-AC)}{C-AC}, \tag{3.18}$$

$$e^{\beta(z)-\alpha(z)} = \frac{e^{\beta(z)}(A-B)+(B-AB)}{A-AB}, \tag{3.19}$$

$$e^{\gamma(z)-\alpha(z)} = \frac{e^{\gamma(z)}(A-C)+(C-AC)}{A-AC}. \tag{3.20}$$

Formulas (3.17) and (3.19) show that 1 is an IM shared value of $e^{\alpha}(z)$ and $e^{\beta}(z)$, while (3.18) and (3.20) imply that 1 is an IM shared value of $e^{\alpha}(z)$ and $e^{\gamma}(z)$. Since 0, ∞ are Picard exceptional values of $e^{\alpha}(z)$, $e^{\beta}(z)$ and $e^{\gamma}(z)$, we see that at least two of $e^{\alpha}(z)$, $e^{\beta}(z)$, $e^{\gamma}(z)$ are the same from Theorem 3.5, which contradicts the assumption. So we complete the proof of Theorem 3.10. ❑

4 Multiplicities and Uniqueness

In this section, we study the relation between multiplicities and uniqueness of two meromorphic functions sharing three values, especially, the result proved by H. X. Yi. Before stating it, we need the following fact.

Theorem 4.1 [11]. *Let $h(z)$ be a non-constant entire function and $f(z) = e^{h(z)}$. Then*

(i) $T(r, h) = S(r, f)$,

(ii) $T(r, h') = S(r, f)$.

We use $n_{k)}(r, \frac{1}{f-a})$ to denote the zeros of $f(z) - a$ in $|\, z\,| \leq r$, whose multiplicities are no greater than k and are counted according to their multiplicities. Likewise, we use $n_{(k+1}\left(r, \frac{1}{f-a}\right)$ denote those zeros of $f(z)-a$ in $|\, z\,| \leq r$, whose multiplicities are greater than k and are counted according to their multiplicities. The corresponding counting functions are denoted by $N_{k)}\left(r, \frac{1}{f-a}\right)$ and $N_{(k+1}\left(r, \frac{1}{f-a}\right)$.

Theorem 4.2 [11]. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing 0, 1, ∞ CM. If*

$$N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, f) \neq S(r, f), \tag{4.1}$$

then $f(z) \equiv g(z)$.

Proof. Suppose $f(z) \not\equiv g(z)$. Theorem 3.3 yields

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1},$$

where $\beta(z)$ and $\gamma(z)$ are entire functions satisfying $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$. Using Theorem 3.3 again, we get

$$T(r, e^{\beta}) = O(T(r, f)), \quad (r \to \infty, r \notin E), \tag{4.2}$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \to \infty, r \notin E),$$

and thus

$$T(r, e^{\beta-\gamma}) = O(T(r, f)), \quad (r \to \infty, r \notin E).$$

It is obvious that the multiple zeros of $f(z)$ must satisfy

$$\begin{cases} e^{\beta(z)} - 1 = 0, \\ \beta'(z) = 0. \end{cases} \tag{4.3}$$

If $\beta(z)$ is a constant, then $N_{(2}\left(r, \frac{1}{f}\right) = 0$. If $\beta(z)$ is not a constant, then from (4.3), we get

$$N_{(2}\left(r, \frac{1}{f}\right) \le 2N\left(r, \frac{1}{\beta'}\right) \le 2T(r, \beta') + O(1). \tag{4.4}$$

From Theorem 4.1, we have $T(r, \beta') = S(r, e^{\beta})$. This and (4.2) lead to

$$T(r, \beta') = S(r, f). \tag{4.5}$$

(4.4) and (4.5) give

$$N_{(2}\left(r, \frac{1}{f}\right) = S(r, f). \tag{4.6}$$

Obviously, the multiple poles of $f(z)$ satisfy

$$\begin{cases} e^{\gamma(z)} - 1 = 0, \\ \gamma'(z) = 0. \end{cases}$$

Similar to the above discussion, we can prove

$$N_{(2}(r, f) = S(r, f). \tag{4.7}$$

Since

$$f(z) - 1 = \frac{e^{\gamma(z)}(e^{\beta(z)-\gamma(z)} - 1)}{e^{\gamma(z)} - 1},$$

we see that the multiple zeros of $f(z)-1$ satisfy

$$\begin{cases} e^{\beta(z)-\gamma(z)}-1=0, \\ \beta'(z)-\gamma'(z)=0. \end{cases}$$

By the similar way, we can obtain

$$N_{(2}\left(r,\frac{1}{f-1}\right)=S(r,f).$$

This together with (4.6) and (4.7) yields

$$N_{(2}\left(r,\frac{1}{f}\right)+N_{(2}\left(r,\frac{1}{f-1}\right)+N_{(2}(r,f)=S(r,f),$$

which contradicts (4.1). Hence $f(z)\equiv g(z)$. ❑

Theorem 4.3 [11]. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing 0, 1, CM and ∞ IM. If*

$$N^*(r,\infty)\neq S(r,f), \tag{4.8}$$

where $N^*(r,\infty)$ is defined to be the counting function of multiple poles of $f(z)$ and $g(z)$, and is counted according to the smaller multiplicity, then $f(z)\equiv g(z)$.

Proof. Let

$$\begin{aligned}\beta(z)&=\left(\frac{f'(z)}{f(z)}-\frac{f'(z)}{f(z)-1}\right)-\left(\frac{g'(z)}{g(z)}-\frac{g'(z)}{g(z)-1}\right)\\ &=-\frac{f'(z)}{f(z)(f(z)-1)}+\frac{g'(z)}{g(z)(g(z)-1)}.\end{aligned} \tag{4.9}$$

If $\beta(z)\not\equiv 0$, then $m(r,\beta)=S(r,f)$. Since $f(z)$ and $g(z)$ share 0, 1 CM, we see that the zeros of $f(z)$ and $f(z)-1$ are not the poles of $\beta(z)$, and that the poles of $f(z)$ are not the poles of $\beta(z)$ from (4.9). Hence $\beta(z)$ is an entire function, and thus

$$T(r,\beta)=S(r,f).$$

Suppose z_0 is a pole of $f(z)$ with multiplicity p (≥ 2), as well as a pole of $g(z)$ with multiplicity q (≥ 2). Equation (4.9) shows that z_0 is a zero of $\beta(z)$ with multiplicity greater than or equal to $\min\{p,q\}-1$. Hence we have

$$N^*(r,\infty)\leq 2N\left(r,\frac{1}{\beta}\right)\leq 2T(r,\beta)+O(1)=S(r,f).$$

This contradicts (4.8), so $\beta(z) \equiv 0$. And thus (4.9) gives

$$\frac{f(z)}{f(z)-1} \equiv c \cdot \frac{g(z)}{g(z)-1}, \tag{4.10}$$

where c ($\neq 0$) is a constant. (4.8) and (4.10) yields $c = 1$, and thus $f(z) \equiv g(z)$.
❑

5 Deficient Values and Uniqueness

In this section, we study the relation between deficient values and uniqueness of two meromorphic functions share three values, especially, the result proved by H. Ueda. Before stating it, we need the following fact.

Theorem 5.1 [5]. *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are linearly independent meromorphic functions satisfying the identity*

$$\sum_{j=1}^{n} f_j(z) \equiv 1.$$

Then, for $1 \leq j \leq n$, we have

$$T(r, f_j) \leq \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) - \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r),$$

where D is the Wronskian determinant $W(f_1, f_2, \cdots, f_n)$,

$$S(r) = o(T(r)) \quad (r \to \infty, r \notin E),$$

and

$$T(r) = \max_{1 \leq k \leq n} \{T(r, f_k)\},$$

and $E \subseteq (0, \infty)$ is a set of finite linear measure.

In the following, we state an important theorem on the combinations of entire functions due to Borel.

Theorem 5.2 [5]. *If $f_j(z)$ $(j = 1, 2, \cdots, n)$ and $g_j(z)$ $(j = 1, 2, \cdots, n)$ $(n \geq 2)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

(ii) *The orders of $f_j(z)$ are less than that of $e^{g_h(z) - g_k(z)}$ for $1 \leq j \leq n$, $1 \leq h < k \leq n$,* *Then* $f_j(z) \equiv 0$, $(j = 1, 2, \cdots, n)$.

The above Borel's theorem plays a very important role in the study of uniqueness of meromorphic functions.

Theorem 5.3 [8, 9]. *Let* $f(z)$ *and* $g(z)$ *be non-constant meromorphic functions sharing 0, 1,* ∞ *CM. If*

$$\overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f}\right) + N(r, f)}{T(r, f)} < \frac{1}{2},$$

then $f(z) \equiv g(z)$ *or* $f(z)g(z) \equiv 1$.

Proof. Since $f(z)$ and $g(z)$ sharing 0, 1 CM, we have

$$\frac{f(z)}{g(z)} = e^{\alpha(z)} \quad \text{and} \quad \frac{f(z)-1}{g(z)-1} = e^{\beta(z)}, \tag{5.1}$$

where $\alpha(z)$ and $\beta(z)$ are entire functions.

Case 1. Suppose that $e^{\beta(z)} \equiv c\ (\neq 0)$ is a constant. If $f(z)$ has at least one zero, then $c = 1$, i.e., $f(z) \equiv g(z)$. If $f(z)$ has no zeros and $c \neq 1$, we have $f(z) - cg(z) = 1 - c \neq 0$. Put $f_1(z) = f(z)^{-1}$, $g_1(z) = g(z)^{-1}$, then $f_1(z)$, $g_1(z)$ are entire functions satisfying $g_1(z) = \frac{cf_1(z)}{1-(1-c)f_1(z)}$. Since $g_1(z)$ is an entire function, $1-(1-c)f_1(z) = e^{\gamma(z)}$, where $\gamma(z)$ is entire. Hence $f(z) = f_1(z)^{-1} = \frac{1-c}{1-e^{\gamma(z)}}$. Thus

$$\begin{aligned} N(r, f) = N\left(r, \frac{1}{e^{\gamma}-1}\right) &= (1+o(1))T(r, e^{\gamma}) \\ &= (1+o(1))T(r, f), \quad (r \to \infty, r \notin E). \end{aligned}$$

This is impossible.

Case 2. Suppose that $e^{\alpha(z)-\beta(z)} \equiv c\ (\neq 0)$ is a constant. If $c = 1$, we have $f(z) \equiv g(z)$. If $c \neq 1$, then $f(z) = \frac{-c(e^{\beta(z)}-1)}{c-1}$. Thus

$$\begin{aligned} N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{e^{\beta}-1}\right) &= (1+o(1))T(r, e^{\beta}) \\ &= (1+o(1))T(r, f), \quad (r \to \infty, r \notin E). \end{aligned}$$

This is impossible.

Case 3. Suppose neither $e^{\beta(z)}$ nor $e^{\alpha(z)-\beta(z)}$ are constants. In this case, we have

$$f(z) = \frac{1-e^{\beta(z)}}{1-e^{\beta(z)-\alpha(z)}} \quad \text{and} \quad g(z) = \frac{1-e^{\beta(z)}}{1-e^{\beta(z)-\alpha(z)}} e^{-\alpha(z)}.$$

Now, we use the argument of impossibility of Borel's identity. Put $\varphi_1(z) = f(z)$, $\varphi_2(z) = -f(z)e^{\beta(z)-\alpha(z)}$ and $\varphi_3(z) = e^{\beta(z)}$. Then

$$\varphi_1 + \varphi_2 + \varphi_3 = 1, \quad \varphi_1^{(n)} + \varphi_2^{(n)} + \varphi_3^{(n)} = 0, \quad (n = 1, 2). \tag{5.2}$$

Further put

$$\triangle = \begin{vmatrix} 1 & 1 & 1 \\ \frac{\varphi_1'}{\varphi_1} & \frac{\varphi_2'}{\varphi_2} & \frac{\varphi_3'}{\varphi_3} \\ \frac{\varphi_1''}{\varphi_1} & \frac{\varphi_2''}{\varphi_2} & \frac{\varphi_3''}{\varphi_3} \end{vmatrix}$$

and

$$\triangle' = \begin{vmatrix} \frac{\varphi_2'}{\varphi_2} & \frac{\varphi_3'}{\varphi_3} \\ \frac{\varphi_2''}{\varphi_2} & \frac{\varphi_3''}{\varphi_3} \end{vmatrix}. \tag{5.3}$$

Assume first that $\triangle = 0$. Then by (5.2)

$$0 = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix} = \begin{vmatrix} \varphi_1 & \varphi_2 & 1 \\ \varphi_1' & \varphi_2' & 0 \\ \varphi_1'' & \varphi_2'' & 0 \end{vmatrix} = \begin{vmatrix} \varphi_1' & \varphi_2' \\ \varphi_1'' & \varphi_2'' \end{vmatrix}.$$

This implies $\varphi_2 = C\varphi_1 + D$ (C, D : constants), i.e., $-f(z)e^{\beta(z)-\alpha(z)} = Cf(z) + D$. If $C \neq 0$, we have $f(z) = \frac{-D}{C+e^{\beta(z)-\alpha(z)}}$, so that

$$N(r,f) = (1+o(1))T(r,f), \quad (r \to \infty, r \notin E).$$

This is impossible. Hence C must vanish, i.e., $f = -De^{\alpha(z)-\beta(z)}$. Substituting this into (5.2), we have $-De^{\alpha(z)-\beta(z)} + e^{\beta(z)} = 1 - D$. By Theorem 3.6, $D = 1$ and $e^{\beta(z)} = e^{\alpha(z)-\beta(z)}$. Thus $f(z)g(z) \equiv 1$. Assume $\triangle \not\equiv 0$, then by (5.3) $\varphi_1 = f = \frac{\triangle'}{\triangle}$, we obtain

$$\begin{aligned} m(r,f) &\leq m(r,\triangle') + m(r,\triangle^{-1}) \\ &\leq m(r,\triangle') + m(r,\triangle) + N(r,\triangle) + O(1). \end{aligned} \tag{5.4}$$

Here we estimate $m(r, \triangle')$ and $m(r, \triangle)$. By (5.1)

$$T(r, e^{\beta}) \leq T(r, f) + T(r, g) + O(1)$$

and

$$\begin{aligned} T(r, e^{\beta-\alpha}) &\leq T(r, e^{\beta}) + T(r, e^{-\alpha}) \\ &\leq 2T(r, f) + 2T(r, g) + O(1). \end{aligned}$$

By Theorem 2.10,

$$\begin{aligned} (1 - o(1))T(r, g) &\leq N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N(r, g) \\ &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N(r, f) \\ &\leq (3 + o(1))T(r, f), \quad (r \to \infty, r \notin E). \end{aligned}$$

Hence

$$T(r, \varphi_3) = T(r, e^{\beta}) \leq (4 + o(1))T(r, f), \quad (r \to \infty, r \notin E)$$

and

$$T(r, \varphi_2) \leq T(r, f) + T(r, e^{\beta-\alpha}) \leq (9 + o(1))T(r, f), \quad (r \to \infty, r \notin E).$$

Therefore

$$m(r, \triangle') = O(\log rT(r, f)), m(r, \triangle) = O(\log rT(r, f)), \quad (r \to \infty, r \notin E).$$

Substituting these into (5.4), we have

$$m(r, f) \leq N(r, \triangle) + O(\log rT(r, f)), \quad (r \to \infty, r \notin E) \tag{5.5}$$

and let $F = \frac{f''}{f} - 2(\frac{f'}{f})^2$, then

$$N(r, F) \leq 2N\left(r, \frac{1}{f}\right) + N(r, f). \tag{5.6}$$

Also, a direct computation shows that

$$\begin{aligned} \triangle = & \left[\frac{f''}{f} - 2\left(\frac{f'}{f}\right)^2\right](\beta' - \alpha') \\ & + \left(\frac{f'}{f}\right)[(\beta')^2 - (\alpha')^2 - 2(\beta' - \alpha') - (\beta'' - \alpha'')] \\ & + \beta'(\beta'' - \alpha'') + \beta'(\beta' - \alpha') - (\beta' - \alpha')[\beta'' + (\beta')^2]. \end{aligned}$$

It follows from this and (5.6)

$$N(r, \triangle) \leq 2N\left(r, \frac{1}{f}\right) + N(r, f). \tag{5.7}$$

By (5.5) and (5.7), we have

$$\begin{aligned} T(r, f) &= m(r, f) + N(r, f) \\ &\leq N(r, \triangle) + O(\log rT(r, f)) + N(r, f) \\ &\leq 2N\left(r, \tfrac{1}{f}\right) + N(r, f) + O(\log rT(r, f)) + N(r, f) \\ &= 2\left[N\left(r, \tfrac{1}{f}\right) + N(r, f)\right] + O(\log rT(r, f)), \quad (r \to \infty, r \notin E). \end{aligned}$$

Hence

$$\overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f}\right) + N(r, f)}{T(r, f)} \geq \frac{1}{2}.$$

This is impossible.

This completes the proof of theorem 5.3. ❑

Theorem 5.4 [12]. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing 0, 1, ∞ CM. If*

$$N_{1)}\left(r, \frac{1}{f}\right) + N_{1)}(r, f) < (\lambda + o(1))T(r), \quad (r \in I), \tag{5.8}$$

where $\lambda < \frac{1}{2}$, $T(r) = max\{T(r, f), T(r, g)\}$, and I is a set in $(0, \infty)$ with infinite linear measure, then $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.

Proof. Let I_1 be the set of $r \in I$ such that $T(r, f) < T(r, g)$, and I_2 be the set of $r \in I$ such that $T(r, g) \leq T(r, f)$. Obviously, at least one of I_1 and I_2 is of infinitely linear measure. Without loss of generality, we assume that

$$T(r, g) \leq T(r, f), \quad (r \in I).$$

Therefore

$$T(r) = T(r, f), \quad (r \in I). \tag{5.9}$$

Suppose that $f(z) \not\equiv g(z)$. By Theorem 4.2, we have

$$N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, f) = S(r, f). \tag{5.10}$$

Theorem 3.3 implies

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1}, \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1}, \tag{5.11}$$

where $\beta(z)$ and $\gamma(z)$ are entire functions, and $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$. Using Theorem 3.3 again, we get

$$T(r, e^{\beta}) = O(T(r, f)), \quad (r \notin E). \tag{5.12}$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \notin E). \tag{5.13}$$

Let $f_1(z) = f(z)$, $f_2(z) = e^{\beta(z)}$ and $f_3(z) = -f(z)e^{\gamma(z)}$. Then (5.11) gives

$$\sum_{j=1}^{3} f_j(z) \equiv 1. \tag{5.14}$$

Let $T^*(r) = \max\{T(r, f_j)\}$, $(j = 1, 2, 3)$. From (5.12) and (5.13), we find

$$T^*(r) = O(T(r, f)), \quad (r \notin E). \tag{5.15}$$

If $f_j(z)$ $(j = 1, 2, 3)$ are linearly independent, then it follows from (5.14), (5.15) and Theorem 5.1 that

$$T(r, f_1) < \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + N(r, D) - N(r, f_2) - N(r, f_3) + S(r, f), \tag{5.16}$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}. \tag{5.17}$$

Obviously, we have

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) = 2N\left(r, \frac{1}{f}\right). \tag{5.18}$$

Formulas (5.14) and (5.17) imply that

$$D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}.$$

Hence

$$N(r,D) - N(r,f_2) - N(r,f_3) \leq N(r,f'') - N(r,f) \leq 2N(r,f). \tag{5.19}$$

From (5.10), (5.16), (5.18) and (5.19), we get

$$T(r,f) < 2N_{1)}\left(r,\frac{1}{f}\right) + 2N_{1)}(r,f) + S(r,f).$$

This together with (5.8) and (5.9) leads to

$$T(r) < 2(\lambda + o(1))T(r), \quad (r \in I).$$

Since $\lambda < \frac{1}{2}$, the above inequality can not hold. Hence $f_j(z)$ $(j = 1,2,3)$ are linearly dependent, that is, there exist constants c_1, c_2 and c_3 (at least one of them is not zero) such that $c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z) = 0$, i.e.,

$$c_1 f(z) + c_2 e^{\beta(z)} - c_3 f(z) e^{\gamma(z)} = 0. \tag{5.20}$$

If $c_2 = 0$, then from (5.20) we get $e^{\gamma(z)} = \frac{c_1}{c_2}$. Substituting the into (5.11) gives

$$f(z) = \frac{e^{\beta(z)} - 1}{\frac{c_1}{c_3} - 1},$$

and thus

$$N_{1)}\left(r,\frac{1}{f}\right) = T(r,f) + S(r,f) = (1 + o(1))T(r), \quad (r \in I).$$

This contradicts (5.8), and so $c_2 \neq 0$. (5.20) leads to

$$e^{\beta(z)} = -\frac{c_1}{c_2} f(z) + \frac{c_3}{c_2} f(z) e^{\gamma(z)}. \tag{5.21}$$

Formulas (5.21) and (5.14) imply that

$$\left(1 - \frac{c_1}{c_2}\right) f(z) - \left(1 - \frac{c_3}{c_2}\right) f(z) e^{\gamma(z)} = 1. \tag{5.22}$$

Note that $f(z)$ is not a constant. If $1 - \frac{c_1}{c_2} \neq 0$, then from (5.22), we see that $1 - \frac{c_3}{c_2} \neq 0$ and

$$f(z) = \frac{1}{\left(1 - \frac{c_1}{c_2}\right) - \left(1 - \frac{c_3}{c_2}\right) e^{\gamma(z)}},$$

which leads to

$$N_{1)}(r,f) = T(r,f) + S(r,f) = (1+o(1))T(r), \quad (r \in I).$$

This contradicts (5.8) too. Hence $1 - \frac{c_1}{c_2} = 0$, i.e., $c_1 = c_2$. Therefore, (5.22) shows $1 - \frac{c_3}{c_2} \neq 0$ and

$$f(z) = \frac{c_2}{c_3 - c_2} e^{-\gamma(z)}. \tag{5.23}$$

Substituting (5.23) into (5.21) gives

$$e^{\beta(z)} = \frac{c_2}{c_2 - c_3}\left(e^{-\gamma(z)} - \frac{c_3}{c_2}\right),$$

which implies that $c_3 = 0$ and $e^{\beta(z)} = e^{-\gamma(z)}$. Hence by (5.11), we get $f(z) = -e^{-\gamma(z)}$ and $g(z) = -e^{\gamma(z)}$, and thus $f(z)g(z) \equiv 1$. ❑

Note that Theorem 5.1 is a generalization of Theorem 5.4.

Theorem 5.5 [10]. *Let $f(z)$ and $g(z)$ be non-constant and distinct meromorphic functions sharing 0,1,∞ CM, and $a \neq 0, 1, \infty$. If $\delta(a,f) > \frac{1}{2}$, then a is a Picard exceptional value of f, furthermore, one and only one of the following cases holds:*

(i) $(f-a)(g+a-1) \equiv a(1-a)$, *and* $f = a(1-e^{\phi}), g = (1-a)(1-e^{-\phi})$;

(ii) $f - (1-a)g \equiv a$, *and* $f = \frac{a}{1-e^{\phi}}, g = \frac{a}{(a-1)(1-e^{-\phi})}$;

(iii) $f \equiv ag$, *and* $f = \frac{ae^{\phi}-1}{e^{\phi}-1}, g = \frac{ae^{\phi}-1}{a(e^{\phi}-1)}$,

where ϕ is a non-constant entire function.

***Proof*.** From Theorem 3.3, we deduce

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1}, \tag{5.24}$$

where $\beta(z)$ and $\gamma(z)$ are entire functions, and $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$. Again from Theorem 3.3, we have

$$T(r, e^{\beta}) = O(T(r,f)), \quad (r \notin E), \tag{5.25}$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \notin E). \tag{5.26}$$

(5.24) implies

$$f(z) - 1 = \frac{e^{\gamma(z)}(e^{\beta(z)-\gamma(z)} - 1)}{e^{\gamma(z)} - 1}. \tag{5.27}$$

We distinguish the following four cases.

Case 1. Suppose that $e^{\beta(z)}$, $e^{\gamma(z)}$, $e^{\beta(z)-\gamma(z)}$ are not constants. (5.24) and (5.25) yield

$$N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{e^{\gamma} - 1}\right) - N(r, f) = N\left(r, \frac{1}{e^{\beta} - 1}\right) = T(r, e^{\beta}) + S(r, f). \tag{5.28}$$

Let

$$f_1(z) = \frac{1}{a-1}(f(z) - a)(e^{\gamma(z)} - 1), \quad f_2(z) = -\frac{1}{a-1}e^{\beta(z)} \quad \text{and} \quad f_3(z) = \frac{a}{a-1}e^{\gamma(z)}.$$

It is obvious that $f_j(z)$ $(j = 1, 2, 3)$ are entire functions, and from (5.24), we have

$$\sum_{j=1}^{3} f_j(z) \equiv 1. \tag{5.29}$$

If $f_j(z)$ $(j = 1, 2, 3)$ are linearly dependent, then there exist constants c_j $(j = 1, 2, 3)$ (at least one of them is not zero) such that

$$c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z) = 0. \tag{5.30}$$

If $c_1 = 0$, then (5.30) gives $c_2 f_2(z) + c_3 f_3(z) = 0$, i.e., $-\frac{c_2}{a-1}e^{\beta(z)} + \frac{c_3 a}{a-1}e^{\gamma(z)} = 0$, which implies that $e^{\beta(z)-\gamma(z)}$ is a constant. This contradicts the assumption. Hence $c_1 \neq 0$, and equation (5.30) can be written as

$$f_1(z) = -\frac{c_2}{c_1}f_2(z) - \frac{c_3}{c_1}f_3(z). \tag{5.31}$$

Substituting (5.31) into (5.29) yields $\left(1 - \frac{c_2}{c_1}\right) f_2(z) + \left(1 - \frac{c_3}{c_1}\right) f_3(z) = 1$, i.e.,

$$-\left(1 - \frac{c_2}{c_1}\right) \cdot \frac{1}{a-1} \cdot e^{\beta(z)} + \left(1 - \frac{c_3}{c_1}\right) \cdot \frac{a}{a-1}e^{\gamma(z)} = 1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $f_j(z)$ $(j = 1, 2, 3)$ must be linearly independent. (5.29) and Theorem 5.1 mean that

$$T(r, f_2) < N\left(r, \frac{1}{f_1}\right) + S(r, f),$$

$$T(r, f_3) < N\left(r, \frac{1}{f_1}\right) + S(r, f).$$

Hence we have

$$T(r, e^{\beta}) < N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) + S(r, f), \tag{5.32}$$

$$T(r, e^{\gamma}) < N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) + S(r, f). \tag{5.33}$$

From (5.28), (5.32) and (5.33), we obtain

$$N\left(r, \frac{1}{f}\right) < N\left(r, \frac{1}{f-a}\right) + S(r, f), \tag{5.34}$$

$$N(r, f) < N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{5.35}$$

(5.25), (5.26) and (5.27) give

$$N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) = N\left(r, \frac{1}{e^{\beta-\gamma}-1}\right) = T(r, e^{\beta-\gamma}) + S(r, f). \tag{5.36}$$

Let

$$g_1(z) = -\frac{1}{a}e^{-\gamma(z)}(f(z)-a)(e^{\gamma(z)}-1), \quad g_2(z) = \frac{1}{a}e^{\beta(z)-\gamma(z)} \quad \text{and} \quad g_3(z) = \frac{a-1}{a}e^{-\gamma(z)}.$$

Obviously, $g_j(z)$ $(j = 1, 2, 3)$ are entire functions, and (5.24) implies that

$$\sum_{j=1}^{3} g_j(z) \equiv 1. \tag{5.37}$$

If $g_j(z)$ $(j = 1, 2, 3)$ are linearly dependent, then there exist constants c_j $(j = 1, 2, 3)$ (at least one of them is not zero) such that

$$c_1 g_1(z) + c_2 g_2(z) + c_3 g_3(z) = 0. \tag{5.38}$$

If $c_1 = 0$, then (5.38) gives $c_2 g_2(z) + c_3 g_3(z) = 0$, i.e., $\frac{c_2}{a}e^{\beta(z)-\gamma(z)} + \frac{c_3(a-1)}{a}e^{-\gamma(z)} = 0$, which implies that $e^{\beta(z)}$ is a constant. This contradicts the assumption. Hence $c_1 \neq 0$, and equation (5.38) can be written as

$$g_1(z) = -\frac{c_2}{c_1}g_2(z) - \frac{c_3}{c_1}g_3(z). \tag{5.39}$$

Substituting (5.39) into (5.37) yields $\left(1-\frac{c_2}{c_1}\right)g_2(z)+\left(1-\frac{c_3}{c_1}\right)g_3(z)=1$, i.e.,

$$\left(1-\frac{c_2}{c_1}\right)\cdot\frac{1}{a}\cdot e^{\beta(z)-\gamma(z)}+\left(1-\frac{c_3}{c_1}\right)\cdot\frac{a-1}{a}e^{-\gamma(z)}=1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $g_j(z)$ $(j=1,2,3)$ must be linearly independent. (5.37) and Theorem 5.1 result in

$$T(r,g_2)<N\left(r,\frac{1}{g_1}\right)+S(r,f),$$

i.e.,

$$T(r,e^{\beta-\gamma})<N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)+S(r,f). \tag{5.40}$$

This together with (5.36) leads to

$$N\left(r,\frac{1}{f-1}\right)<N\left(r,\frac{1}{f-a}\right)+S(r,f). \tag{5.41}$$

From (5.34), (5.35), (5.41) and Theorem 2.10 (the second fundamental theorem), we get

$$\begin{aligned}2T(r,f)&<N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)+N(r,f)+N\left(r,\frac{1}{f-a}\right)+S(r,f)\\&<4N\left(r,\frac{1}{f-a}\right)+S(r,f)\\&<4(1-\delta(a,f)+o(1))T(r,f),\quad (r\notin E),\end{aligned} \tag{5.42}$$

which is impossible due to $\delta(a,f)>\frac{1}{2}$.

Case 2. Suppose that $e^{\gamma(z)}\equiv k_1$, where k_1 $(\neq 0,1)$ is a constant. It follows from (5.24) that

$$f(z)-a=\frac{1}{k_1-1}\left\{e^{\beta(z)}-[1+a(k_1-1)]\right\}. \tag{5.43}$$

Noticing that $\delta(a,f)>\frac{1}{2}$, if $1+a(k_1-1)\neq 0$, then

$$\begin{aligned}N\left(r,\frac{1}{f-a}\right)&=N\left(r,\frac{1}{e^{\beta}-[1+a(k_1-1)]}\right)\\&=T(r,e^{\beta})+S(r,f)\\&=T(r,f)+S(r,f).\end{aligned} \tag{5.44}$$

From (5.44), we obtain $\delta(a, f) = 0$, it is a contradiction. Hence $1 + a(k_1 - 1) = 0$, and thus $k_1 = \frac{a-1}{a}$. Substituting $e^{\gamma(z)} = \frac{a-1}{a}$ into (5.24) gives

$$f(z) = a(1 - e^{\beta(z)}), \quad g(z) = (1 - a)(1 - e^{-\beta(z)}).$$

Hence

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

That means (i) holds.

Case 3. Suppose that $e^{\beta(z)} \equiv k_2$, where k_2 ($\neq 0, 1$) is a constant. By (5.24), we get

$$f(z) - a = \frac{(k_2 + a - 1) - ae^{\gamma(z)}}{e^{\gamma(z)} - 1}. \tag{5.45}$$

Since $\delta(a, f) > \frac{1}{2}$, the above equation leads to $k_2 + a - 1 = 0$, i.e., $k_2 = 1 - a$. Substituting $e^{\beta(z)} = 1 - a$ into (5.24) gives

$$f(z) = \frac{a}{1 - e^{\gamma(z)}}, \quad g(z) = \frac{a}{a-1} \cdot \frac{1}{1 - e^{-\gamma(z)}}.$$

Hence $f - (1 - a)g \equiv a$. That is (ii).

Case 4. Suppose that $e^{\beta(z)-\gamma(z)} \equiv k_3$, where k_3 ($\neq 0, 1$) is a constant. (5.24) means that

$$f(z) - a = \frac{(k_3 - a)e^{\gamma(z)} - (1 - a)}{e^{\gamma(z)} - 1}. \tag{5.46}$$

Since $\delta(a, f) > \frac{1}{2}$, the above equation leads to $k_3 - a = 0$, and thus $k_3 = a$. Substituting $e^{\beta(z)-\gamma(z)} = a$ into (5.24) gives

$$f(z) = \frac{ae^{\gamma(z)} - 1}{e^{\gamma(z)} - 1}, \quad g(z) = \frac{e^{-\gamma(z)} - a}{a(e^{-\gamma(z)} - 1)},$$

and thus $f = ag$. Hence we obtain the conclusion (iii) ❑

Theorem 5.6 [10]. *Let $f(z)$ and $g(z)$ be non-constant and distinct entire functions sharing 0, 1 CM, and $a \neq 0, 1, \infty$. If $\delta(a, f) > \frac{1}{3}$, then a and $1 - a$ are the Picard exceptional values of $f(z)$ and $g(z)$, respectively. Furthermore*

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

Proof. From Theorem 3.3, we deduce

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1}, \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1}, \tag{5.47}$$

where $\beta(z)$ and $\gamma(z)$ are entire functions, and $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$. Again from Theorem 3.3, we have

$$T(r, e^{\beta}) = O(T(r, f)), \quad (r \notin E), \tag{5.48}$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \notin E). \tag{5.49}$$

(5.47) implies

$$f(z) - 1 = \frac{e^{\gamma(z)}(e^{\beta(z)-\gamma(z)} - 1)}{e^{\gamma(z)} - 1}. \tag{5.50}$$

If $e^{\beta(z)}$, $e^{\gamma(z)}$, $e^{\beta(z)-\gamma(z)}$ are not constants. (5.47) and (5.48) yield

$$N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{e^{\gamma} - 1}\right) - N(r, f) = N\left(r, \frac{1}{e^{\beta} - 1}\right) = T(r, e^{\beta}) + S(r, f). \tag{5.51}$$

Let

$$f_1(z) = \frac{1}{a-1}(f(z) - a)(e^{\gamma(z)} - 1), \quad f_2(z) = -\frac{1}{a-1}e^{\beta(z)}, \quad f_3(z) = \frac{a}{a-1}e^{\gamma(z)}.$$

It is obvious that $f_j(z)$ $(j = 1, 2, 3)$ are entire functions, and from (5.47), we have

$$\sum_{j=1}^{3} f_j(z) \equiv 1. \tag{5.52}$$

If $f_j(z)$ $(j = 1, 2, 3)$ are linearly dependent, then there exist constants c_j $(j = 1, 2, 3)$ (at least one of them is not zero) such that

$$c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z) = 0. \tag{5.53}$$

If $c_1 = 0$, then (5.53) gives $c_2 f_2(z) + c_3 f_3(z) = 0$, i.e., $-\frac{c_2}{a-1}e^{\beta(z)} + \frac{c_3 a}{a-1}e^{\gamma(z)} = 0$, which implies that $e^{\beta(z)-\gamma(z)}$ is a constant. This contradicts the assumption. Hence $c_1 \neq 0$, and equation (5.53) can be written as

$$f_1(z) = -\frac{c_2}{c_1} f_2(z) - \frac{c_3}{c_1} f_3(z). \tag{5.54}$$

Substituting (5.54) into (5.52) yields $(1-\frac{c_2}{c_1})f_2(z)+(1-\frac{c_3}{c_1})f_3(z)=1$, i.e.,

$$-\left(1-\frac{c_2}{c_1}\right)\cdot\frac{1}{a-1}\cdot e^{\beta(z)}+\left(1-\frac{c_3}{c_1}\right)\cdot\frac{a}{a-1}e^{\gamma(z)}=1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $f_j(z)$ $(j=1,2,3)$ must be linearly independent. (5.52) and Theorem 5.1 mean that

$$T(r,f_2)<N\left(r,\frac{1}{f_1}\right)+S(r,f),$$

$$T(r,f_3)<N\left(r,\frac{1}{f_1}\right)+S(r,f).$$

Hence we have

$$T(r,e^{\beta})<N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)+S(r,f), \tag{5.55}$$

$$T(r,e^{\gamma})<N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)+S(r,f). \tag{5.56}$$

From (5.51), (5.55) and (5.56), we obtain

$$N\left(r,\frac{1}{f}\right)<N\left(r,\frac{1}{f-a}\right)+S(r,f). \tag{5.57}$$

(5.48), (5.49) and (5.50) give

$$N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)=N\left(r,\frac{1}{e^{\beta-\gamma}-1}\right)=T(r,e^{\beta-\gamma})+S(r,f). \tag{5.58}$$

Let

$$g_1(z)=-\frac{1}{a}e^{-\gamma(z)}(f(z)-a)(e^{\gamma(z)}-1),\quad g_2(z)=\frac{1}{a}e^{\beta(z)-\gamma(z)}\quad\text{and}\quad g_3(z)=\frac{a-1}{a}e^{-\gamma(z)}.$$

Obviously, $g_j(z)$ $(j=1,2,3)$ are entire functions, and (5.47) implies that

$$\sum_{j=1}^{3}g_j(z)\equiv 1. \tag{5.59}$$

If $g_j(z)$ $(j=1,2,3)$ are linearly dependent, then there exist constants c_j $(j=1,2,3)$ (at least one of them is not zero) such that

$$c_1g_1(z)+c_2g_2(z)+c_3g_3(z)=0. \tag{5.60}$$

If $c_1 = 0$, then (5.60) gives $c_2 g_2(z) + c_3 g_3(z) = 0$, i.e., $\frac{c_2}{a} e^{\beta(z)-\gamma(z)} + \frac{c_3(a-1)}{a} e^{-\gamma(z)} = 0$, which implies that $e^{\beta(z)}$ is a constant. This contradicts the assumption. Hence $c_1 \neq 0$, and equation (5.60) can be written as

$$g_1(z) = -\frac{c_2}{c_1} g_2(z) - \frac{c_3}{c_1} g_3(z). \tag{5.61}$$

Substituting (5.61) into (5.59) yields $\left(1 - \frac{c_2}{c_1}\right) g_2(z) + \left(1 - \frac{c_3}{c_1}\right) g_3(z) = 1$, i.e.,

$$\left(1 - \frac{c_2}{c_1}\right) \cdot \frac{1}{a} \cdot e^{\beta(z)-\gamma(z)} + \left(1 - \frac{c_3}{c_1}\right) \cdot \frac{a-1}{a} e^{-\gamma(z)} = 1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $g_j(z)$ $(j = 1, 2, 3)$ must be linearly independent. (5.59) and Theorem 5.1 result in $T(r, g_2) < N\left(r, \frac{1}{g_1}\right) + S(r, f)$, i.e.,

$$T(r, e^{\beta-\gamma}) < N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) + S(r, f). \tag{5.62}$$

This together with (5.58) leads to

$$N\left(r, \frac{1}{f-1}\right) < N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{5.63}$$

From (5.57), (5.63) and Theorem 2.10 (the second fundamental theorem), we get

$$\begin{aligned} 2T(r, f) &< N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &< 3N\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &< 3(1 - \delta(a, f) + o(1))T(r, f), \quad (r \notin E), \end{aligned} \tag{5.64}$$

which is impossible due to $\delta(a, f) > \frac{1}{3}$. Hence at least one of $e^{\beta(z)}$, $e^{\gamma(z)}$, $e^{\beta(z)-\gamma(z)}$ is a constant. Since $f(z)$ is a non-constant entire function, we see that $e^{\beta(z)}$, $e^{\beta(z)-\gamma(z)}$ are not constants from (5.47), and so $e^{\gamma(z)}$ is a constant. Suppose that $e^{\gamma(z)} \equiv k_1$, where k_1 $(\neq 0, 1)$ is a constant. It follows from (5.47) that

$$f(z) - a = \frac{1}{k_1 - 1} \left\{ e^{\beta(z)} - [1 + a(k_1 - 1)] \right\}. \tag{5.65}$$

Noticing that $\delta(a, f) > \frac{1}{3}$, if $1 + a(k_1 - 1) \neq 0$, then

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &= N\left(r, \frac{1}{e^{\beta} - [1 + a(k_1 - 1)]}\right) \\ &= T(r, e^{\beta}) + S(r, f) \\ &= T(r, f) + S(r, f). \end{aligned} \tag{5.66}$$

From (5.66), we obtain $\delta(a, f) = 0$, it is a contradiction. Hence $1 + a(k_1 - 1) = 0$, and thus $k_1 = \frac{a-1}{a}$. Substituting $e^{\gamma(z)} = \frac{a-1}{a}$ into (5.47) gives

$$f(z) = a(1 - e^{\beta(z)}) \quad \text{and} \quad g(z) = (1 - a)(1 - e^{-\beta(z)}).$$

Hence

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

The proof of Theorem 5.6 is completed. ❑

Theorem 5.7 [7]. *Let $f(z)$ and $g(z)$ are non-constant entire functions with finite lower order. If $f(z)$ and $g(z)$ share 0 and 1 CM, and if $\delta(0, f) > 0$, $\delta(1, f) > 0$, then $f(z) \equiv g(z)$.*

Theorem 5.8 [10]. *Let $f(z)$ and $g(z)$ be non-constant and distinct entire functions with finite order, and $a \neq 0$, 1, ∞. If $f(z)$ and $g(z)$ share $0, 1$ CM and $\delta(a, f) > 0$, then a and $1 - a$ are the Picard exceptional values of $f(z)$ and $g(z)$, respectively. Furthermore*

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

Proof. From Theorem 3.3, we deduce

$$f(z) = \frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1} \quad \text{and} \quad g(z) = \frac{e^{-\beta(z)} - 1}{e^{-\gamma(z)} - 1}, \tag{5.67}$$

where $\beta(z)$ and $\gamma(z)$ are entire functions, and $e^{\beta(z)} \not\equiv 1$, $e^{\gamma(z)} \not\equiv 1$, $e^{\beta(z)} \not\equiv e^{\gamma(z)}$. Again from Theorem 3.3, we have

$$T(r, e^{\beta}) = O(T(r, f)), \quad (r \notin E), \tag{5.68}$$

$$T(r, e^{\gamma}) = O(T(r, f)), \quad (r \notin E). \tag{5.69}$$

(5.67) implies

$$f(z) - 1 = \frac{e^{\gamma(z)}(e^{\beta(z)-\gamma(z)} - 1)}{e^{\gamma(z)} - 1}. \tag{5.70}$$

If $e^{\beta(z)}, e^{\gamma(z)}, e^{\beta(z)-\gamma(z)}$ are not constants. (5.67) and (5.68) yield

$$N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)=N\left(r,\frac{1}{e^{\beta}-1}\right)=T(r,e^{\beta})+S(r,f). \quad (5.71)$$

Let

$$f_1(z)=\frac{1}{a-1}(f(z)-a)(e^{\gamma(z)}-1), \quad f_2(z)=-\frac{1}{a-1}e^{\beta(z)} \quad \text{and} \quad f_3(z)=\frac{a}{a-1}e^{\gamma(z)}.$$

It is obvious that $f_j(z)$ $(j=1,2,3)$ are entire functions, and from (5.67), we have

$$\sum_{j=1}^{3} f_j(z)\equiv 1. \quad (5.72)$$

If $f_j(z)$ $(j=1,2,3)$ are linearly dependent, then there exist constants c_j $(j=1,2,3)$ (at least one of them is not zero) such that

$$c_1f_1(z)+c_2f_2(z)+c_3f_3(z)=0. \quad (5.73)$$

If $c_1=0$, then (5.73) gives $c_2f_2(z)+c_3f_3(z)=0$, i.e., $-\frac{c_2}{a-1}e^{\beta(z)}+\frac{c_3a}{a-1}e^{\gamma(z)}=0$, which implies that $e^{\beta(z)-\gamma(z)}$ is a constant. This contradicts the assumption. Hence $c_1\neq 0$, and equation (5.73) can be written as

$$f_1(z)=-\frac{c_2}{c_1}f_2(z)-\frac{c_3}{c_1}f_3(z). \quad (5.74)$$

Substituting (5.74) into (5.72) yields $\left(1-\frac{c_2}{c_1}\right)f_2(z)+\left(1-\frac{c_3}{c_1}\right)f_3(z)=1$, i.e.,

$$-\left(1-\frac{c_2}{c_1}\right)\cdot\frac{1}{a-1}\cdot e^{\beta(z)}+\left(1-\frac{c_3}{c_1}\right)\cdot\frac{a}{a-1}e^{\gamma(z)}=1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $f_j(z)$ $(j=1,2,3)$ must be linearly independent. (5.72) and Theorem 5.1 mean that

$$T(r,f_2)<N\left(r,\frac{1}{f_1}\right)+S(r,f),$$

$$T(r,f_3)<N\left(r,\frac{1}{f_1}\right)+S(r,f).$$

Hence we have

$$T(r,e^{\beta})<N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{e^{\gamma}-1}\right)-N(r,f)+S(r,f), \quad (5.75)$$

$$T(r, e^{\gamma}) < N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) + S(r, f). \tag{5.76}$$

From (5.71), (5.75) and (5.76), we obtain

$$N\left(r, \frac{1}{f}\right) < N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{5.77}$$

(5.68), (5.69) and (5.70) give

$$N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) = N\left(r, \frac{1}{e^{\beta-\gamma}-1}\right) = T(r, e^{\beta-\gamma}) + S(r, f). \tag{5.78}$$

Let

$$g_1(z) = -\frac{1}{a}e^{-\gamma(z)}(f(z)-a)(e^{\gamma(z)}-1), \quad g_2(z) = \frac{1}{a}e^{\beta(z)-\gamma(z)} \quad \text{and} \quad g_3(z) = \frac{a-1}{a}e^{-\gamma(z)}.$$

Obviously, $g_j(z)$ $(j = 1, 2, 3)$ are entire functions, and (5.67) implies that

$$\sum_{j=1}^{3} g_j(z) \equiv 1. \tag{5.79}$$

If $g_j(z)$ $(j = 1, 2, 3)$ are linearly dependent, then there exist constants c_j $(j = 1, 2, 3)$ (at least one of them is not zero) such that

$$c_1 g_1(z) + c_2 g_2(z) + c_3 g_3(z) = 0. \tag{5.80}$$

If $c_1 = 0$, then (5.80) gives $c_2 g_2(z) + c_3 g_3(z) = 0$, i.e., $\frac{c_2}{a}e^{\beta(z)-\gamma(z)} + \frac{c_3(a-1)}{a}e^{-\gamma(z)} = 0$, which implies that $e^{\beta(z)}$ is a constant. This contradicts the assumption. Hence $c_1 \neq 0$, and equation (5.80) can be written as

$$g_1(z) = -\frac{c_2}{c_1}g_2(z) - \frac{c_3}{c_1}g_3(z). \tag{5.81}$$

Substituting (5.81) into (5.79) yields $\left(1 - \frac{c_2}{c_1}\right) g_2(z) + \left(1 - \frac{c_3}{c_1}\right) g_3(z) = 1$, i.e.,

$$\left(1 - \frac{c_2}{c_1}\right) \cdot \frac{1}{a} \cdot e^{\beta(z)-\gamma(z)} + \left(1 - \frac{c_3}{c_1}\right) \cdot \frac{a-1}{a}e^{-\gamma(z)} = 1.$$

Applying Theorem 3.6, we can get a contradiction. Therefore $g_j(z)$ $(j = 1, 2, 3)$ must be linearly independent. (5.79) and Theorem 5.1 result in $T(r, g_2) < N\left(r, \frac{1}{g_1}\right) + S(r, f)$, i.e.,

$$T(r, e^{\beta-\gamma}) < N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{e^{\gamma}-1}\right) - N(r, f) + S(r, f). \tag{5.82}$$

This together with (5.78) leads to

$$N\left(r, \frac{1}{f-1}\right) < N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{5.83}$$

From (5.77) and (5.83), we obtain

$$\delta(0, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f}\right)}{T(r, f)} \geq 1 - \underline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} = \delta(a, f) > 0$$

and

$$\delta(1, f) = 1 - \overline{\lim_{r\to\infty}} \frac{m\left(r, \frac{1}{f-1}\right)}{T(r, f)} \geq 1 - \delta(a, f) = \underline{\lim_{r\to\infty}} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = \delta(a, f) > 0.$$

Then by theorem 5.7, we get $f(z) \equiv g(z)$, which contradicts the assumption of Theorem 5.8. Since $f(z)$ is non-constant, (5.67) imply $e^{\beta(z)}$ and $e^{\beta(z)-\gamma(z)}$ are not constants. Hence $e^{\gamma(z)}$ is a constant. Suppose that $e^{\gamma(z)} \equiv k_1$, where k_1 $(\neq 0, 1)$ is a constant. It follows from (5.67) that

$$f(z) - a = \frac{1}{k_1 - 1}\left\{e^{\beta(z)} - [1 + a(k_1 - 1)]\right\}. \tag{5.84}$$

Noticing that $\delta(a, f) > 0$, if $1 + a(k_1 - 1) \neq 0$, then

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &= N\left(r, \frac{1}{e^{\beta} - [1 + a(k_1 - 1)]}\right) \\ &= T(r, e^{\beta}) + S(r, f) \\ &= T(r, f) + S(r, f). \end{aligned} \tag{5.85}$$

From (5.85), we obtain $\delta(a, f) = 0$, it is a contradiction. Hence $1 + a(k_1 - 1) = 0$, and thus $k_1 = \frac{a-1}{a}$. Substituting $e^{\gamma(z)} = \frac{a-1}{a}$ into (5.67) gives

$$f(z) = a(1 - e^{\beta(z)}), \quad g(z) = (1 - a)(1 - e^{-\beta(z)}).$$

Hence

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

The proof of Theorem 5.8 is completed. ❑