Chapter 3

Dynamic Asset Allocation under Learning about Inflation

3.1 Introduction

How should an investor who has a long-term horizon and can invest only in nominal assets construct his/her portfolio when the inflation rate is stochastic and the investor has learning capability about the inflation rate process? Despite the importance of this question to both academics and investors, within the rich literature about portfolio selection we find no papers that address it directly. The tactical asset allocation strategy based on the classical static, single-period framework introduced by Markowitz (1959) is a popular way to construct investment portfolios across broad asset classes such as bonds, stocks, and cash. Tactical asset allocation, however, might not be appropriate when the investor's investment opportunity set will shift in the future (e.g., changes in expected returns or covariance). Mossin (1968) showed that a myopic objective function underlying tactical asset allocation is appropriate only if the investor has a logarithmic utility function. For non-log utility functions the investor would like to hedge against adverse shifts in the future investment opportunity set.

Merton (1971) was the first to consider the effect of a stochastic investment opportunity set, and this consideration creates a set of hedge demands in addition to the standard myopic demand. However, further work on optimal strategies under stochastic investment opportunities has languished due to the lack of empirical findings on time variation in expected returns, as well as the lack of computing power to solve realistic problems with time-varying expected returns.

One of the striking empirical findings in the 1990's is the evidence of predictability for asset returns, which revived the work on optimal strategies under stochastic investment opportunities.[1] Brennan, Schwartz, and Lagnado (1997) made the first attempt to address the portfolio choice problem in the presence of time-varying expected returns, using a numerical analysis of the portfolio problem of a long-lived investor when the interest rate is stochastic and the equity premium is predictable by the interest rate and dividend yield. Motivated by their results, Campbell and Viceira (1999) were able to find an analytical approximation to optimal consumption and portfolio rules for an infinite-horizon investor with Epstein-Zin utility. Kim and Omberg (1996) and Wachter (2002) made related theoretical contributions by analyzing the optimal investment strategy when the interest rate is constant but the equity premium follows an Ornstein-Uhlenbeck process. Omberg (1999), Sorensen (1999), and Brennan and Xia (2000) computed optimal strategies when the interest rate follows a Vasicek (1977) process and risk premiums are constant.

Although there is a growing body of evidence that asset returns are predictable, the true extent of predictability is highly uncertain. The effect of parameter uncertainty on portfolio selection was fist investigated by Bawa, Brown, and Klein (1979) in a single-period context. Kandel and Stambaugh (1996) were the first to explore the economic importance of parameter uncertainty when asset returns are partially predictable and the coefficients of the predictive relation are estimated rather than known. Barberis (2000) derived a dynamic strategy in a discrete time setting with

[1] The improvement in computing power over the same period is also striking. On the other hand, Pliska (1986), Karatzas et al. (1987), and Cox and Huang (1989) bypassed the difficulties in solving the Bellman equation and employed the martingale pricing approach to solve the investor's optimal portfolio problem in the complete market setting.

estimation risk. Examining a wider range of horizons rather than the one-month horizon of Kandel and Stambaugh (1996), he demonstrated the substantial role played by parameter uncertainty. He also demonstrated that predictability in returns is important even after incorporating parameter uncertainty.

The effects of parameter uncertainty are rather different from those found in the discrete-time model, as shown by Detemple (1986), Dothan and Feldman (1986), Gennotte (1986), and Feldman (1992). Parameter uncertainty must be considered by an investor with non-logarithmic utility in a continuous time setting in which security prices follow diffusion processes because he/she will learn more about the parameters as time passes. The estimates of the unknown parameter values become state variables in the dynamic optimization problem, and the unanticipated changes in these state variables need to be hedged. In a two-asset world with an unknown but constant expected return from a risky asset, Brennan (1998) derived the optimal investment from the risky asset when investors possess learning capability about the mean return. He showed that the effect of learning on the optimal portfolio allocation increases with the uncertainty over the true value of the mean parameter and the investor's time horizon, whereas it decreases with the volatility of the market return. Rogers (2001) demonstrated that the effect of parameter uncertainty on the investor's expected utility, after taking learning into account, is far more significant than the effect of discrete trading. Xia (2001) derived the evolution of the investor's beliefs about the unknown parameters when the asset prices are possibly predictable and the predictive variables themselves are stochastic.[2] She showed that stochastic covariance between stock return and dynamic learning created substantial market timing in the optimal hedge demands.

[2]This feature of the model distinguished the learning effects with return predictability analyzed by Xia (2001) from the learning effect without return predictability analyzed by Brennan (1998).

However, these papers do not allow for stochastic inflation, which is an essential issue to all long-term investors. Campbell and Viceira (2001) were the first to incorporate inflation into the optimal portfolio strategies. They developed a model of optimal consumption and portfolio choice for an infinitely lived investor with recursive utility in a discrete time setting in which the real interest rate and the expected rate of inflation are stochastic. They solved this model using an approximate analytical method and analyzed the hedging effects of long-term bonds in the presence of inflation. Brennan and Xia (2002) analyzed a finite-horizon investor's asset allocation problem under inflation with the consideration of short sales constraints. They developed closed form expressions for the investor's optimal policy and provided a rationale for the bond maturity choice in a setting where the expected rate of inflation and interest rate follow correlated Ornstein-Uhlenbeck processes and the risk premiums are constant. They showed that hedge demands depend on the maturities of the bonds included in the portfolio as well as the investor's time horizon.

This Chapter contributes to the literature of asset allocation by analyzing how learning about inflation may affect the portfolio selection of a long-term investor. Uncertainty about inflation is a major issue in long-term financial planning. Our contribution to the line of the literature about time variation in asset returns including Brennan et al. (1997) and Campbell and Viceira (1999), therefore, is to incorporate uncertain inflation into the optimal portfolio choice problem. On the other hand, investors will learn more about the stochastic inflation rate process as time passes, and this dynamic learning will certainly affect the optimal portfolio choice. Our contribution to the line of the literature about asset allocation under inflation, such as Campbell and Viceira (2001) and Brennan and Xia (2002), is hence to examine the effects of learning about inflation rate on the optimal dynamic portfolio choice. In short, we apply the results from two lines of the literature about asset allocation to solve a more general question: how should a long-term investor who can invest only in nominal assets construct his/her portfolio when the inflation rate is stochastic and the investor possesses learning capability about the inflation rate process?

Assuming that the inflation rate process is not directly observable, we employ information from the consumer price index (CPI) to predict the unobservable inflation rates using the filtering mechanism of Lipster and Shiryayev (1978). The investment opportunity set is characterized by four nominal assets including nominal instantaneously risk-free cash, a stock index, and bonds with two different maturities. Assuming that the investor is to maximize the expected utility of the terminal wealth and the financial market is complete, we then use the martingale method developed by Cox and Huang (1989) to solve for the optimal asset allocation. We obtain a closed-form solution and perform numerical analyses to demonstrate the economic importance of learning about inflation.

The rest of this chapter is organized as follows. In Section 3.2 we explain how to infer the hidden inflation rate dynamics using the filtering algorithm. We present the market structure and price dynamics of the available assets in Section 3.3, and formulate the portfolio optimization problem characterized by the agent's criteria in Section 3.4. Under these settings, there exists a uniquely determined stochastic discount factor, and the martingale methods can be implemented to formulate the wealth constraints to solve the optimal portfolio. We derive the closed-form solution of the optimal investment strategy that fully incorporates the hedging demands for the inflation risk in this section. Section 3.5 provides numerical illustrations to explore the effects of learning about inflation on the welfare and utility valueof the investor. Section 3.6 concludes this chapter.

3.2 Learning about Inflation

In this chapter we assume that no instantaneously risk-free real asset exists[3] and focus on finding the optimal asset allocation in the presence of inflation rate risk, interest rate risk, and stock market risk. More specifically, the investor is assumed to invest in four nominal assets: cash, a stock index, and two bonds with different maturities. The investor can buy or sell these assets continuously without any restrictions (e.g., short sales constraints) and incurring no trading costs. We further assume that the financial market is complete[4] and free of arbitrage opportunities.

Moreover, we further assume that the inflation rate process is not observable, but the information from CPI can properly infer future inflation rates, and for this we employ CPI as the only inflation forecaster.[5] This inference is complicated by the facts that the observed system is typically driven by inputs other than our own known controls and that the relations among state variables and measured outputs are known with some degree of uncertainty. Furthermore, any measurement will be corrupted to certain degree by noise and bias. To provide some means to extract valuable information from a noisy signal, we first construct a prediction model for the inflation rate using a Bayesian approach and define the filter mechanism as a data processing algorithm. The Bayesian filtering method incorporates all provided

[3]Although countries such as Canada, the United Kingdom, and the United States have inflation-indexed bonds to hedge the inflation rate risk, only a few maturities are available.

[4]Brennan and Xia (2002) show that the complete market case is sustained whenever the expected rate of inflation is not observable, but that it must be inferred from observing the price level itself and not from any other information. The market completeness implies that the change in the expected rate of inflation is perfectly correlated with the realized rate of inflation.

[5]We assume that there are no other predictive variables for the inflation rate. This assumption is along the lines of Gennotte (1986) and Brennan (1998), in which no return predictability is assumed for learning. Assuming that the inflation rate is predictable by some financial variables such as GDP, interest rates, or exchange rates will be in line with Xia (2001) who investigated learning effects with predictability. Since the evidence on the return predictability as well as the inflation rate predictability is not conclusive, both assumptions have their merits.

information and processes all available measurements to estimate the current value of the interested variables.

Following Brennan and Xia (2002) and Basel, Ahmad and Wafaa (2004), we assume that the consumer price index (CPI), Π, follows a stationary Ito process:

$$\frac{d\Pi(t)}{\Pi(t)} = \pi(t)dt + \xi_\pi(t)dz_\pi(t), \tag{3.1}$$

where

$\pi(t)$ is the instantaneous expected rate of inflation,

$\xi_\pi(t)$ is the standard deviation of CPI,

$z_\pi(t)$ is a standard Wiener process,

and $dz_\pi(t)$ is the associated white noise.

In other words, the CPI process, $d\Pi/\Pi$, is written as a linear combination of the instantaneous expected price growth πdt and the innovations from the inflation rate risk dz_π.

Since the process of the instantaneous expected inflation rate is not observable, we proceed to estimate the latent process through the learning process based on the information generated by $\Pi(s)$ from $s = 0$ to $s = t$. Let $\Pi_0^t \equiv \{\Pi(s) \mid 0 \leq s \leq t\}$ be the filtration generated by Π. The instantaneous latent inflation rate process is assumed to follow the following stochastic diffusion process:

$$d\pi(t) = \mu_\pi(t)dt + \sigma_\pi(t)dz_\pi(t) + \sigma_\tau(t)dz_\tau(t), \tag{3.2}$$

$\mu_\pi(t)$ is the instantaneous expected inflation growth rate at time t,

$\sigma_\pi(t)$ is the standard deviation of the instantaneous inflation growth rate at time t,

$\sigma_\tau(t)$ is the standard deviation of the noise process at time t,

$z_\tau(t)$ is a standard Wiener process,

and $dz_\tau(t)$ is the associated white noise such that $dz_\pi(t)$ and $dz_\tau(t)$ are independent.

Noting that $\pi(t)$ is affected by dz_π and dz_τ; however, $\Pi(t)$ is only influenced by dz_π.This presents that we could not use the information of CPI process to perfectly predict the inflation process. For tractability, the instantaneous growth of the inflation rate and the inflation rate are assumed to satisfy the linear relation as follows:

$$\mu_\pi(t) = a_0(t) + a_\pi(t)\pi(t), \tag{3.3}$$

where $a_0(t)$ is the intercept and $a_\pi(t)$ is the coefficient in the associated regression model. If $a_\pi(t) = 0$, then the instantaneous latent inflation rate process in Eq.(3.2) corresponds to independently distributed growths with deterministic drifts. Let $\pi(0)$ represent the investor's prior distribution regarding the instantaneous expected inflation rate process. We assume $\pi(0)$ to be the normal distribution $N(\hat{\pi}(0), v(0))$, in which $\hat{\pi}(0)$ and $v(0)$ denote the mean and variance of the prior process at time 0 respectively.

Using the optimal filtering equations developed by Lipster and Shiryayev (1978), we compute the posterior mean and variance of the unobservable stochastic process $\pi(t)$. The posterior mean $\hat{\pi}(t) = E\left[\pi(t) \mid F_t^\Pi\right]$ is the optimal Bayesian estimator for $\pi(t)$, where $F_t^\Pi = \sigma\left\{\omega : \Pi(s), s \leq t\right\}$ is the σ-algebra generated by Π_0^t. $v(t) = Var\left[\pi(t) \mid F_t^\Pi\right]$ represents the variance of the estimation error for the latent inflation rate at time t and provides measurements of the uncertainty through the learning process. The instantaneous changes in the drift and the variance of $\hat{\pi}(t)$ can be written as:

$$\begin{aligned} d\hat{\pi}(t) &= [a_0(t) + a_\pi(t)\hat{\pi}(t)]dt + \frac{\sigma_\pi \xi_\pi(t) + v(t)}{\xi_\pi^2(t)} \left[\frac{d\Pi(t)}{\Pi(t)} - \hat{\pi}(t)dt \right] \qquad (3.5) \\ \frac{dv(t)}{dt} &= 2a_\pi(t)v(t) + \sigma_\pi^2 + \sigma_\tau^2 - \left[\frac{\sigma_\pi \xi_\pi(t) + v(t)}{\xi_\pi(t)} \right]^2. \end{aligned}$$

To simplify the structure in this economy, we assume that the inference reaches a steady state. In other words, the variance of the posterior mean processes does not change with the updated information. Thus $dv(t) = 0$ and the steady state variance $v(t)$ would not be a significant state variable in our portfolio choice problem. It then follows that the variance $v(t)$ of is positive and satisfies:

$$2a_\pi(t)v(t) + \sigma_\pi^2 + \sigma_\tau^2 - \left[\frac{\sigma_\pi \xi_\pi(t) + v(t)}{\xi_\pi(t)} \right]^2 = 0 \qquad (3.6)$$

The new innovation process $\hat{z}_\pi$, defined as the normalized deviation of the inflation rate from its posterior mean, is given by:

$$\xi_\pi(t)d\hat{z}_\pi = (\pi(t) - \hat{\pi}(t))\, dt + \xi_\pi(t)dz_\pi \qquad (3.7)$$

Although z_π is not observable, $\hat{z}_\pi$ can be computed conditionally from the observable CPI process. The process $\hat{z}_\pi$ does not contain the noise process dz_τ because dz_π and dz_τ are independent noise processes. Eq.(3.7) implies that $d\Pi/\Pi$ satisfies the following Ito process:

$$\frac{d\Pi}{\Pi} = \pi dt + \xi_\pi \left(d\hat{z}_\pi - \frac{\pi - \hat{\pi}}{\xi_\pi} dt \right) = \hat{\pi} dt + \xi_\pi d\hat{z}_\pi \qquad (3.8)$$

The dynamics of the inflation rate process now become predictable based on the optimal filtering equations. The estimated inflation rate dynamics is as follows:

$$d\hat{\pi}(t) = [a_0(t) + a_\pi(t)\hat{\pi}(t)]dt + \frac{\sigma_\pi \xi_\pi(t) + v(t)}{\xi_\pi(t)} d\hat{z}_\pi(t) \qquad (3.9)$$

For further simplicity, we assume that $a_0(t)$ and $a_\pi(t)$ are constant with values of a_0 and a_π respectively. We then have:

$$\begin{aligned} d\hat{\pi}(t) &= -a_\pi \left[-\frac{a_0}{a_\pi} - \hat{\pi}(t) \right] dt + \frac{\sigma_\pi \xi_\pi(t) + v(t)}{\xi_\pi(t)} d\hat{z}_\pi(t) \qquad (3.10) \\ &= \alpha(\bar{\pi} - \hat{\pi}(t))dt + \hat{\sigma}_\pi d\hat{z}_\pi(t). \qquad (3.11) \end{aligned}$$

where $\alpha = -a_\pi$, $\bar{\pi} = -a_0/a_\pi$ and $\hat{\sigma}_\pi = [\sigma_\pi \xi_\pi(t) + v(t)]/\xi_\pi(t)$.

3.3 Dynamics of Investment Opportunities

Consider an investor who can trade nominal instantaneous risk-free cash, a stock index, and two nominal bonds with different maturities. Let dz_m adopt the innovation from the market risk, dz_r adopt the innovation from the interest rate risk, and $d\hat{z}_\pi$ adopt the innovation from the inflation rate risk. Following Brennan and Xia (2002), we express the real pricing kernel of the economy, $M(t)$, that determines the expected returns on all securities as:

$$\begin{aligned} \frac{dM(t)}{M(t)} &= -r(t)dt + \phi_m dz_m + \phi_r dz_r + \phi_\pi d\hat{z}_\pi \\ &= -r(t)dt + \phi' dZ \end{aligned} \tag{3.12}$$

where $\phi = \begin{bmatrix} \phi_m & \phi_r & \phi_\pi \end{bmatrix}'$ and $dZ = \begin{bmatrix} dz_m & dz_r & d\hat{z}_\pi \end{bmatrix}'$. Here ϕ represents the constant loadings on the innovations in the economy and determines the associated prices of risks λ_m, λ_r, and λ_π respectively. We denote the drift of the pricing kernel by $-r(t)$ because the instantaneous real riskless interest rate is equal to the negative of the drift of the real pricing kernel in an economy with a fully indexed riskless asset.[6] The real pricing kernel can be rewritten as:

$$\frac{M(t)}{M(s)} = \exp\left\{ -\int_s^t r(u)du + \int_s^t \phi' dZ(u) - \frac{1}{2}\int_s^t \phi'\rho\phi du \right\}, \tag{3.13}$$

where ρ is the correlation matrix of dz_m, dz_r, and $d\hat{z}_\pi$ with rows $\begin{bmatrix} 1 & \rho_{mr} & \rho_{m\pi} \end{bmatrix}$, $\begin{bmatrix} \rho_{m\pi} & 1 & \rho_{r\pi} \end{bmatrix}$, and $\begin{bmatrix} \rho_{m\pi} & \rho_{r\pi} & 1 \end{bmatrix}$.

The investment opportunity set have to depend on the instantaneous real interest rate process $r(t)$ that is assumed to follow the Ornstein-Uhlenbeck process:

$$dr(t) = \kappa(\bar{r} - r)dt + \sigma_r dz_r, \tag{3.14}$$

[6] In other words, the instantaneous real rate of return should be r if an instantaneously riskless asset exists.

where κ, $\bar{r}$, and σ_r are positive constants and dz_r is a standard Brownian motion. The dynamic of zero coupon bond $B(t,T)$ with $r(t)$ is in **Appendix 5**.

We assume there exists the following four underlying assets in the financial market:

1. The price process of the nominal risk-free asset, i.e., cash, is:

$$\frac{dS_0(t)}{S_0(t)} = R(t)dt, \tag{3.15}$$

   where $R(t)$ represents the instantaneous nominal interest rate (see **Appendix 5**).

2. The price process of the stock index $S(t)$ is assumed to be:

$$\frac{dS(t)}{S(t)} = R(t)dt + \sigma_S(dz_m + \lambda_m dt), \tag{3.16}$$

   where σ_S denotes the volatility of the stock index.

3. The price dynamics of rolling nominal bond $B_K(t)$ with different maturities $K = T_1$ and $K = T_2$:

$$\frac{dB_K(t)}{B_K(t)} = R(t)dt + \sigma_B^K(dz_r + \lambda_r dt) + \sigma_\pi^K(d\hat{z}_\pi + \lambda_\pi dt), \tag{3.17}$$

   where $\sigma_B^K = -\kappa^{-1}(1 - e^{-\kappa K})\sigma_r$ and $\sigma_\pi^K = -\alpha^{-1}(1 - e^{-\alpha K})\hat{\sigma}_\pi$. The rolling nominal bond $B_K(t)$ could be traded with cash $S_0(t)$ and zero coupon bond $B(t,T)$.(see **Appendix 5**)

Combining Eq.(3.16) and Eq.(3.17), the asset dynamics of our market can be written as:

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ \frac{dB_{T_1}(t)}{B_{T_1}(t)} \\ \frac{dB_{T_2}(t)}{B_{T_2}(t)} \end{bmatrix} = R(t)dt + \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_B^{T_1} & \sigma_\pi^{T_1} \\ 0 & \sigma_B^{T_2} & \sigma_\pi^{T_2} \end{bmatrix} \begin{bmatrix} \lambda_m \\ \lambda_r \\ \lambda_\pi \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_B^{T_1} & \sigma_\pi^{T_1} \\ 0 & \sigma_B^{T_2} & \sigma_\pi^{T_2} \end{bmatrix} \begin{bmatrix} dz_m \\ dz_r \\ d\hat{z}_\pi \end{bmatrix}^P$$

$$= R(t)dt + \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_B^{T_1} & \sigma_\pi^{T_1} \\ 0 & \sigma_B^{T_2} & \sigma_\pi^{T_2} \end{bmatrix} \left\{ \begin{bmatrix} \lambda_m \\ \lambda_r \\ \lambda_\pi \end{bmatrix} dt + \begin{bmatrix} dz_m \\ dz_r \\ d\hat{z}_\pi \end{bmatrix}^P \right\}$$

$$= R(t)dt + \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_B^{T_1} & \sigma_\pi^{T_1} \\ 0 & \sigma_B^{T_2} & \sigma_\pi^{T_2} \end{bmatrix} \begin{bmatrix} dz_m \\ dz_r \\ d\hat{z}_\pi \end{bmatrix}^Q$$

where $\begin{bmatrix} dz_m \\ dz_r \\ d\hat{z}_\pi \end{bmatrix}^Q = \begin{bmatrix} \lambda_m \\ \lambda_r \\ \lambda_\pi \end{bmatrix} dt + \begin{bmatrix} dz_m \\ dz_r \\ d\hat{z}_\pi \end{bmatrix}^P .$

Let Λ be the column vector of nominal risk premiums

$$\Lambda = \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_B^{T_1} & \sigma_\pi^{T_1} \\ 0 & \sigma_B^{T_2} & \sigma_\pi^{T_2} \end{bmatrix} \begin{bmatrix} \lambda_m \\ \lambda_r \\ \lambda_\pi \end{bmatrix},$$

and denote the market price of risk vector λ as $\lambda' = \begin{bmatrix} \lambda_m & \lambda_r & \lambda_\pi \end{bmatrix}$. For any time $t \geq 0$, $H(t)$ is defined as:

$$H(t) = \exp\left\{ \int_0^t R(s)ds + \int_0^t \lambda' dZ(s) + \frac{1}{2}\int_0^t \lambda'\rho\lambda ds \right\}. \tag{3.18}$$

The process $H(t)$ is called the nominal growth-optimal portfolio, while $H^{-1}(t)$ represents the shadow state-price density. It is known that:

$$E_t^Q \left[\exp\left\{ -\int_t^T R(s)ds \right\} \right] = E_t \left[\frac{H(t)}{H(T)} \right], \tag{3.19}$$

where Q denotes the unique equivalent martingale measure defined by:

$$\frac{dQ}{dP} = \exp\left\{ -\int_0^T R(s)ds - \frac{1}{2}\int_0^T \lambda'\rho\lambda ds \right\}. \tag{3.20}$$

3.4 Optimization Problem

3.4.1 Optimization Criterion

We assume that the investor is a constant relatively risk-averse individual with the following increasing and strictly concave utility function as follows:

$$U(W) = \frac{W^\gamma}{\gamma}, \qquad \gamma \in (-\infty, 1) \setminus \{0\}, \tag{3.21}$$

The proportion of wealth invested in the bond with maturity T_1, the bond with maturity T_2, the stock index, and cash is denoted by w_{B_1}, w_{B_2}, w_S, and $(1 - w_{B_1} - w_{B_2} - w_S)$ respectively. The investor's problem is to maximize the expected utility of terminal nominal wealth:

$$\max_{(w(t))_{t\in[0,T)}\in\mathcal{A}} EU[W(T)], \tag{3.22}$$

where $\mathcal{A}$ is the set of admissible controls and the wealth process $(W(T))_{t\in[0,T]}$ is defined as:

$$\frac{dW(t)}{W(t)} = (1 - w_{B_1} - w_{B_2} - w_S)\frac{dS_0(t)}{S_0(t)} + w_{B_1}\frac{dB_{T_1}(t)}{B_{T_1}(t)} + w_{B_2}\frac{dB_{T_2}(t)}{B_{T_2}(t)} + w_S\frac{dS(t)}{S(t)}.$$

We can write an equivalent representation of the wealth in Eq.(3.22) as:

$$\max_{W\in L_+^P(\Omega,F,P)} EU[W(T)] \tag{3.23}$$

$$s.t. \quad : \quad W \text{ is financed by some } w(t) \in \mathcal{A} \text{ and } E_0^P\left[\frac{W(T)}{H(T)}\right] = W_0.$$

3.4.2 Solutions to the Optimization Problem

The above complete-market setting allows us to solve the investor's optimal portfolio problem using the martingale pricing approach of Cox and Huang (1989), in which the original dynamic optimization problem is mapped into a static variational problem. Eq.(3.23) can thus be solved in a standard way according to Lagrangian theory. The optimal terminal wealth is given by:

$$W^* = I\left(\frac{\theta}{H(T)}\right) = \left(\frac{\theta}{H(T)}\right)^{1/(\gamma-1)}, \tag{3.24}$$

where θ is the Lagrangian multiplier associated with Eq.(3.21) and $I(x)$ can be written as (see Deelstra et al., 2000):

$$I(x) := (U'_+)^{-1}(x) = x^{1/(\gamma-1)}. \tag{3.25}$$

The optimal wealth at time t is associated with the following function and boundary conditions:

$$\begin{aligned} F(H(t),t) &= E_t\left[I\left(\frac{\theta}{H(T)}\right)\frac{H(t)}{H(T)}\right] \\ &= E_t\left[\left(\frac{\theta}{H(T)}\right)^{1/(\gamma-1)}\frac{H(t)}{H(T)}\right] \\ &= \theta^{\frac{1}{\gamma-1}}H(t)E_t\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] \\ &= \theta^{\frac{1}{\gamma-1}}H(t)^{\frac{1}{1-\gamma}}E_t\left[\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}}\right], \\ F(H(0),0) &= \theta^{\frac{1}{\gamma-1}}E_0\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] = W_0, \text{ and} \\ \theta &= \left(\frac{E_0\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}{W_0}\right)^{1-\gamma}. \end{aligned} \tag{3.26}$$

In order to compute Eq.(3.26), we need an expression for $E_t\left[\left(\frac{H(T)}{H(t)}\right)^{\frac{1}{\gamma-1}}\right]$. The following technical computation is used to obtain this.

Lemma 1 *There exist three deterministic functions $k_1(t,c)$, $k_2(t,c)$, and $k_3(t,c)$ such that*

$$E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] = k_1(t,c)\exp\{-r(t)k_2(T-t,c)\}\exp\{-\hat{\pi}(t)k_3(T-t,c)\} \tag{3.27}$$

Proof. (Deelstra et al. (2003)) For $c = 0$ the statement is obvious. We deal with the case of $c \neq 0$ below. From Eq.(3.18), we write:

$$
\begin{aligned}
E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] &= E_t \exp c\left\{\int_t^T R(s)ds + \int_t^T \theta' dZ(s) + \frac{1}{2}\int_t^T \theta'\rho\theta ds\right\} \\
&= E_t \exp c \left\{ \begin{array}{c} \int_t^T \left(R(s) + \frac{1}{2}\theta'\rho\theta\right) ds \\ + \int_t^T \lambda_m dz_m(s) + \int_t^T \lambda_r dz_r(s) + \int_t^T \lambda_\pi d\hat{z}_\pi(s) \end{array} \right\}
\end{aligned}
$$

Since $c\int_t^T \lambda_m dz_m(s) + c\int_t^T \lambda_r dz_r(s) + c\int_t^T \lambda_\pi d\hat{z}_\pi(s)$ is a multivariate normal distribution, its moment generating function leads us to obtain:

$$
\begin{aligned}
E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] &= E_t \exp\left\{\int_t^T \left(R(s) + \frac{1}{2}\theta'\rho\theta\right) ds\right\} \cdot E_t \exp\left\{c\int_t^T \lambda_m dz_m(s)\right\} \\
&\quad \cdot E_t \exp\left\{c\int_t^T \lambda_r dz_r(s)\right\} \cdot E_t \exp\left\{c\int_t^T \lambda_\pi d\hat{z}_\pi(s)\right\} \\
&\quad \cdot \exp\left\{\Psi \cdot c^2 \cdot (T-t)\right\}
\end{aligned}
$$

where $\Psi = \lambda_r\lambda_m\rho_{mr} + \lambda_\pi\lambda_m\rho_{m\pi} + \lambda_r\lambda_\pi\rho_{\pi r}$. From Eq.(3.11) and Eq.(3.14),

$$
\begin{aligned}
\int_t^T \lambda_\pi d\hat{z}_\pi(s) &= \frac{\lambda_\pi}{\hat{\sigma}_\pi}\left[\hat{\pi}(T) - \pi(t) - \alpha\bar{\pi}(T-t) + \alpha\int_t^T \hat{\pi}(s)ds\right] \text{ and} \\
\int_t^T \lambda_\pi d\hat{z}_\pi(s) &= \frac{\lambda_\pi}{\hat{\sigma}_\pi}\left[\hat{\pi}(T) - \pi(t) - \alpha\bar{\pi}(T-t) + \alpha\int_t^T \hat{\pi}(s)ds\right]
\end{aligned}
$$

Substituting these into the above equation, we obtain

$$
E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] = f(T-t, r(t), \hat{\pi}(t), c) E_t\left[e^{\eta_r r(t) - \mu_r \int_t^T r(s)ds}\right] E_t\left[e^{\eta_\pi \hat{\pi}(t) - \mu_\pi \int_t^T \hat{\pi}(s)ds}\right]
$$

where

$$
\begin{aligned}
f(T-t, r(t), \hat{\pi}(t), c) &= \exp c \left\{ \begin{array}{c} (T-t)\left(\frac{1}{2}\left(\theta'\rho\theta + c\lambda_m^2\right) + c\Psi - \xi_\pi\lambda_\pi - \frac{\lambda_r}{\sigma_r}\kappa\bar{r} - \frac{\lambda_\pi}{\hat{\sigma}_\pi}\alpha\bar{\pi}\right) \\ -\frac{\lambda_r}{\sigma_r} r(t) - \frac{\lambda_\pi}{\hat{\sigma}_\pi}\hat{\pi}(t) \end{array} \right\}, \\
\eta_r &= c\frac{\lambda_r}{\sigma_r},\ \mu_r = -c\left(1 + \frac{\lambda_r}{\sigma_r}\kappa\right),\ \eta_\pi = c\frac{\lambda_\pi}{\hat{\sigma}_\pi}, \text{ and } \mu_\pi = -c\left(1 + \frac{\lambda_\pi}{\hat{\sigma}_\pi}\alpha\right)
\end{aligned}
$$

We then follow Deelstra et al. (2003) to compute $E_t\left[e^{\eta_r r(t) - \mu_r \int_t^T r(s)ds}\right]$ and $E_t\left[e^{\eta_\pi \hat{\pi}(t) - \mu_\pi \int_t^T \hat{\pi}(s)ds}\right]$.

(i) Since $I(\eta_r, \mu_r, T-t) \equiv \eta_r r(T) - \mu_r \int_t^T r(s)ds$ is Gaussian, it follows that:

$$E_t\left[e^{\eta_r r(t)-\mu_r \int_t^T r(s)ds}\right] = e^{E_t[I(\eta_r,\mu_r,T-t)]+\frac{1}{2}V_t[I(\eta_r,\mu_r,T-t)]}$$

For $s \geq t$, $r(s) = e^{-\kappa(s-t)}r(t) + \kappa\bar{r}\int_t^s e^{-\kappa(s-u)}du + \sigma_r \int_t^s e^{-\kappa(s-u)}dz_r(u)$. Then

$$\begin{aligned}
E_t\left[r(T)\right] &= r(t)e^{-\kappa(T-t)} + \bar{r}\left(1-e^{-\kappa(T-t)}\right), \\
E_t\left[r(T)\right] &= \frac{\sigma_r^2}{2\kappa}(1-e^{-2\kappa(T-t)}), \\
E_t\left[\int_t^T r(s)ds\right] &= \frac{1-e^{-\kappa(T-t)}}{\kappa}r(t) + \bar{r}(T-t) - \frac{\bar{r}}{\kappa}\left(1-e^{-\kappa(T-t)}\right), \\
V_t\left[\int_t^T r(s)ds\right] &= \sigma_r^2\int_t^T\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du, \\
Cov\left(r(T), \int_t^T r(s)ds\right) &= \sigma_r^2\int_t^T e^{-\kappa(T-u)}\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_t\left[e^{\eta_r r(t)-\mu_r \int_t^T r(s)ds}\right] &= \exp\left\{\begin{array}{c} \eta_r E_t\left[r(t)\right] - \mu_r E_t\left[\int_t^T r(s)ds\right] + \frac{\eta_r^2}{2}V_t\left[r(T)\right] \\ +\frac{\mu_r^2}{2}V_t\left[\int_t^T r(s)ds\right] - \eta_r\mu_r Cov\left(r(T), \int_t^T r(s)ds\right) \end{array}\right\} \\
&= k_r(\eta_r, \mu_r, T-t)e^{-r(t)h_2(\eta_r,\mu_r,T-t)}
\end{aligned}$$

where $h_2(\eta_r, \mu_r, T-t) = -\eta_r e^{-\kappa(T-t)} + \frac{\mu_r}{\kappa}\left(1-e^{-\kappa(T-t)}\right)$ and $k_r(\eta_r, \mu_r, T-t) =$ $\exp\left\{\begin{array}{c} \bar{r}\left(\frac{\mu_r}{\kappa}+\eta_r\right)\left(1-e^{-\kappa(T-t)}\right) - \mu_r\bar{r}(T-t) + \frac{\eta_r^2\sigma_r^2}{4\kappa}\left(1-e^{-2\kappa(T-t)}\right) \\ +\frac{\mu_r^2\sigma_r^2}{2}\int_t^T\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du - \eta_r\mu_r\sigma_r^2\int_t^T e^{-\kappa(T-u)}\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)du \end{array}\right\}$

(ii)By a very similar procedure, we obtain:

$$\begin{aligned}
E_t\left[e^{\eta_\pi \hat{\pi}(t)-\mu_\pi \int_t^T \hat{\pi}(s)ds}\right] &= \exp\left\{\begin{array}{c} \eta_\pi E_t\left[\hat{\pi}(t)\right] - \mu_\pi E_t\left[\int_t^T \hat{\pi}(s)ds\right] + \frac{\eta_\pi^2}{2}V_t\left[\hat{\pi}(T)\right] \\ +\frac{\mu_\pi^2}{2}V_t\left[\int_t^T \hat{\pi}(s)ds\right] - \eta_\pi\mu_\pi Cov\left(\hat{\pi}(T), \int_t^T \hat{\pi}(s)ds\right) \end{array}\right\} \\
&= k_\pi(\eta_\pi, \mu_\pi, T-t)e^{-\hat{\pi}(t)h_3(\eta_\pi,\mu_\pi,T-t)}
\end{aligned}$$

where $h_3(\eta_\pi, \mu_\pi, T-t) = -\eta_\pi e^{-\alpha(T-t)} + \frac{\mu_\pi}{\alpha}\left(1-e^{-\alpha(T-t)}\right)$ and

$$k_\pi(\eta_\pi, \mu_\pi, T-t) = \exp\left\{\begin{array}{c} \bar{\pi}\left(\frac{\mu_\pi}{\alpha}+\eta_\pi\right)\left(1-e^{-\alpha(T-t)}\right) - \mu_\pi\bar{\pi}(T-t) + \frac{\eta_\pi^2\hat{\sigma}_\pi^2}{4\kappa}\left(1-e^{-2\alpha(T-t)}\right) \\ +\frac{\mu_\pi^2\hat{\sigma}_\pi^2}{2}\int_t^T\left(\frac{1-e^{-\alpha(T-u)}}{\alpha}\right)^2 du - \eta_\pi\mu_\pi\hat{\sigma}_\pi^2\int_t^T e^{-\alpha(T-u)}\left(\frac{1-e^{-\alpha(T-u)}}{\alpha}\right)du \end{array}\right\}$$

Finally,

$$E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] = f(T-t, r(t), \hat{\pi}(t), c)k_r(\eta_r, \mu_r, T-t)e^{-r(t)h_2(\eta_r,\mu_r,T-t)}$$
$$k_\pi(\eta_\pi, \mu_\pi, T-t)e^{-\hat{\pi}(t)h_3(\eta_\pi,\mu_\pi,T-t)}$$
$$= k_1(t,c)\exp\{-r(t)k_2(T-t,c)\}\exp\{-\hat{\pi}(t)k_3(T-t,c)\}$$

where

$$k_2(T-t,c) = h_2(\eta_r, \mu_r, T-t) + \eta_r,\ k_3(T-t,c) = h_3(\eta_\pi, \mu_\pi, T-t) + \eta_r, \text{ and}$$
$$k_1(t,c) = k_r(\eta_r, \mu_r, T-t)k_\pi(\eta_\pi, \mu_\pi, T-t)$$
$$\exp c\left\{(T-t)\left(\frac{1}{2}(\theta'\rho\theta + c\lambda_m^2) + c\Psi - \xi_\pi\lambda_\pi - \frac{\lambda_r}{\sigma_r}\kappa\bar{r} - \frac{\lambda_\pi}{\hat{\sigma}_\pi}\alpha\bar{\pi}\right)\right\}$$

■

Remark 2 *Given that the pricing kernel satisfies Eq.(3.12) and that the interest rates follow Eq.(3.14), the risk neutral price at time t of a default-free bond paying 1 dollar at time T (also called the Arrow-Debreu prices of the bond with maturity $t \to T$) is:*

$$E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] = k_1(t,c)\exp\{-r(t)k_2(T-t,c)\}\exp\{-\hat{\pi}(t)k_3(T-t,c)\} \quad \textit{when } c = -1.$$

Equipped with Eq.(3.27), we are able to solve the investment problem in two steps. First, we compute the trading strategy that replicates the portfolio $H(t)$ using the basic assets $S_0(t)$, $B_{T_1}(t)$, $B_{T_2}(t)$ and $S(t)$. Then the strategy replicating $F(H(t),t)$ is written in terms of $H(t)$. Denote the proportions invested at time t in $S_0(t)$, $B_{T_1}(t)$, $B_{T_2}(t)$ and $S(t)$ to duplicate the self-financing portfolio $H(t)$ as:

$$\left(w^H(t)\right)_{t\in[0,T]} = \left[\begin{array}{cccc} 1 - w^H_{B_1} - w^H_{B_2} - w^H_S & w^H_{B_1} & w^H_{B_2} & w^H_S \end{array}\right], t \geq 0$$

We then present the solutions to the above two steps in the following Propositions.

Proposition 3 *According to Lemma 1 and the dynamics of the investment opportunity set in Chapter 3.3, the strategy* $\left(w^H(t)\right)_{t\in[0,T]}$ *is given by:*

$$\begin{aligned}
w^H_{B_1} &= \frac{-\lambda_r\sigma^{T_2}_\pi(t)+\lambda_\pi\sigma^{T_2}_B(t)}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)} \\
w^H_{B_2} &= \frac{\lambda_r\sigma^{T_1}_\pi(t)-\lambda_\pi\sigma^{T_1}_B(t)}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)} \\
w^H_S &= \frac{\lambda_m}{\sigma_S}
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{dH(t)}{H(t)} &= R(t)dt+\lambda_m dz_m(t)+\lambda_r dz_r(t)+\lambda_\pi d\hat{z}_\pi(t) \\
&= [\cdot]\,dt+\frac{\lambda_m}{\sigma_S}\frac{dS(t)}{S(t)} \\
&\quad+\frac{-\lambda_r\sigma^{T_2}_\pi(t)+\lambda_\pi\sigma^{T_{2(t)}}_B}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)}\frac{dB_{T_1}(t)}{B_{T_1}(t)} \\
&\quad+\frac{\lambda_r\sigma^{T_1}_\pi(t)-\lambda_\pi\sigma^{T_1}_B(t)}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)}\frac{dB_{T_2}(t)}{B_{T_2}(t)}
\end{aligned}$$

■

Applying the Itô's formula to $F(H(t),t)$, differentiating both sides of Eq.(3.26), and grouping the coefficients with respect to the dynamic combination of the processes $S_0(t)$, $B_{T_1}(t)$, $B_{T_2}(t)$ and $H(t)$, we obtain the following proposition.

Proposition 4 *The trading strategy replicating* $F(H(t),t)$ *consists of* $S_0(t)$, $B_{T_1}(t)$, $B_{T_2}(t)$ *and* $H(t)$ *with weights as follows:*

$$\begin{aligned}
w^F_{S_0} &= 1-w^F_{B_1}-w^F_{B_2}-w^F_H \\
w^F_{B_1} &= \frac{k_2(T-t,\frac{\gamma}{1-\gamma})\sigma_r\sigma^{T_2}_\pi(t)-k_3(T-t,\frac{\gamma}{1-\gamma})\hat{\sigma}_\pi\sigma^{T_2}_B(t)}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)} \\
w^F_{B_2} &= \frac{k_3(T-t,\frac{\gamma}{1-\gamma})\hat{\sigma}_\pi\sigma^{T_1}_B(t)-k_2(T-t,\frac{\gamma}{1-\gamma})\sigma_r\sigma^{T_1}_\pi(t)}{\sigma^{T_2}_B(t)\sigma^{T_1}_\pi(t)-\sigma^{T_1}_B(t)\sigma^{T_2}_\pi(t)} \\
w^F_H &= \frac{1}{1-\gamma}
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
F(H(t),t) &= \theta^{\frac{1}{\gamma-1}} H(t)^{\frac{1}{1-\gamma}} E_t\left[\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}}\right] \\
&= \theta^{\frac{1}{\gamma-1}} H(t)^{\frac{1}{1-\gamma}} k_1(T-t,\frac{\gamma}{1-\gamma}) \\
&\quad \exp\left\{-r(t)k_2(T-t,\frac{\gamma}{1-\gamma})\right\}\exp\left\{-\hat{\pi}(t)k_3(T-t,\frac{\gamma}{1-\gamma})\right\} \\
\frac{dF}{F} &= \frac{1}{1-\gamma}\frac{dH(t)}{H(t)} - k_2(T-t,\frac{\gamma}{1-\gamma})dr(t) - k_3(T-t,\frac{\gamma}{1-\gamma})d\hat{\pi}(t) + [\cdot]\,dt
\end{aligned}
$$

Proposition 5 *From Proposition 3 and Proposition 4, the optimal dynamic strategy for the optimization problem of Eq.(3.22) is given by:*

$$
\begin{aligned}
w_{S_0} &= 1 - w_{B_1} - w_{B_2} - w_S \\
w_{B_1} &= \frac{k_2(T-t,\frac{\gamma}{1-\gamma})\sigma_r\sigma_\pi^{T_2}(t) - k_3(T-t,\frac{\gamma}{1-\gamma})\hat{\sigma}_\pi\sigma_B^{T_2}(t)}{\sigma_B^{T_2}(t)\sigma_\pi^{T_1}(t) - \sigma_B^{T_1}(t)\sigma_\pi^{T_2}(t)} \\
&\quad + \frac{1}{1-\gamma}\frac{-\lambda_r\sigma_\pi^{T_2}(t) + \lambda_\pi\sigma_B^{T_2}(t)}{\sigma_B^{T_2}(t)\sigma_\pi^{T_1}(t) - \sigma_B^{T_1}(t)\sigma_\pi^{T_2}(t)} \\
w_{B_2} &= \frac{k_3(T-t,\frac{\gamma}{1-\gamma})\hat{\sigma}_\pi\sigma_B^{T_1}(t) - k_2(T-t,\frac{\gamma}{1-\gamma})\sigma_r\sigma_\pi^{T_1}(t)}{\sigma_B^{T_2}(t)\sigma_\pi^{T_1}(t) - \sigma_B^{T_1}(t)\sigma_\pi^{T_2}(t)} \\
&\quad + \frac{1}{1-\gamma}\frac{\lambda_r\sigma_\pi^{T_1}(t) - \lambda_\pi\sigma_B^{T_1}(t)}{\sigma_B^{T_2}(t)\sigma_\pi^{T_1}(t) - \sigma_B^{T_1}(t)\sigma_\pi^{T_2}(t)} \\
w_S &= \frac{1}{1-\gamma}\frac{\lambda_m}{\sigma_S}
\end{aligned}
$$

Proposition 5 shows that the allocation to stock holdings is not related to the covariance matrix. Instead, it depends on the market risk, the associated risk premium, and the risk aversion of the investor. Furthermore, the inflation rate risk can be managed by rebalancing the fixed income portfolio using the two bonds with different maturities.

3.5 Numerical Analyses

This section provides estimates on how significant the effects of learning about the inflation rate influence an investor's terminal wealth and its expected utility.[7] In the first part, we simulate the processes of an optimal dynamic investment portfolio derived by Section 3.4. Secondly, we provide the sensitivity tests for some important parameters of the learning process to show the importance of learning ability. The parameter estimates of the stock return and real interest rate processes are from Brennan and Xia (2002), in which they used monthly data from January 1970 to December 1995 to perform maximum likelihood estimations. The parameter values of the CPI and latent inflation rate processes are chosen to be close to those in Brennan and Xia (2002).[8] They are listed in Table 3.1. The maturities of rolling bonds are chosen to be one year and ten years. The investment horizon is 30 years and the number of simulations is $50,000$.

3.5.1 Optimal Investment Strategy

The optimal asset allocation for different horizons under $\gamma = -1.5$ is plotted in Figure 3.1. The optimal stock allocation (0.868) is independent of the horizon, as **Proposition 5** shows. Figure 3.1 further demonstrates the significant difference between myopic and optimal asset allocation for a long term horizon investor. For instance, the myopic allocation of a one-year nominal rolling bond is about 3.25 while the optimal allocation is more than 5 for $T > 16$. The possible reason is that the maturity of one-year bond is short and easy to sell for the investor, thus, the weight of one-year bond is high. The weight of the ten-year nominal rolling bond is about

[7] The importance of inflation to asset allocation is not repeatedly addressed here since Brennan and Xia (2002) have provided a clear assessment.

[8] The parameters associated with the inflation rate process cannot be directly borrowed from Brennan and Xia (2002) due to the learning mechanism introduced in this paper.

Table 3.1: Model Parameters used in the Numerical Analyses

Notation	Value	Notation	Value
Real interest rate		**Rolling nominal bond process**	
Mean reversion κ	0.631	Market price of interest rate risk λ_r	-0.209
Mean rate $\bar{r}$	0.012	Market price of inflation risk λ_π	-0.105
Volatility σ_r	0.026		
		Stock index process	
Inflation rate		Volatility σ_S	0.158
Mean reversion a_0	0.001458	Market price of market risk λ_m	0.343
Mean ratea_π	-0.027		
Volatility σ_π	0.05	**Correlation coefficient matrix**	
EstimatedVariance v	0.001	ρ_{mr}	-0.129
Volatility ξ_π	0.05	$\rho_{m\pi}$	-0.024
		$\rho_{r\pi}$	-0.061

1.11 while the optimal allocation is negative for $T > 21$. The short position for the ten-year bond is for reducing risk when the time t is close to the end of investment. However, the weight of nominal cash is negative which means, the return rate of nominal cash is too low to invest in for long term investors. Moreover, the short sale ratio of nominal cash increases as time goes by.

3.5.2 Sensitivity Tests

The focus of this part is to demonstrate how important learning about the inflation rate is since the major difference between our paper and Brennan and Xia (2002) is that we assume the investor is not able to observe the instantaneous expected inflation rate perfectly, but has learning capability. We assume that there are two kinds of investors. The investors with learning ability that estimate the parameters of the inflation rate through learning processes and develop investment strategies

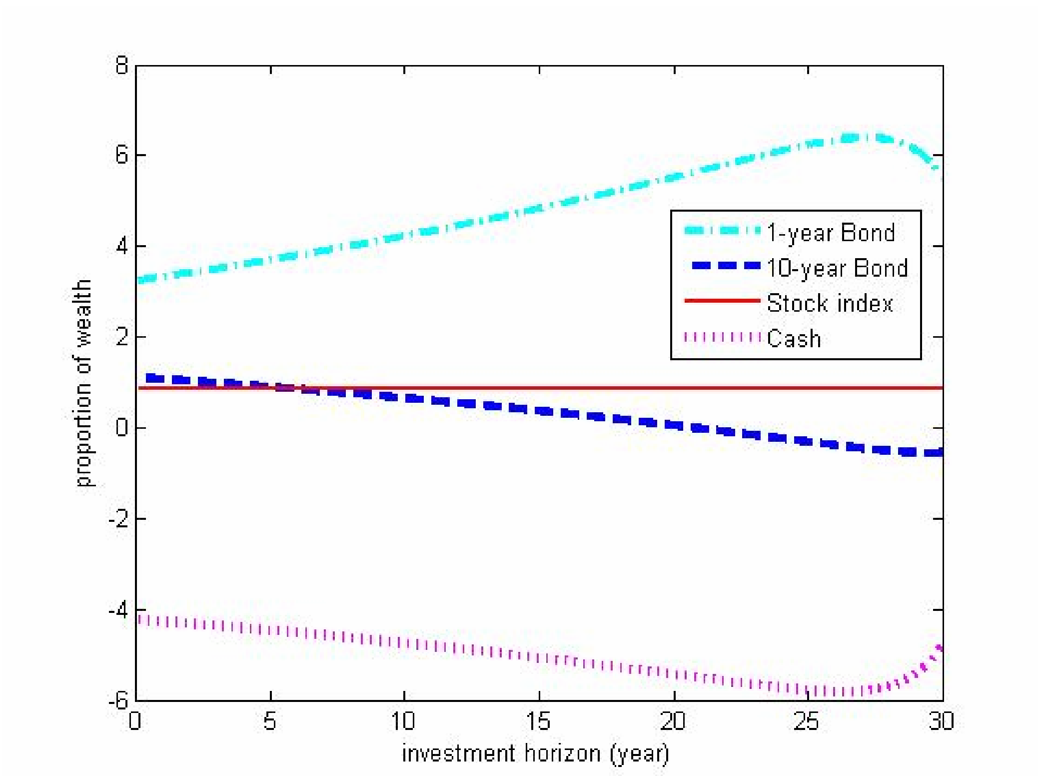

Figure 3.1: The optimal asset allocation strategy ($\gamma = -1.5$and $T = 30$)

(according to **Proposition 5**). The other investors guess the value of the parameters of inflation dynamics and use their conjecture to calculate investment weights. For the investors without learning ability, we assume that a_0 and a_π follow $U(0, 0.002916)$ and $U(-0.054, 0)$ respectively. We compare the mean, variance, and the fifth percentile of terminal wealth and the expected utility value between learning and no-learning investors. We analyze five important parameters which are; the investment horizon (T), the risk-averse parameter (γ), the standard deviation of CPI (ξ_π), the standard deviation of the inflation rate (σ_π), and the variance of the estimation error (ν).

(1) Investment horizon (T)

In Figure 3.1, the twilled and dotted bars represent terminal wealth under the no-learning effect and the learning effect respectively, and the line shows the wealth ratio of the learning effect to the no-learning effect. We find that terminal wealth with the learning effect is larger than terminal wealth without the learning effect. Moreover, the influence is more significant with an increasing investment time horizon. For example, the wealth ratio of the leaning effect to the no-learning effect is 1.02 when T is 5 years; however this ratio becomes 1.09 when T is 30 years.

We compare the variance and the fifth percentile of terminal wealth under the no-learning effect and the learning effect. When the investment horizon is less than 5 years, the learning effect is not significant for the variance of investors' terminal wealth. However, when T is 30 years, the learning method reduces almost a half of variance of the investors' terminal wealth. Similarly, the fifth percentile of terminal wealth under the learning effect is larger than the no-learning effect. This shows that the learning method precisely helps investors to hedge downside risks. Moreover, we calculate the improvement rate of the expected utility value from no-learning to learning. Similarly, the effect of learning on the expected utility value increases with

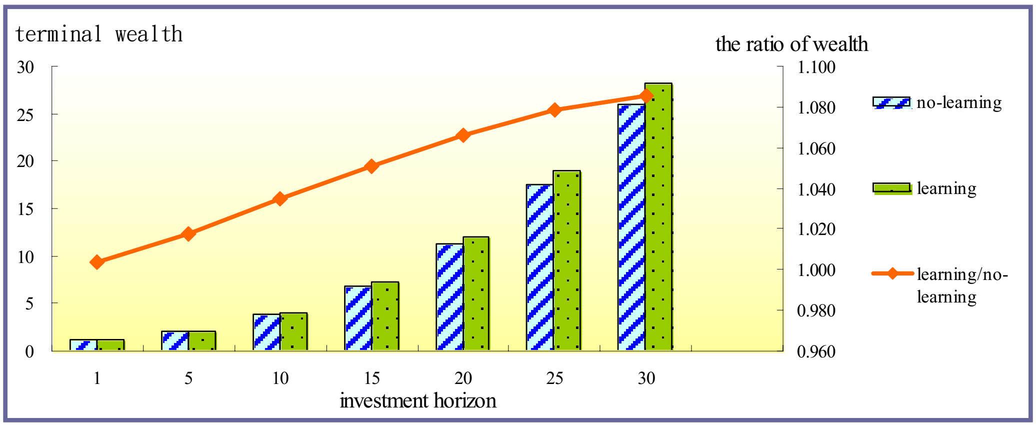

Figure 3.2: The effect on wealth of different investment horizons under the learning method

the investor's time horizon. The reduction in the expected utility is over 22% for the investor with an investment horizon of 30 years, while the utility loss from no-learning is about 0.5% only for the one-year-horizon investor. This is consistent with Brennan (1998), in which he showed that the effect of learning about mean returns on optimal portfolio allocation increases with the investor's time horizon. We present the results on Table 3.2.

(2) Risk averse parameter (γ)

Learning is also more important to investors who are risk averse. γ represents the risk attitude of investors. More risk-averse investors have lower γ. Figure 3.3 exhibits the influence of terminal wealth with learning capability under different risk-averse parameters, and the twilled and dotted bars represent terminal wealth under the no-learning effect and the learning effect respectively, and the line shows the wealth ratio of the learning effect to the no-learning effect. The terminal wealth of more

Table 3.2: The analysis of learning effect under different investment horizons

Investment Horizon (T)	Variance of terminal wealth (learning)	Variance of terminal wealth (no-learning)	The 5th percentile of terminal wealth $\left(\frac{\text{learning}}{\text{no learning}}\right)$	Increment of utility value with learning capability (%)
1	0.001	0.001	1.127 1.121	0.537
5	0.012	0.015	1.932 1.873	2.735
10	0.125	0.168	3.511 3.289	5.565
15	0.857	1.164	5.837 5.263	8.593
20	4.420	6.205	8.986 7.752	12.081
25	17.646	26.812	12.891 10.567	16.367
30	59.200	101.526	17.503 13.369	22.016

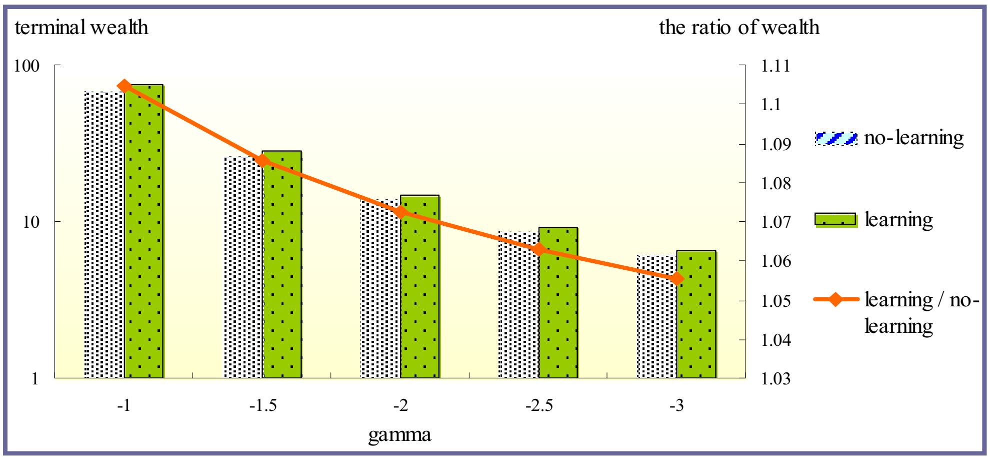

Figure 3.3: The effect on wealth of different risk-averse parameters under the learning method

risk-averse investors decreases because they prefer less risky portfolios. On the other hand, Figure 3.3 shows that learning capability could enhance the mean of terminal wealth under different risk-averse parameters.

In the part on variance and the fifth percentile, learning capability precisely reduces the variance and increases the value of the fifth percentile of terminal wealth. The variance of terminal wealth decreases when investors are more risk-averse because they are more conservative. The influence of utility values is also presented in Table 3.3. The decrease in the expected utility is almost 35% for investors with $\gamma = -3$; whereas, for investors with $\gamma = -1$, the utility loss from no-learning is 16.743%. This is reasonable because learning is no longer important when the investor acts myopically and the hedge demands are close to zero.

(3) Standard deviation of CPI (ξ_π)

Table 3.3: The analysis of the learning effect under different risk-aversion parameters

Risk averse parameter (γ)	-1	-1.5	-2	-2.5	-3
Variance of terminal wealth (learning)	508.927	59.200	13.984	4.970	2.284
Variance of terminal wealth (no-learning)	835.939	101.526	24.678	8.950	4.175
The 5th percentile of terminal wealth ($\frac{\text{learning}}{\text{no learning}}$)	43.896 31.894	17.503 13.369	9.459 7.446	6.079 4.886	4.359 3.562
Increment of utility value with learning capability (%)	16.743	22.016	26.629	30.893	34.949

In this part, we simulate $50,000$ instances of terminal wealth under different ξ_π between 0.04 and 0.08. Firstly, the terminal wealth increases when ξ_π goes up. This is because when ξ_π increases, investors hold more ten-year nominal rolling bonds and fewer one-year nominal rolling bonds. This more risky portfolio raises terminal wealth. However, we find that the wealth ratio of the learning effect to the no-learning effect declines with increasing ξ_π. This is because as ξ_π increases investors find it more difficult to predict the inflation rate with CPI information.

Table 3.4 presents the variance and fifth percentile of terminal wealth and improvements on expected utility value. The learning capability accurately reduces the volatility of accumulated wealth and improves downside risks for diminishing variance while increasing the fifth percentile. This mechanism also positively increases the investor's utility value; however, this improvement decreases with an increasing value of the standard deviation of CPI. (When ξ_π raises from 0.04 to 0.08, the improved ratio decreases over 3% to 19.912%). This is reasonable because

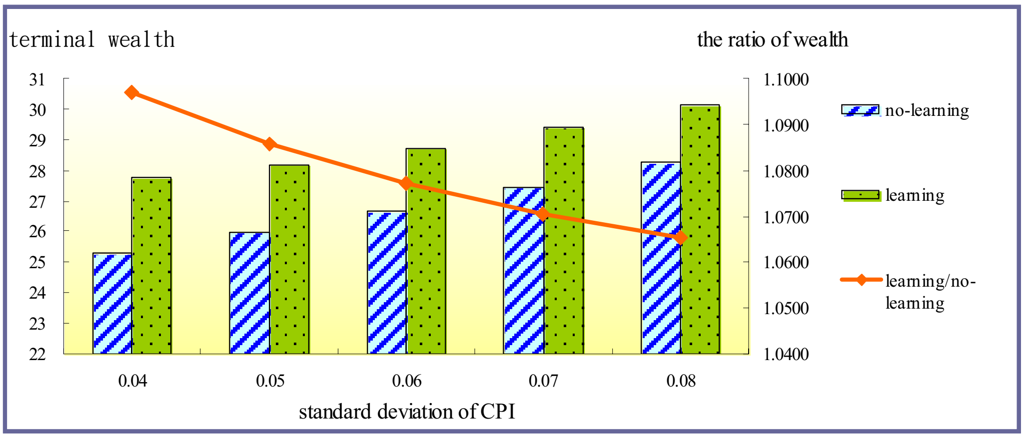

Figure 3.4: The effect on wealth of different standard deviations of CPI under the learning method

the larger ξ_π represents CPI as more volatile and investors seldom use it to predict the inflation rate.

(4) Standard deviation of the inflation rate (σ_π)

As σ_π increases, the optimal weights of one-year and ten-year nominal rolling bonds slightly increase and decrease respectively. Therefore, terminal wealth declines with increasing σ_π as presented in Figure 3.5. Then, we compare the terminal wealth between learning ability and no-learning ability. We find that investors with learning capability have enhanced terminal wealth.

Learning capability also deduces the variance of terminal wealth and the improved margin is larger when the inflation rate is more volatile. In Table 3.4, we find that the reduction in the variance is over 65% when σ_π is 0.08, while the variance improvement from no-learning is about 41% only when σ_π is 0.04. Table 3.4 also shows that investors with learning ability could hedge downside risk. In

Table 3.4: The analysis of the learning effect under different standard deviations of CPI

Standard Deviation of CPI (ξ_π)	0.04	0.05	0.06	0.07	0.08
Variance of terminal wealth (learning)	67.918	59.200	54.398	51.675	50.210
Variance of terminal wealth (no-learning)	105.770	101.526	100.650	101.694	104.004
The 5th percentile of terminal wealth ($\frac{\text{learning}}{\text{no learning}}$)	16.514 12.584	17.503 13.369	18.401 14.067	19.239 14.713	20.056 15.349
Increment of utility value with learning capability (%)	23.152	22.016	21.149	20.465	19.912

other words, they enhance the value of the fifth percentile of terminal wealth. Subsequently, we calculate the improvement ratio of the expected utility value from no-learning ability to learning ability. When the value of σ_π is doubled to 0.08, the increment of utility value with learning capability also becomes two-fold to 44.411%. This means that investors need more learning mechanisms to predict the inflation rate when the volatility of the inflation rate is high.

(5) Variance of the estimation error (ν)

In Figure 3.6, we illustrate the terminal wealth of learning and no-learning investors under different values of the variance of the estimation error. We find that the learning capability of investors increases their terminal wealth. Table 3.6 displays that learning capability also reduces the variance of terminal wealth. Moreover, the higher fifth percentile of terminal wealth under the learning process shows the importance of learning mechanisms for long term investors. However, the utility

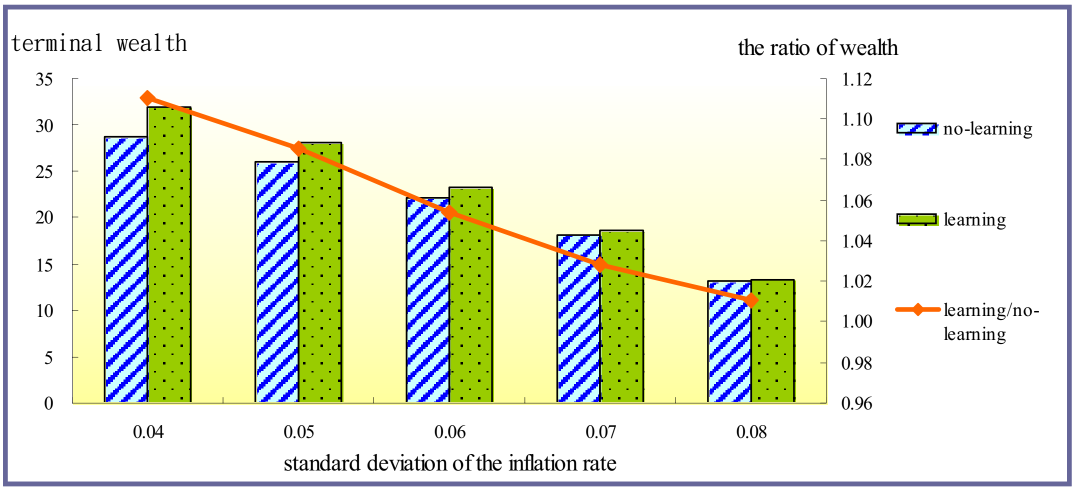

Figure 3.5: The effect on wealth of different standard deviations of the inflation rate under the learning method

Table 3.5: The analysis of the learning effect under different standard deviations of the inflation rate

Standard Deviation of inflation rate (σ_π)	0.04	0.05	0.06	0.07	0.08
Variance of terminal wealth (learning)	44.630	59.200	67.807	68.552	70.332
Variance of terminal wealth (no-learning)	76.424	101.526	136.002	164.922	198.376
The 5th percentile of terminal wealth $(\frac{\text{learning}}{\text{no learning}})$	22.278 15.671	17.503 13.369	12.494 8.967	8.118 5.256	4.869 2.435
Increment of utility value with learning capability (%)	20.617	22.016	26.745	34.677	44.411

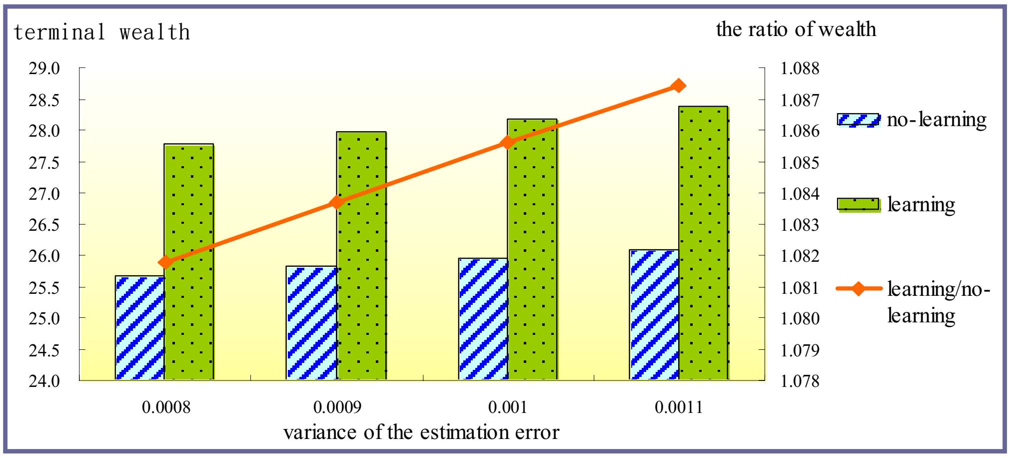

Figure 3.6: The effect on wealth of different variances of the estimation error under the learning method

loss from no-learning investors decreases from 22.824% to 21.725% when ν increases from 0.0008 to 0.0011. When ν increases, the estimation of the inflation rate becomes more volatile and lowers the improvement of the learning process.

3.6 Discussions

Determining a dynamic investment strategy to hedge against inflation uncertainty is a crucial issue for all long term investors. The goal of these investors is no longer to beat the market index, but rather to devise appropriate long term strategies that will move them to their objectives with controlled risks. Campbell and Viceira (2001) and Brennan and Xia (2002) developed models and strategies for these investors and examined how inflation risk affects dynamic asset allocation. In this paper, we extend their studies by introducing learning about inflation into allocation decisions. More specifically, we assume that the investor is not able to perfectly observe the

Table 3.6: The analysis of the learning effect under different variances of the estimation error

Variance of the estimation error (ν)	0.0008	0.0009	0.001	0.0011
Variance of terminal wealth (learning)	49.341	54.184	59.200	64.398
Variance of terminal wealth (no-learning)	96.201	98.731	101.526	104.573
The 5th percentile of terminal wealth $(\frac{\text{learning}}{\text{no learning}})$	17.912 13.287	17.702 13.350	17.503 13.369	17.307 13.360
Increment of utility value with learning capability (%)	22.824	22.376	22.016	21.725

instantaneous expected inflation rate but has learning capabilities to changes his/her assessment of the rate.

The investor's problem is to find the optimal asset allocation in the presence of inflation rate risk, interest rate risk, and market risk when there is no instantaneously riskless asset. We assume that the investor employs CPI information to project inflation rates through optimal filtering equations. More specifically, the inflation process is inferred from the learning process based on a Bayesian approach. Assuming that the financial market is complete, we are then able to use the martingale pricing approach to solve the investor's optimal portfolio problem.

We find that the optimal strategy consists of two main components: a stock index portfolio and a fixed income portfolio with various maturities. The stock index holding remains constant, and its weight relies heavily on the risk attitude parameter. The dynamic optimal fixed income portfolio is a combination of two different zero-coupon bonds to hedge inflation risk and the interest rate risk. These can be considered as trading vehicles to manage the uncertainty observed in the market over time. The hedging is created by fine-tuning the maturity of the fixed income portfolio. In short, the optimal strategy for investors to hedge against inflation uncertainty is to incorporate a dynamic fixed income portfolio with different maturities. The hedge demand can be rather significant, according to our numerical analyses.

More importantly, we find that the welfare loss from no-learning investors is significant. We compare the differences in mean, variance, and the fifth percentile of terminal wealth and the improvement ratio of the expected utility between the investor with learning capability and the investor who never updates his/her assessment of the inflation rate conditionally by observing the realized CPI. We analyze five important parameters which are investment horizon (T), risk-averse parameter (γ), the standard deviation of CPI (ξ_π), the standard deviation of the instantaneous inflation rate (σ_π), and the variance of the estimation error (ν). The differences are found to be large. The effect of learning on the expected utility value increases with the investor's time horizon, the investor's risk averseness, and the standard deviation of the instantaneous inflation rate. Taking the learning effect into account is, therefore, important for conservative long-term investors. Moreover, investors need to use the learning process to predict the inflation rate under more volatile inflation dynamics. On the other hand, the learning process also enhances the expected utility value under different values of the standard deviation of CPI and the variance of the estimation error. However, the improvement ratio decreases with increasing standard deviation of CPI and the variance of the estimation error. This shows that when the estimated error and volatility of predicted variable raise, the effects of the learning process reduce.