行政院國家科學委員會補助專題研究計畫成果報告

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Existence and Uniqueness of Solutions of quasilinear Wave Equations (II)

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Abstract In this work we try to prove the existence and uniqueness of solutions of quasilinear wave equations and we consider also their trivial solutions.

Introduction

We consider the following initial-boundary value problem for the nonlinear wave equation of the form

$$\text{(QL)} \qquad \Box u + f(u) + g(\dot{u}) = 0 \quad \text{in} \quad [0,T) \times \Omega$$

with initial values $u_0 = u(0,\cdot), u_1 = \dot{u}(0,\cdot)$ and boundary vale null, that is, $u(t,x) = 0$ on $[0,T] \times \partial\Omega$. Where $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^n, (n \in \mathbb{N})$ is a bounded domain on which the divergent theorem can be applied. $\left(L^2(\Omega), \|\cdot\|_2\right), \left(H^1(\Omega), \|\cdot\|_{1,2}\right)$ are denoted the usual spaces of Lebesgue and Sobolev - functions. Further we use the following notations:

$$\begin{aligned}
\cdot \quad &: = \frac{\partial}{\partial t}, \nabla := \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \Box := \partial^2/\partial t^2 - \triangle, Du := (\dot{u}, \nabla u), \\
|Du|^2 \quad &: = \dot{u}^2 + |\nabla u|^2, C_\Omega := \inf\left\{\|\nabla u\|_2 / \|u\|_2 : u \in H_0^1(\Omega)\right\}, \\
F(s) \quad &: =_0^s f(r)\,dr, E(t) :=_\Omega \left(|Du(t,x)|^2 + 2F(u(t,x))\right) dx.
\end{aligned}$$

For a Banach space X and $0 < T \leq \infty$ we set

$$\begin{aligned}
W^{k,p}(0,T,X) \quad &: \quad = \text{ Sobolev space of } W^{k,p} - \text{functions } [0,T] \to X. \\
C^k([0,T],X) \quad &: \quad = \text{ space of } C^k\text{–functions } [0,T] \to X. \\
C^k(0,T,X) \quad &: \quad = \text{space of } C^k - \text{functions } [0,T) \to X. \\
H1 \quad &: \quad = C^0\left(0,T,H_0^1(\Omega)\right) \cap C^1\left(0,T,L^2(\Omega)\right), \\
H2 \quad &: \quad = C^1\left([0,T],H_0^1(\Omega)\right) \cap C^2\left([0,T],L^2(\Omega)\right).
\end{aligned}$$

To the related problem about the semilinear wave equations; that is, $g \equiv 0$, there are many results, we point out some important of them to the equation

$$\text{(SL)} \qquad \Box u + f(u) = 0.$$

We have proved the non-existence of global solutions of IBVP (SL) [Li2] under the assumptions $E(0) < 0$, $a'(0) > 0$ and

$$\eta f(\eta) - 2(1+2\alpha)\, F(\eta) \leq 2\alpha C_\Omega^2 |\eta|^2 \quad \forall \eta \in \mathbb{R}.$$

The result of Li [Li2] allows the managements for IBVP (SL) for $f(u) = u^p$, $p \in [1, n/n-2]$. In this case Li [Li1] showed the uniqueness of the solutions. Further contributions to the theme "blow-up" see Racke [R].

Segal [S2] applied the semi-group theory to get the global existence of solutions under the case $f(0)=0, F(\eta)\geq -c\eta^2\ \ \forall\eta\in\mathbb{R},\ c>0$ and $lim_{|\eta|\to\infty}|F(\eta)|/|f(\eta)|=\infty$.

In this work we consider the existence, uniqueness and triviality of the solutions of quasilinear and semilinear wave equations through the classical energy method.

I. Existence and Uniqueness of global Solutions of damping Wave Equations

Although there is an existence and uniqueness theorem for the problem (QL) in $[H, p.91]$ and where Haraux A. has present an elegant proof, but the method is not elementary and it is not easy to read. That theorem says that:

Suppose that $u_1\in H_0^1(\Omega)$, $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$ *and* $h\in W^{1,1}(0,T,L^2(\Omega))$. *If* $f(u)\in L^2(\Omega)$, $g(u)\in L^2(\Omega)$ *for each* $\forall u\in H_0^1(\Omega)$ *and* $\|g(v)\|\leq c(M)$ *for each* $\|v\|\leq M$. *Let* g *be a monotone increasing function and* f *be local Lipschitz-bounded; that is, there exists a function* $B\left(\|u\|_{1,2},\|v\|_{1,2}\right)\leq K$ *for* $\|u\|_{1,2},\ \|v\|_{1,2}\leq K$ *with*

$$\|f(u)-f(v)\|_2\leq B\left(\|u\|_{1,2},\|v\|_{1,2}\right)\|u-v\|_{1,2}.$$

Then there exists exactly a function $u:[0,T]\to H_0^1(\Omega)$ *with* $u(0,\cdot)=u_0, \dot{u}(0,\cdot)=u_1; \dot{u}(t)\in H_0^1(\Omega), u(t)\in H^2(\Omega)\ \forall t\in[0,T]$ *and*

$$\Box u+f(u)+g(\dot{u})=h(t,x)\qquad a.e.\ in\ (0,T)\times\Omega.$$

Further, $\|\dot{u}(t)\|_{1,2},\|u(t)\|_{2,2}$ *are bounded in* $[0,T]$.

The following Lemma 1 is take from $[LM, p.95; H, p.96]$.

Lemma 0.1 (1). *For* $h\in W^{1,1}(0,T,L^2(\Omega))$, *the linear wave equation*

$$\begin{cases}\Box u=h(t,x) & in\ \times\Omega\\ u(0,\cdot):=u_0\in H^2(\Omega)\cap H_0^1(\Omega),\\ \dot{u}(0,\cdot):=u_1\in H_0^1(\Omega),\end{cases}$$

posesses exactly one solution $u\in H2$ *with* $u(t)\in H^2(\Omega)\ \forall t\in[0,T]$.

Further,

$$\frac{d}{dt}\|Du(t)\|_2^2+2_\Omega h\dot{u}(t,x)\,dx=0\qquad a.e.\ in\ [0,T]. \tag{1}$$

We have the following result

Theorem 0.2 (2). *If* $g\in C^1(\mathbb{R})$ *is local Lipschitz and monotone increasing with* $g(\dot{u})\in W^{1,1}(0,T,L^2(\Omega))$ *for each* $u\in H2$, *then the BVP for the damping wave equation*

$$\begin{cases}\Box u+\ g(\dot{u})=0 & in\quad \mathbb{R}^+\times\Omega\\ u(0,\cdot):=u_0\in H^2(\Omega)\cap H_0^1(\Omega),\\ \dot{u}(0,\cdot):=u_1\in H_0^1(\Omega),\end{cases} \tag{DG}$$

with $u(t,x)=0$ *on* $[0,T]\times\partial\Omega$ *posses exactly one global solution in* $H2$, *i.e.* $T=\infty$.

Proof (i) Proof the locale existence in $H1$.

1) For a $T>0$ and $v\in H2$, we have $g(\dot{v})\in W^{1,1}(0,T,L^2(\Omega))$. According to Lemma 1, let $w:=Sv$ be the existing solution of BVP for the equation

$$\begin{cases} \Box w+g(\dot{v})=0 \\ w(0,\cdot):=u_0\in H^2(\Omega)\cap H_0^1(\Omega), \\ \dot{w}(0,\cdot):=u_1\in H_0^1(\Omega), \end{cases}$$

so we have $w\in H2, w(t)\in H^2(\Omega)\quad \forall t\in[0,T]$ and

$$\frac{d}{dt}\|Dw\|_2^2(t)+2_{\Omega}\dot{w}g(\dot{v})(t,x)\,dx=0.$$

Suppose that $v_1:=t\,u_0$, then we get $g(\dot{v}_1)=g(u_0)\in L^2(\Omega)\subset W^{1,1}(0,T,L^2(\Omega))$ and therefore, there exists a function $v_2\in H2$ which satisfies the BVP for the equation

$$\begin{cases} \Box w+g(\dot{v}_1)=0, \\ w(0,\cdot):=u_0\in H^2(\Omega)\cap H_0^1(\Omega), \\ \dot{w}(0,\cdot):=u_1\in H_0^1(\Omega). \end{cases}$$

Let $v_{m+1}:=Sv_m$ be the solution of the BVP for the linear equation

$$\begin{aligned} &\Box v_{m+1}+g(\dot{v}_m)=0 \quad \text{in} \quad [0,T)\times\Omega, \\ &v_{m+1}(0,\cdot)=u_0\in H^2(\Omega)\cap H_0^1(\Omega), \\ &\dot{v}_{m+1}(0,\cdot)=u_1\in H_0^1(\Omega). \end{aligned}$$

Therefore, by Lemma 1, we have $v_{m+1}(t)\in H^2(\Omega)\ \forall t\in[0,T]$, $v_{m+1}\in H2, m\in\mathbb{N}$ and

$$\frac{d}{dt}_{\Omega}|Dv_{m+1}(t,x)|^2\,dx+2_{\Omega}\dot{v}_{m+1}(t,x)\,g(\dot{v}_m(t,x))\,dx=0 \quad a.e.\ in\ [0,T].$$

Set

$$A_{m+1}(t):=\|Dv_{m+1}(t)\|_2.$$

Then we find that

$$\left(A_{m+1}(t)^2\right)'\leq 2A_{m+1}(t)\,\|g(\dot{v}_m(t))\|_2 \quad a.e.\ in\,[0,T]$$

and

(2) $$A_{m+1}(t)\leq A_{m+1}(0)+_0^t\|g(\dot{v}_m)(r)\|_2\,dr$$

for every $m\in\mathbb{N}$, almost everywhere in $[0,T]$, besides $A_{m+1}(t)=0$.

2) Since that $g(u_0)\in L^2(\Omega)$, we get $v_2\in H2$ and by the inequality (2), we obtain that

$$A_2(t)\leq\|Du_0\|_2+t\,\|g(u_0)\|_2\leq const..$$

Set

$$M \; : \; = \text{constant} > \|Du_0\|_2 ,$$

$$k(M) \; : \; = \text{local Lipschitz constant of } g,$$

$$\|v\|_{\infty,T} \; : \; = \sup\left\{\|Dv(t)\|_2 \; : \; 0 \leq t \leq T\right\},$$

$$T \; : \; = (M - \|Du(0)\|_2) \,/\, (k(M)\,M \; + \; \|g(0)\|_2).$$

Then

$$k(M)\,T = \frac{k(M)\,M - k\,\|Du_0\|_2}{k(M)\,M + \|g(0)\|_2} < 1$$

and

$$\begin{aligned} A_2(t) &\leq A_2(0) +_0^t \|g(u_0)\|_2(r)\,dr \\ &= \|Du_0\|_2 + \frac{M - \|Du(0)\|_2}{k(M)\,M + \|g(0)\|_2} \cdot (k(M)\,M + \|g(0)\|_2) \\ &= M \quad \forall t \in [0, T], \end{aligned}$$

consequently

$$\|v_2\|_{\infty,T} \leq M.$$

Suppose that $\|v_m\|_{\infty,T} \leq M$, then we have

$$A_{m+1}(t) \leq \|Du_0\|_2 + (k(M)\,M + \|g(0)\|_2)\,T = M.$$

Thus we get $\|v_{m+1}\|_{\infty,T} \leq M \quad \forall m \in \mathbb{N}$.

3) We claim thatv_m is a Cauchy sequence in $H1$. By using inequality (2)

$$|A_{m+1} - A_m|(t) \leq k(M)\,t\,\|v_m - v_{m-1}\|_{\infty,T} \quad \forall t \in [0, T).$$

So it is,

$$\|v_{m+1} - v_m\|_{\infty,\,T} \leq (k(M)\,T)^{m-2}\,\|v_2 - v_1\|_{\infty,\,T}$$

and herewith it follows

$$\|v_{m+k} - v_m\|_{\infty,T} \leq \frac{(k(M)\,T)^{m-1}\,\|v_2 - v_1\|_{\infty,T}}{1 - k(M)\,T} \to 0 \tag{4}$$

for m goes to ∞.

(ii) We prove the uniqueness of the solutions in $H1$

Suppose that u is the limit of v_m, and $v \in H1$ is an another solution for (DG), then

$$\frac{d}{dt}_{\Omega} |Dv_{m+1} - Dv|^2(t,x)\,dx \leq 2\,\|D\dot{v}_{m+1}(t) - Dv(t)\|_2\,\|g(v_m(t)) - g(v(t))\|_2 .$$

From the inequality (3), we obtain

$$\|v_{m+1} - v\|_{\infty,\,T} \leq k(M)\,T\,\|v_m - v\|_{\infty,\,T} \leq (k(M)\,T)^{m-2}\,\|v_2 - v\|_{\infty,\,T} \to 0$$

for m goes to ∞, so $u \equiv v$ in $H1$.

(iii) We show the global existence of a solution in $H1$.

Suppose that u is the limit of v_m, then by the monotone of g and Fato Lemma we get

$$\begin{aligned}\frac{d}{dt}{}_\Omega |Du(t,x)|^2 dx &\leq \lim_{m\to\infty} \frac{d}{dt}{}_\Omega |Dv_{m+1}(t,x)|^2 dx \\ &\leq -2_\Omega \dot{u}(t,x)\, g(u(t,x))\, dx \leq 0\end{aligned}$$

and

$$_\Omega |Du(t,x)|^2 dx \leq_\Omega |Du(0,x)|^2 dx < M.$$

We set now

$$\overline{u}_0(\cdot) := u(T/2,\cdot) \in H_0^1(\Omega),\ \overline{u}_1(\cdot) := \dot{u}(T/2,\cdot) \in L^2(\Omega)$$

and construct the solutions with those initial data in an interval $[T/2, T)$.

According to the vorstanding estimations and the contraction property in 3) we can choose $T^\wedge = 3T/2$. The on the interval $[0,T)$ and $[T/2, 3T/2)$ construct the solutions are equal since the uniqueness of the solutions on $[T/2, T)$. Those further conditions reach us a solution on $[0,\infty)$.

(iv) Now we show the local existence u in $H2$ and $u(t) \in H^2(\Omega)$
Suppose that T, M, k are the same given in (i-2).

1) For a $T > 0$ and $v_m \in H2$, we have $g(\dot{v}_m) \in W^{1,1}(0, T, L^2(\Omega))$. According to Lemma 1, we have $v_{m+1}(t) \in H^2(\Omega)\ \forall t \in [0,T],\ v_{m+1} \in H2, m \in \mathbb{N}$ and

$$\frac{d}{dt}{}_\Omega |Dv_{m+1}(t,x)|^2 dx = -2_\Omega \dot{v}_{m+1}(t,x)\, g(\dot{v}_m(t,x))\, dx \quad a.e.\ in\ [0,T]$$

also

$$\begin{aligned}&\frac{d}{dt}\|Dv_{m+1}\|_2^2(t) + \frac{d^2}{dt^2}\int_\Omega v_{m+1}(t,x)^2 dx \\ \leq\ & 2\int_\Omega \dot{v}_{m+1}^2(t,x)\,dx + 2\left(\|v_{m+1}(t)\|_2 + \|\dot{v}_{m+1}(t)\|_2\right)\|g(\dot{v}_m(t))\|_2\end{aligned}$$

Then we find that

$$A_{m+1}(t) \leq A_{m+1}(0) +_0^t \|g(\dot{v}_m)(r)\|_2\, dr \tag{2}$$

for every $m \in \mathbb{N}$, almost everywhere in $[0,T]$, besides $A_{m+1}(t) = 0$.

2) Since that $g(u_0) \in L^2(\Omega)$, we get $v_2 \in H2$ and by the inequality (2), we obtain that

$$A_2(t) \leq A_2(0) +_0^t \|g(u_0)\|_2\, dr = \|Du_0\|_2 + t\,\|g(u_0)\|_2 \leq const..$$

Set

$$\begin{aligned}M &: = \text{constant} > \|Du_0\|_2, \\ k(M) &: = \text{local Lipschitz constant of } g, \\ \|v\|_{\infty,T} &: = \sup\left\{\|Dv(t)\|_2\ :\ 0 \leq t \leq T\right\}, \\ T &: = \left(M - \|Du(0)\|_2\right)/\left(k(M)\,M\ +\ \|g(0)\|_2\right).\end{aligned}$$

Then

$$k(M)T = \frac{k(M)M - k\|Du_0\|_2}{k(M)M + \|g(0)\|_2} < 1$$

and

$$\begin{aligned} A_2(t) &= \|Dv_2\|_2(t) \le A_2(0) +_0^t \|g(u_0)\|_2(r)\,dr \\ &= \|Du_0\|_2 + \frac{M - \|Du(0)\|_2}{k(M)M + \|g(0)\|_2} \cdot (k(M)M + \|g(0)\|_2) \\ &= M \quad \forall t \in [0,T], \end{aligned}$$

consequently

$$\|v_2\|_{\infty,T} \le M.$$

Suppose that $\|v_m\|_{\infty,T} \le M$, then we have

$$A_{m+1}(t) \le \|Du_0\|_2 + (k(M)M + \|g(0)\|_2)T = M.$$

Thus we get $\|v_{m+1}\|_{\infty,T} \le M \quad \forall m \in \mathbb{N}$.

3) We claim thatv_m is a Cauchy sequence in $H2$. By using inequality (2)

$$|A_{m+1} - A_m|(t) \le_0^t \|g(\dot{v}_m) - g(\dot{v}_{m-1})\|_2(r)\,dr \le k(M)t\|v_m - v_{m-1}\|_{\infty,T} \quad \forall t \in [0,T).$$

So it is,

$$\text{(1)} \qquad \begin{aligned} \|v_{m+1} - v_m\|_{\infty,T} &\le k(M)T\|v_m - v_{m-1}\|_{\infty,T} \\ &\le (k(M)T)^{m-2}\|v_2 - v_1\|_{\infty,T} \end{aligned}$$

and herewith it follows

$$\text{(4)} \qquad \|v_{m+k} - v_m\|_{\infty,T} \le \frac{(k(M)T)^{m-1}\|v_2 - v_1\|_{\infty,T}}{1 - k(M)T} \to 0$$

for m goes to ∞.

(ii) We prove the uniqueness of the solutions in $H2$

Suppose that u is the limit of v_m, and $v \in H2$ is an another solution for (DG), then

$$\frac{d}{dt_\Omega}|Dv_{m+1}(t,x) - Dv(t,x)|^2\,dx$$

$$\le 2\|D\dot{v}_{m+1}(t) - Dv(t)\|_2 \cdot \|g(v_m(t)) - g(v(t))\|_2 .$$

From the inequality (3) we obtain

$$\|v_{m+1} - v\|_{\infty,T} \le k(M)T\|v_m - v\|_{\infty,T} \le (k(M)T)^{m-2}\|v_2 - v\|_{\infty,T} \to 0$$

for m goes to ∞, so $u \equiv v$ in $H1$.

(iii) We show the global existence of a solution.

Suppose that u is the limit of v_m, then by the monotone of g and Fato Lemma we get

$$\frac{d}{dt_\Omega}|Du(t,x)|^2\,dx \le -2_\Omega\dot{u}(t,x)\,g(u(t,x))\,dx \le 0$$

and

$$\Omega \left|Du\left(t,x\right)\right|^{2}dx \leq_{\Omega} \left|Du\left(0,x\right)\right|^{2}dx < M.$$

We set now

$$\overline{u}_{0}\left(\cdot\right) := u\left(T/2,\cdot\right) \in H_{0}^{1}\left(\Omega\right), \overline{u}_{1}\left(\cdot\right) := \dot{u}\left(T/2,\cdot\right) \in L^{2}\left(\Omega\right)$$

and construct the solutions with those initial data in an interval $[T/2, T)$.

According to the vorstanding estimations and the contraction property in 3) we can choose $T^{\wedge} = 3T/2$. The on the interval $[0,T)$ and $[T/2, 3T/2)$ construct the solutions are equal since the uniqueness of the solutions on $[T/2,T)$. Those further conditions reach us a solution on $[0,\infty)$. ∎

II. Generalization for theorem 2

For the semilinear wave equation we have the following simple form result.

Theorem 0.3 (3). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1}-$ function and there exist constant $k > 0$ and $p \in \mathbb{R}$ with*

$$\left|f'\left(\eta\right)\right| \leq k\left(1+\left|\eta\right|^{p-1}\right) \ \forall \eta \in \mathbb{R},$$

then the BVP for the semilinear wave equation (SL) posses mostly one solution u in $H1$ with $T=\infty$, if one of the following two conditions holds:

case 1 *:* $p > 1$ and $u \in C^{0}\left(\mathbb{R}^{+}, H^{1}\left(\Omega\right)\right) \cap C^{1}\left(\mathbb{R}^{+}, L^{2}\left(\Omega\right) \ \cap \ L^{\infty}\left(\Omega\right)\right)$

case 2: *$1 \leq p \leq n/\left(n-2\right)$ and $u \in H1$ with $T=\infty$.*

Proof of case 1 Suppose that w is the solution of the BVP for the linear wave equation

$$\Box w + h\left(t,x\right) = 0, w\left(0,\cdot\right) = 0 = \dot{w}\left(0,\cdot\right),$$

so it brings to the inequality

$$\left(\left\|Dw\right\|^{2}\left(t\right)\right)' \leq \left\|Dw\right\|_{2}^{2}\left(t\right) + \left\|h\right\|_{2}^{2}\left(t\right).$$

Multiplication with e^{-t}, it yields

$$\left(e^{-t}\left\|Dw\right\|_{2}^{2}\left(t\right)\right)' \leq \left\|h\right\|_{2}^{2}\left(t\right) \quad \forall t \geq 0.$$

By the fact that $w\left(0,\cdot\right) = 0 = \dot{w}\left(0,\cdot\right)$ and $\left\|Dw\right\|_{2}\left(0\right) = 0$ follows

$$\left\|Dw\right\|_{2}^{2}\left(t\right) \leq e^{t}\int_{0}^{t}\left\|h\right\|_{2}^{2}\left(s\right)ds \quad \forall t \geq 0.$$

Suppose that u, v are two solutions of (SL), then $w := u - v$ satisfies the problem in *1)* with $h := f\left(u\right) - f\left(v\right)$, thus

$$\left|f\left(u\right)-f\left(v\right)\right|^{2} \leq 3k^{2}2^{2(p-2)}\left(1+\left(\left|u\right|^{2(p-1)}+\left|v\right|^{2(p-1)}\right)\right)w^{2}.$$

For $A\left(t\right) := \left\|Dw\right\|_{2}^{2}\left(t\right)$ and $t \geq 0$ we get

$$A\left(t\right) \leq 3k^{2}2^{2(p-2)}e^{t}\int_{0}^{t}\int_{\Omega}\left(1+\left|u\right|^{2(p-1)}+\left|v\right|^{2(p-1)}\right)w^{2}\left(s,x\right)dxds.$$

Now, we fix $T > 0$, there exists a positive k_1 with

$$3 \cdot 2^{2(p-2)} k^2 e^T sup_{t\in[0,T]} \left\{ \left\| 1 + |u|^{2(p-1)} + |v|^{2(p-1)} \right\|_\infty (t) \right\} \leq k_1 (T)$$

and

$$A(t) \leq k_1 \int_0^t A(s)\, ds, A(t) \leq A(0)\, e^{k_1(T)t} = 0 \quad \forall t \in [0,T].$$

This means yet $u \equiv v$ in $[0,T] \times \Omega$.

Proof of case 2

Proof for $1 + 1/n \leq p \leq n/(n-2)$.

1) We set $r = n/2$. By the Sobolev-imbedding $H_0^1(\Omega) \subset L^{2r(p-1)/(r-1)}(\Omega)$, there are constants k_2 and $\delta := 2r(p-1)/(r-1)$ so that the inequality holds

$$1 + \|u\|_\delta^\delta (t) + \|v\|_\delta^\delta (t) \leq k_2 \left(1 + \|u\|_{1,2}^\delta + \|v\|_{1,2}^\delta\right)(t) \qquad \forall t \geq 0.$$

There exists a constant $k_3 > 0$ with

$$\|u - v\|_{2r}^2 (s) \leq k_3 \|D(u-v)\|_2^2 (s).$$

By the fact that u, v in $H1$ with $T = \infty$ the following supremum exists

$$k_4(T) := Sup_{t\in[0,T)} \left\{ \left(1 + \|u\|_{1,2}^\delta + \|v\|_{1,2}^\delta\right)^{(r-1)/r} (t) \right\}.$$

Let us set

$$B(t) := \int_0^t A(r)\, dr.$$

From the Hölder inequality follows

$$\begin{aligned} B'(t) &\leq 2ke^T \int_0^T \left\| 1 + u^{2(p-1)} + v^{2(p-1)} \right\|_{r/(r-1)} \|u - v\|_{2r}^2 (s)\ ds \\ &\leq 2kk_3 e^T B(t) \int_0^T \left(2^{r/(r-1)} \int_\Omega \left(1 + |u|^\delta + |v|^\delta\right)(s,x)\, dx \right)^{1-1/r} ds \\ &\leq 4kk_3(T)\, k_4(T)\, e^T B(t) \qquad t \in [0,T]. \end{aligned}$$

Using the Gronwall inequality we obtain

$$B(t) \leq B(0)\, e^{k_5(T)\, t} = 0 \quad \forall t \in [0,T]$$

with $k_5 := 4kk_3k_4(T)\, e^T$.

By the definition of $B(t) := \int_0^t A(r)\, dr$ and $A(t) := \|D(u-v)\|_2^2 (t)$ we find that

$$A(t) \equiv 0 \quad \forall t \in [0,T].$$

This means that however $u(t,x) = v(t,x)$ $\forall t \in [0,T]$, a.e. in Ω.

2) Proof for the case: $1 + 1/n$. Suppose that u, v are two solutions of (SL). From the condition for f for $w := u - v$ with

$$n = \frac{k}{p-1},\ \alpha = \frac{2n}{n-2}, k \in \left[1, \frac{2n}{n-2}\right],$$

there exists $k_6 > 0$ such that

$$\|w\|_\alpha (t) \leq k_6 \|\nabla w\|_2 (t),\ k_6 > 0$$

and according to $u,\ v \in H$ with $T = \infty$ there exists the supremum

$$k_8(T) := \sup_{t\in[0,T)} \left\{ \left[|\Omega| + k_7 \left(\|\nabla u\|_2^{(p-1)n} + \|\nabla v\|_2^{(p-1)n} \right)(t) \right]^{1/n} \right\},$$

thus

$$A'(t) \le 4k_6 k_8(T) A(t)$$

and consequently

$$A(t) \le A(0)\, e^{4\, k_6 k_8 (T)\cdot t} = 0 \quad \forall t \in [0,T).$$

This means that $u(t,x) = v(t,x)$ for $t \in [0,T]$.

For the case $p = 1$ we have

$$|f'(\eta)| \le 2k \left(1 + |\eta|^{q-1}\right) \quad \forall \eta \in \mathbb{R}, q \in \left[\frac{n+1}{n}, \frac{n}{n-2}\right].$$

In this case, the proof is similar to the proof in step 2). ∎

III. Uniqueness of the Solutions of qusilinear Wave Equations

For some particular quasilinear wave equations we have also uniqueness result below:

Theorem 0.4 (4). *Suppose that $f\ , g : \mathbb{R} \to \mathbb{R}$ are monotone increasing with $f(0) = 0, g(0) = 0, F(u_0) \in L^1(\Omega)$. Then the BVP for the wave equation (QL) has at most one solution u in $H1$, if f is local Lipschitz, that is, $M(u) := f(u) : H_0^1(\Omega) \to L^2(\Omega)$ with*

$$\|M(u) - M(v)\|_2 \le B(K)\, \|u - v\|_{1,2},$$

B is bounded for $\|u\|_{1,2}, \|v\|_{1,2} \le K$. Further, for local Lipschitz g there is a constant b with

$$\|Du\|_2(t) \le b \quad \forall t \in [0,T).$$

Proof Suppose that u, v are two solutions of the wave equation (QL), according to the monotone of g we get

$$\frac{d}{dt} \|D(u-v)\|_2^2(t) \le 2 \int_\Omega \left(|D(u-v)|^2 + |f(u) - f(v)|^2 \right)(t,x)\, dx.$$

By the similar way proving theorem 3, we conclude the uniqueness of the solutions of the wave equation (QL). We show the global boundness. By the fact that $s\, g(s) \ge 0$ and $F(u_0) \in L^1(\Omega)$ we obtain

$$\frac{d}{dt} \int_\Omega \left(|Du|^2 + 2F(u) \right)(t,x)\, dx \le -2 \int_\Omega \dot{u} g(\dot{u})(t,x)\, dx \le 0$$

Using $F(s) \geq 0$, we conculde that

$$\int_\Omega |Du|^2(t,x)\,dx \leq const. < \infty \ \ \forall t \in [0,T). \ \blacksquare$$

Remark 0.1 (1). *The theorem 4 valids particularly for the case*

$$f(u) + g(\dot{u}) = au^p + b\dot{u}, \ a, b = const., \ p \in [1, n/(n-2)].$$

For $f : \mathbb{R}^{n+2} \to \mathbb{R}$ Lipschitz in its variables, the following BVP for the wave equation has at most one solution in H1.

$$\begin{cases} \Box u + f(u, \dot{u}, \nabla u) = 0 \qquad in \qquad \mathbb{R}^+ \times \Omega \\ u(0,\cdot) := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ \dot{u}(0,\cdot) := u_1 \in H_0^1(\Omega). \end{cases}$$

Since we have

$$\frac{d}{dt}\|Dw\|_2^2(t) \leq \left(1 + 2k + 2kC_\Omega^2\right)\|Dw\|_2^2(t) \quad \forall t \in [0,T].$$

for each $T > 0$ and $w := u - v, t \in [0,T]$.

Where $\sqrt{k}$ is the Lipschitz $-$ constante of f. Integration we obtain

$$\int_\Omega \left(\dot{w}^2 + |\nabla w|^2\right)(t,x)\,dx$$

$$\leq \int_\Omega \left(\dot{w}^2 + |\nabla w|^2\right)(0,x)\,dx \cdot \exp\left(1/2 + k + kC_\Omega^{-2}\right)t = 0.$$

By $w := u - v \in H1$ with $T = \infty$ it follows $w(t,\cdot) = 0$.

IV. Trivial Solution of some semilinear Wave Equations

In [Li2] we have an interesting result,the solutions of (SL) must be trivial under the conditions $u_0 \equiv 0 \equiv u_1 \equiv f(0)$ and $\eta\ f(\eta) + 2F(\eta) \geq -k|\eta|^p\ \forall \eta \in \mathbb{R}$. in this section we want generalize this result.

Theorem 0.5 (5). *The BVP for the wave equation (SL) has only trivial solution $u \equiv 0$ as global solution in $H1$ for $u_0 \equiv 0 \equiv u_1$, if the following valids*

$$f(\eta)^2 \leq k|\eta|^p \quad \forall \eta \in \mathbb{R}.$$

Proof. Let us set

$$t_1 := \sup\{t \geq 0 : A(t) \leq 1\}.$$

According to the conditions on f and using the lemma 1 we get

$$A'(t) \leq A(t) + kA(t) + k\int_\Omega |u|^p(t,x)\,dx.$$

For $2 \leq p \leq 2n/(n-2)$, we have Sobolev inequality

$$\|u\|_p^p(t) \leq k_1\|Du\|_2^p(t) = k_1 A(t)^{\frac{p}{2}}.$$

These together we obtain that for t in $[0,t_1]$

$$A'(t) \leq (1+k)\,A(t) + kk_1 A(t)^{\frac{p}{2}} \leq k_2 A(t),$$

where $k_2 := 1 + k + 2^{-1}pkk_1$. By Gronwall inequality, the assertion of this theorem follows. ∎

Corollary 0.6 (6). *The theorem 5 is applieable particular to the well-defined functions*

$$f(u) = u^{p/2} + u^{q/2},\ u^{p/2} - u^{q/2}\ ,\ p\ ,\ q \in [2, 2n/n-2]$$

or under the condition

$$f(\eta)^2 \leq \sum_{i=1}^{m} k_i\,|\eta|^{p_i} \qquad \forall \eta \in \mathbb{R},$$

$k_i =$ *positiv constants* , $p_i \in [2, 2n/n-2]$.

Theorem 0.7 (7). *Suppose that u in $H1$ is a solution of the BVP for the semilinear wave equation (SL). Then $u \equiv 0$ and $f(0) = 0$, if $u_0 \equiv 0 \equiv u_1$ and*

$$F(\eta) \geq -k\eta^2 \quad \forall \eta \in \mathbb{R},\ k = const.$$

Proof By the lemma 1 we have

$$A(t) = -2\int_\Omega F(u)(t,x)\,dx \leq 2ka(t),$$

$$\begin{aligned} C_\Omega a'(t) &= 2C_\Omega \int_\Omega u\dot{u}(t,x)\,dx \\ &\leq C_\Omega^2 a(t) + \int_\Omega \dot{u}(t,x)^2\,dx \\ &\leq A(t) \leq 2ka(t) \quad \forall t \geq 0. \end{aligned}$$

Using Granwall inequality, we conclude that

$$a(t) \leq a(0)\exp(2kt/C_\Omega) = 0 \quad \forall t \geq 0,$$

since that $u_0 \equiv 0 \equiv u_1$. And this implies that $u \equiv 0$ and $f(0) = 0$. ∎

Corollary 0.8 (8). *Theorem 7 valids specially for the monotone increasing function f with $f(0) = 0$. For instance $f(u) = u^{2p-1}$, $-1 + \exp u$.*

Theorem 0.9 (9). *Let $\Omega := B_{r_2}(0) - \overline{B_{r_1}(0)}$, $r_2 > r_1 > 0$ be an annulus in R^n and there exist constants $k > 0$ and $p \geq 1$ with*

$$\eta f(\eta) + 2F(\eta) \geq -k\,|\eta|^p \qquad \forall \eta \in \mathbb{R}.$$

Then the BVP for (SL) has only $u \equiv 0$ as global radial solution in $H1$, if $u_0 \equiv 0 \equiv u_1 \equiv f(0)$ valid.

Remark 0.2 (2). *Here we put no condition on p. $f(u) = -mu + ku^q$ is a typical example.*

Proof Suppose that $u(t,|x|) = u(t,r)$, $r = |x|$ is a radial solution of (SL). Setting

$$u(t,r) = v(t,r)\, r^{(1-n)/2},\ G(v) := \int_0^v g(s)\, ds,$$

$$g(v) := 4^{-1}(n-1)(n-3)\, r^{-2} v + r^{(n-1)/2} f\left(r^{(n-1)/2} v\right),$$

then the equation (SL) is transformed into the following form

$$\ddot{v} - \partial^2 v / \partial r^2 + g(v) = 0 \quad in \quad [0,T) \times (r_1, r_2),$$

$$v(0,r) = u_0 r^{(n-1)/2} \equiv 0 \equiv \dot{v}(0,r),$$

$$v(t,r_1) \equiv 0 \equiv v(t,r_2).$$

Choosing $\eta = r^{(1-n)/2} v$ in the condition in theorem 9 on f and F, then we have

$$r^{(n-1)/2} v f\left(r^{-(n-1)/2} v\right) + 2F\left(r^{-(n-1)/2} v\right) \geq -k r^{-(n-1)p/2} |v|^p.$$

From this, we conclude

$$vg(v) + 2G(v) \geq 2^{-1}(n-1)(n-3)\, r^{-2} v^2 - k r^{(n-1)p/2} |v|^p$$

$$\geq \quad -k\left(v^2 \ + \ |v^p|\right)$$

with $k := Max_{r \in [r_1, r_2]} \left\{-2^{-1}(n-1)(n-3)\, r^{-2} + k r^{(1-n)p/2}\right\}$.

From Satz 2 in Li 2 for two variables t and r the assertion of this theorem follows.

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