Appendix

1. Replication of Rolling Bond under Vasciek Interest Rate Model

In reality, given the instantaneous short interest rate (1.1), we could assume that there exists tradable zero coupon bonds. Then, the return of a zero coupon bond with maturity T is given by:

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= (r(t) - \beta(t,T)\lambda_r)\, dt + \beta(t,T) dz_r(t), \\ \beta(t,T) &= \frac{1 - e^{-a(T-t)}}{a} \sigma_r. \end{aligned}$$

Nevertheless, as pointed out in Boulier et al. (2001), it is quite unrealistic to assume the existence of infinite zero coupon bonds. Furthermore, a rolling bond seems to be very useful for fund managers and they argue that the asset allocation problem can be solved by just taking into account this bond without any loss of generality. In fact, the values $B(t,T)$ and $B_K(t)$ are linked (through the cash) by the following equation:

$$\frac{dB(t,T)}{B(t,T)} = \left(1 - \frac{\beta(t,T)\sigma_r}{\sigma_B^K}\right) \frac{dS_0(t)}{S_0(t)} + \frac{\beta(t,T)\sigma_r}{\sigma_B^K} \frac{dB_K(t)}{B_K(t)},$$

This means that the original bond can be obtained through a suitable portfolio (i.e. a linear combination) of the cash and the $B_K(t)$ bond. The diffusion matrix for the considered financial market is given by:

$$\Sigma = \begin{bmatrix} \sigma_{Sr} & \sigma_{Sm} \\ -\sigma_B^K & 0 \end{bmatrix},$$

and since σ_{Sr} and σ_{Sm} are different from zero by hypothesis, and $\sigma_B^K \neq 0$ by construction, it follows that:

$$\det \Sigma = \sigma_{Sm}\sigma_B^K \neq 0.$$

2. DEVIATION OF *Eq.*(1.15)

Now we are interested in the second component $w^*_{G(2)}$ of the optimal portfolio, which is the state variables hedge portfolio. Since the quadratic term $M'(\Gamma\Gamma')^{-1}M$ does not depend on the state variables, this term is deleted. We rearrange the $w^*_{G(2)}$ as the following equation.

$$w^*_{G(2)} = -\frac{1}{\beta_2}\frac{1}{W_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega \cdot \int_t^T \frac{\partial}{\partial \nu} E_t\left[Q_1 + Q_2\right] ds,$$

where

$$\begin{aligned} Q_1 &= [W_N(r - re - \mu_\pi)] - W_N\Phi\Gamma'(\Gamma\Gamma')^{-1}M + \frac{1}{2}\beta_2 W_N^2\sigma_\pi^2, \\ Q_2 &= \gamma L\mu_L - \gamma L\Lambda\Gamma'(\Gamma\Gamma')^{-1}M + \frac{1}{2}\beta_2[-2\gamma L W_N \sigma_L\sigma_\pi + \gamma^2 L^2\sigma_L^2]. \end{aligned}$$

Accordingly, the derivative of the expected value in Eq.(1.15) can be written as follows:

$$\begin{bmatrix} \frac{\partial}{\partial r(t)} E_t[Q_1 + Q_2] \\ \frac{\partial}{\partial L(t)} E_t[Q_1 + Q_2] \end{bmatrix} = \begin{bmatrix} (1-e)W_N(t) \\ \gamma\mu_L - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi W_N(t) + \beta_2\gamma^2\sigma_L^2 E_t[\widetilde{L}] \end{bmatrix}.$$

The only term we have to compute is the expected value of the modified process of labor incomes, that is to say the modified real contribution. Now we carry out the necessary computation for the modified process of L. In particular, we need to compute the matrix product:

$$[-M'(\Gamma\Gamma')^{-1}\Gamma\Omega + \beta_2(W_N\Phi' + \gamma L\Lambda')(I - \Gamma'(\Gamma\Gamma')^{-1}\Gamma)\Omega]'.$$

For simplicity, we assume that $\Gamma'(\Gamma\Gamma')^{-1}\Gamma = I$. According to what has already presented in the previous sections, we can write:

$$[-M'(\Gamma\Gamma')^{-1}\Gamma\Omega]' = \begin{bmatrix} \delta_1 \\ L\delta_2 \end{bmatrix},$$

where δ_1 and δ_2 are given by

$$\begin{aligned}
\delta_1 &= \sigma_r \cdot \lambda_r, \\
\delta_2 &= \gamma\beta_2 L\sigma_L^2 - \beta_2 W_N \sigma_L \sigma_\pi - \frac{2\sigma_{Lm}\sigma_{Sr}\lambda_r}{\sigma_{Sm}} - \sigma_{Lm}\lambda_m + \sigma_{Lr}\lambda_r.
\end{aligned}$$

Thus, the modified differential of the state variables $\widetilde{\nu}_s$ can be written as:

$$\begin{bmatrix} d\widetilde{r} \\ \frac{d\widetilde{L}}{\widetilde{L}} \end{bmatrix} = \begin{bmatrix} a(b-\widetilde{r}) - \delta_1 \\ \mu_L - \delta_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_r & 0 & 0 \\ \sigma_{Lr} & \sigma_{Lm} & \sigma_L \end{bmatrix} \begin{bmatrix} dz_r \\ dz_m \\ dz_L \end{bmatrix}.$$

In particular, for $s \geq t$, the solution of the interest rate process is

$$\widetilde{r}(s) = \widetilde{r}(t)e^{a(t-s)} + \frac{ab-\delta_1}{a}(1-e^{a(t-s)}) + \sigma_r e^{-as}\int_t^s e^{a\tau} dz_r(\tau).$$

The solution of the labor incomes process is

$$\begin{aligned}
\widetilde{L}(s) &= \widetilde{L}(t)\exp[(\mu_L - \delta_2 - \frac{1}{2}\sigma_{Lr}^2 - \frac{1}{2}\sigma_{Lm}^2 - \frac{1}{2}\sigma_L^2)(s-t) \\
&\quad + \sigma_{Lr}(z_r(s) - z_r(t)) + \sigma_{Lm}(z_m(s) - z_m(t)) + \sigma_L(z_L(s) - z_L(t))].
\end{aligned}$$

Then, according to the boundary equation ($\widetilde{\nu}(s) = \nu(s)$), we can get the expected value:

$$E_t[\widetilde{L}(s)] = L(t)e^{R(s-t)},$$

where

$$\begin{aligned}
R(s-t) &= (\mu_L - \delta_2 - \frac{1}{2}\sigma_{Lr}^2 - \frac{1}{2}\sigma_{Lm}^2 - \frac{1}{2}\sigma_L^2)(s-t) \\
&\quad + \sigma_{Lr}(z_r(s) - z_r(t)) + \sigma_{Lm}(z_m(s) - z_m(t)) + \sigma_L(z_L(s) - z_L(t)).
\end{aligned}$$

Thus, the integral defining the second component of the optimal portfolio is given by:

$$\int_t^T \frac{\partial}{\partial \nu} E_t\left[Q_1 + Q_2\right] ds = \begin{bmatrix} (1-e)W_N(t) \\ \gamma\mu_L - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi W_N(t) + \beta_2\gamma^2\sigma_L^2 L(t)e^{R(s-t)} \end{bmatrix}.$$

Finally, we rewrite the second optimal portfolio component as

$$\begin{aligned} w^*_{G(2)} &= \frac{-1}{\beta_2 W_N(1-e)}(\Gamma\Gamma')^{-1}\Gamma\Omega \\ &\times \begin{bmatrix} (1-e)W_N(t) \\ \gamma\mu_L - \gamma\Lambda\Gamma'(\Gamma\Gamma')^{-1}M - \beta_2\gamma\sigma_L\sigma_\pi W_N(t) + \beta_2\gamma^2\sigma_L^2 L(t)e^{R(s-t)} \end{bmatrix}. \end{aligned}$$

3. Replication of Rolling Bond under CIR Interest Rate Model

In Chapter 2, we apply the CIR interest rate model, then according to Cox et al. (1985), we have the dynamics and price processes of the zero coupon bond with respect to Eq.(2.1).

1. The stochastic process of zero coupon bond $B(t,T)$ is as follows:

$$\frac{dB(t,T)}{B(t,T)} = \left[r(t) + \sigma_B(T-t, r(t))\lambda_r\sqrt{r(t)}\right]dt + \sigma_B(T-t, r(t))dz_r(t), \quad (2.19)$$

   with volatility $\sigma_B(T-t, r(t)) = \sigma_r h(T-t)\sqrt{r(t)}$, where

$$h(t) = \frac{2(e^{\delta t}-1)}{\delta - (b-\sigma_r\lambda_r) + e^{\delta t}(\delta + b - \sigma_r\lambda_r)}, \; \delta = \sqrt{(b-\sigma_r\lambda_r)^2 + 2\sigma_r^2}. \quad (2.20)$$

2. The price of the bond is given by the explicit formula:

$$B(t,T) = \exp\left\{-a\phi(T-t) - r(t)h(T-t)\right\},$$

   where $\phi(t) = -\frac{2}{\sigma_r^2}\log\left[\frac{2\delta\exp\left(\frac{1}{2}(\delta+b-\sigma_r\lambda_r)t\right)}{\delta-(b-\sigma_r\lambda_r)+(\delta+b-\sigma_r\lambda_r)\exp(\delta t)}\right]$, $t \geq 0$. Note that the function $h(t)$ is the derivation of $\phi(t)$.

3. We assume that there exists tradable zero coupon bonds, then the following equation gives the relationship between $B(t,T)$ and $B_K(t)$ through the cash $S_0(t)$:

$$\frac{dB(t,T)}{B(t,T)} = \left(1 - \frac{\sigma_B(T-t,r(t))}{\sigma_B^K}\right)\frac{dS_0(t)}{S_0(t)} + \frac{\sigma_B(T-t,r(t))}{\sigma_B^K}\frac{dB_K(t)}{B_K(t)}.$$

4. Deviation of $Eq.(2.18)$

In this part, the explicit solutions $(w(t))_{t\in[0,T]}$ of Eq.(2.10) are derived. First, according to Deelstra et al. (2000) and Eq.(2.9), we deduce:

$$I(x) : (U'_+)^{-1}(x) = x^{1/(\gamma-1)},\ 0 < x < \infty$$

so that $Y^* = \left(\frac{\theta}{H(T)}\right)^{1/(\gamma-1)}$.

The optimal wealth at time t is associated with the following function and boundary conditions:

$$F(H(t),t) = \theta^{1/(\gamma-1)}H(t)E_t[H(T)^{\gamma/(1-\gamma)}]$$

where θ is determined by $\theta^{1/(\gamma-1)}E[H(T)^{\gamma/(1-\gamma)}] = Y_0$. Therefore, θ is $\left(\frac{E[H(T)^{\gamma/(1-\gamma)}]}{Y_0}\right)^{1-\gamma}$.

Now, we need to compute $E_t\left[\left(\frac{H(T)}{H(t)}\right)^{\gamma/(1-\gamma)}\right]$, and we apply Deelstra et al. (2000) to derive this term, which progress is as the following **Proposition 6**.

Proposition 6 *(Deelstra et al. (2003)) Suppose that c is a real number such that* $c\left(1+\frac{\lambda_r^2}{2}-\frac{\lambda_r b}{\sigma_r}\right) \le \frac{b^2}{2\sigma_r^2}$*,then there exist two deterministic functions* $k_1(t,c)$, $k_2(t,c)$ *such that*

$$E_t\left[\left(\frac{H(T)}{H(t)}\right)^c\right] = k_1(T-t,c)\exp\{-r(t)k_2(T-t,c)\}. \tag{2.21}$$

with

$$\phi_{\eta,\mu}(t) = -\frac{2}{\sigma_r^2}\log\left(\frac{2\alpha e^{(\alpha+b)t/2}}{\sigma_r^2\eta(e^{\alpha t}-1)+\alpha-b+e^{\alpha t}(\alpha+b)}\right),$$

$$\phi'_{\eta,\mu}(t) = \psi_{\eta,\mu}(t) = \frac{\eta(\alpha+b+e^{\alpha t}(\alpha-b))+2\mu(e^{\alpha t}-1)}{\sigma_r^2\eta(e^{\alpha t}-1)+\alpha-b+e^{\alpha t}(\alpha+b)},$$

and $\alpha = \sqrt{b^2 + 2\sigma_r^2 \mu}$, $\mu = -c\left(1 + \frac{\lambda_r^2}{2} - \frac{\lambda_r b}{\sigma_r}\right)$, $\eta = c\frac{\lambda_r}{\sigma_r}$

$$k_1(t, c) = \exp\left\{c\left(c\frac{\lambda_m^2}{2} + \frac{\lambda_m^2}{2} + \frac{\lambda_r a}{\sigma_r}\right)t - a\phi_{\eta,\mu}(t)\right\}, \qquad (2.22)$$

$$k_2(t, c) = -\frac{\lambda_r}{\sigma_r}c + \psi_{\eta,\mu}(t). \qquad (2.23)$$

According to Eq.(2.21), we could to solve the asset allocation problem with two steps. First, we have to replicate the portfolio $H(t)$ in terms of the main securities $(S_0(t), B_K(t), S(t))$. Second, we compute the investment strategy to replicate $F(H(t), t)$ in terms of $H(t)$, that is the current wealth $Y(t)$. We denote by $y^H(t)_{t\in[0,T]} = [(1 - y_B^H - y_S^H), y_B^H, y_S^H]$, $t \geq 0$, which are proportions invested into $S_0(t)$, $B_K(t)$ and $S(t)$ respectively necessary to duplicate $H(t)$.

Proposition 7 *The strategy* $\left(y^H(t)\right)_{t\in[0,T]}$ *is given by:*

$$\left[y_B^H, y_S^H\right] = \left[\frac{\sigma_{Sm}\lambda_r - \sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)}, \frac{\lambda_m}{\sigma_{Sm}}\right] \qquad (2.24)$$

where the function $h(K)$ *is defined by (2.20).*

Proof. Following for example Karatzas (1989), the weights of the strategy are given by $\begin{bmatrix} y_B^H \\ y_S^H \end{bmatrix} = \left(\sigma(t, r(t))'\right)^{-1} \lambda(t, r(t))$. ■

Notice that Eq.(2.24) is the solution of Eq.(2.16) where $U(y) = \log(y)$, and for this reason the process $H(t)$ is also called the logarithmic portfolio in Long (1990).

Applying the Itô's formula to $F(H(t), t)$ and grouping the coefficients with respect to the dynamic combination of the processes $S_0(t)$, $B_K(t)$ and $S(t)$, we obtain the trading strategy as following which replicates $F(H(t), t)$:

$$\begin{aligned} y_{S_0} &= 1 - y_B - y_S \\ y_B &= \frac{k_2(T-t, \gamma/(1-\gamma))}{h(K)} + \frac{1}{1-\gamma}\frac{\sigma_{Sm}\lambda_r - \sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)} \\ y_S &= \frac{1}{1-\gamma}\frac{\lambda_m}{\sigma_{Sm}} \end{aligned} \qquad (2.25)$$

Hence the solution $(w(t))_{t\in[0,T]}$ is given by the following equation:

$$W(t)\,\hat{w}(t) = Y(t)\,\hat{y}(t) + G(t)\left[\sigma(t, r(t))'\right]^{-1}\lambda(t) + H(t)\left[\sigma(t, r(t))'\right]^{-1}\underline{\rho}(t) \quad (2.26)$$

where $\hat{w}'(t) = (w_B, w_S)$, $\hat{y}'(t) = (y_B, y_S)$, $\sigma = \begin{bmatrix} 0 & \sigma_B^K \\ \sigma_{Sm} & \sigma_{Sr}\sqrt{r(t)} \end{bmatrix}$, and $\lambda(t) = \left[\lambda_{Sm}, \lambda_{Sr}\sqrt{r(t)}\right]'$.

In order to find $\hat{w}(t)$, we need to find the process $(G(t), \underline{\rho}(t))$. The following part will introduce this part.

For a given process $N(t)$ let us denote $\tilde{N}(t) = H^{-1}(t)N(t)$. We have $d\tilde{Y}(t) = d\tilde{W}(t) - d\tilde{G}(t)$. From previous equations, easy computations leads

$$\begin{aligned} d\left(\tilde{W}(t)\right) &= W(t)dH^{-1}(t) + H^{-1}(t)dW(t) + dH^{-1}(t)dW(t) \\ &= \tilde{W}(t)\left(\frac{dH^{-1}(t)}{H^{-1}(t)} + \frac{dW(t)}{W(t)} + \frac{dH^{-1}(t)}{H^{-1}(t)}\frac{dW(t)}{W(t)}\right) \\ &= \tilde{W}(t)\left(\hat{w}'(t)\sigma(t, r(t)) - \lambda'(t)\right)dZ(t) \end{aligned} \quad (2.27)$$

Using the martingale representation theorem for the Brownian motion (Karatzas and Shreve (1991)), we have the following proposition.

Proposition 8 *There exists a unique square integrable process* $\left(\underline{\rho}(t)\right)_{t\in[0,T]}$ *with* $\underline{\rho}(t) = (\rho(t), \rho^r(t))'$, *satisfying* $\int_0^T \left|\underline{\rho}(t)\right|^2 dt < +\infty \quad P - a.e.$ *such that:*

$$d\left(\tilde{G}_t(t)\right) := d\left(E_t\left[\tilde{G}_T\right]\right) = \underline{\rho}'(t)dZ(t) \quad (2.28)$$

where $\left(\underline{\rho}(t)\right)_{t\in[0,T]}$ *is given by:*

$$\begin{aligned} \rho(t) &= -H^{-1}(t)\,G(t)\lambda_m \\ \rho^r(t) &= H^{-1}(t)\,G(t)\left(\sigma_B(T-t, r(t)) - \lambda_r\sqrt{r(t)}\right) \end{aligned}$$

Proof. (1) We have known that

$$G(t) = E_t\left[\frac{H(t)}{H(T)}G_T\right] = W_0 \exp\left\{\int_0^T g(\tau)d\tau\right\} E_t\left[\frac{H(t)}{H(T)}\right], \text{ where } 0 \le \tau,\ t \le T$$

Let us consider $B(t,T) = E_t\left[\frac{H(t)}{H(T)}\right]$, so that $G(t) = G_T B(t,T)$, and $dG(t) = G(t)\left[r(t)dt + \sigma_B(T-t,r(t))\left(dz_r(t) + \lambda_r\sqrt{r_t}dt\right)\right]$

(2) We want to derive $d\tilde{G}(t) = d\left(H^{-1}(t)G(t)\right) = \tilde{G}(t)\left(\frac{dH^{-1}(t)}{H^{-1}(t)} + \frac{dG(t)}{G(t)} + \frac{dH^{-1}(t)}{H^{-1}(t)}\frac{dG(t)}{G(t)}\right)$.

From part (1), we get that:

$$d\tilde{G}(t) = \tilde{G}(t)\left\{\left[\begin{array}{cc} -\lambda_m & -\lambda_r\sqrt{r(t)} + \sigma_B(T-t,r(t)) \end{array}\right]\left[\begin{array}{c} dz_m(t) \\ dz_r(t) \end{array}\right]\right\}.$$

Therefore, $\rho(t) = -H^{-1}(t)\,G(t)\lambda_m$, $\rho^r(t) = H^{-1}(t)\,G(t)\left(\sigma_B(T-t,r(t)) - \lambda_r\sqrt{r(t)}\right)$.

■

After getting the process of $(G(t),\underline{\rho}(t))_{t\in[0,T]}$, we could compute $\hat{w}(t)$. First, from Eq.(2.25), we have:

$$\begin{aligned} d\tilde{Y}(t) &= \left(\tilde{W}(t)\left(\hat{w}'(t)\sigma(t,r(t)) - \lambda'(t)\right) - \underline{\rho}'(t)\right)dZ(t) \\ &= \tilde{Y}(t)\left(\hat{y}'(t)\sigma(t,r(t)) - \lambda'(t)\right)dZ(t) \end{aligned}$$

Thus, $\left(\tilde{W}(t)\,\hat{w}'(t) - \tilde{Y}(t)\,\hat{y}'(t)\right)\sigma t, r(t) - \left(\tilde{W}(t) - \tilde{Y}(t)\right)\lambda'(t) - \underline{\rho}'(t) = 0$, then we rewrite above equation as following:

$$\left(\tilde{W}(t)\,\hat{w}'(t) - \tilde{Y}(t)\,\hat{y}'(t)\right) = \tilde{G}(t)\sigma^{-1}(t,r(t))\,\lambda'(t) + \sigma^{-1}(t,r(t))\,\underline{\rho}'(t)$$

so we can show it by the following equations:

$$\left(W(t)\,\hat{w}'(t) - Y(t)\,\hat{y}'(t)\right) = \tilde{G}(t)\sigma^{-1}(t,r(t))\,\lambda'(t) + H(t)\,\sigma^{-1}(t,r(t))\,\underline{\rho}'(t) \quad (2.29)$$

$$Y(t)\,\hat{y}'(t) = W(t)\,\hat{w}'(t) - \tilde{G}(t)\sigma^{-1}(t,r(t))\,\lambda'(t) - H(t)\,\sigma^{-1}(t,r(t))\,\underline{\rho}'(t) \quad (2.30)$$

According to **Proposition 8**, we apply $\underline{\rho}'(t)$ in to Eq.(2.29), then:

$$
\begin{aligned}
\left(W\left(t\right)\hat{w}'(t)-Y\left(t\right)\hat{y}'(t)\right) &= G(t)\sigma^{-1}\begin{bmatrix}\lambda_m\\ \lambda_r\sqrt{r(t)}\end{bmatrix}\\
&\quad +H\left(t\right)\sigma^{-1}\begin{bmatrix}-H^{-1}\left(t\right)G(t)\lambda_m\\ H^{-1}\left(t\right)G(t)\left(-\lambda_r\sqrt{r(t)}+\sigma_B\left(T-t,r(t)\right)\right)\end{bmatrix}\\
&= G(t)\sigma^{-1}\left\{\begin{bmatrix}\lambda_m\\ \lambda_r\sqrt{r(t)}\end{bmatrix}+\begin{bmatrix}-\lambda_m\\ \left(-\lambda_r\sqrt{r(t)}+\sigma_B\left(T-t,r(t)\right)\right)\end{bmatrix}\right\}\\
&= G(t)\begin{bmatrix}\frac{h(T-t)}{h(K)}\\ 0\end{bmatrix}
\end{aligned}
$$

From above equations, we have $\begin{bmatrix}\frac{h(T-t)}{h(K)} & 0\end{bmatrix}' = \hat{v}'(t) = [v_B, v_S]'$, where $\hat{v}'(t)$ is the proportions of $G(t)$ invested into $(B_K(t), S(t))$. Since $\left(W\left(t\right)\hat{w}'(t)-Y\left(t\right)\hat{y}'(t)\right) = G(t)\hat{v}'(t)$ and $W\left(t\right)\hat{w}'(t) = W\left(t\right)\hat{y}'(t)+G(t)(\hat{v}'(t)-\hat{y}'(t))$, the optimal portfolio $\hat{w}(t)$ is obtained:

$$
\begin{aligned}
\hat{w}'(t) &= \hat{y}'(t)+W^{-1}\left(t\right)G(t)\left(\hat{v}'(t)-\hat{y}'(t)\right)\\
&= \begin{bmatrix}\frac{k_2(T-t,\gamma/(1-\gamma))}{h(K)}+\frac{1}{1-\gamma}\frac{\sigma_{Sm}\lambda_r-\sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)}\\ \frac{1}{1-\gamma}\frac{\lambda m}{\sigma_{Sm}}\end{bmatrix}\\
&\quad +\frac{G(t)}{W\left(t\right)}\left\{\begin{bmatrix}\frac{h(T-t)}{h(K)}\\ 0\end{bmatrix}-\begin{bmatrix}\frac{k_2(T-t,\gamma/(1-\gamma))}{h(K)}+\frac{1}{1-\gamma}\frac{\sigma_{Sm}\lambda_r-\sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)}\\ \frac{1}{1-\gamma}\frac{\lambda_m}{\sigma_{Sm}}\end{bmatrix}\right\}\\
&= \left\{\begin{bmatrix}\frac{k_2(T-t,\gamma/(1-\gamma))}{h(K)}\\ 0\end{bmatrix}+\frac{1}{1-\gamma}\begin{bmatrix}\frac{\sigma_{Sm}\lambda_r-\sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)}\\ \frac{\lambda m}{\sigma_{Sm}}\end{bmatrix}\right\}\\
&\quad +\frac{G(t)}{W\left(t\right)}\left\{\begin{bmatrix}\frac{h(T-t)}{h(K)}-\frac{k_2(T-t,\gamma/(1-\gamma))}{h(K)}-\frac{1}{1-\gamma}\frac{\sigma_{Sm}\lambda_r-\sigma_{Sr}\lambda_m}{\sigma_{Sm}\sigma_r h(K)}\\ -\frac{1}{1-\gamma}\frac{\lambda_m}{\sigma_{Sm}}\end{bmatrix}\right\}
\end{aligned}
$$

5. Dynamics of Zero Coupon Bond of *Eq.*(3.14)

The prices of the nominal bonds vary with the real interest rate and inflation rate. The definition of the pricing kernel implies that $B(t,T)$, the nominal price at time

t of a default-free bond paying 1 dollar at time T, is:

$$B(t,T) = E_t\left[\frac{M(T)/M(t)}{\Pi(T)/\Pi(t)}\right]$$

According to Brennan and Xia (2002),

$$\begin{aligned} B(t,T) &= E_t\left[\frac{M(T)/M(t)}{\Pi(T)/\Pi(t)}\right] \\ &= \exp\left\{E_t\left[\ln\left(\frac{M(T)}{M(t)}\right) - \ln\left(\frac{\Pi(T)}{\Pi(t)}\right)\right] + \frac{1}{2}Var_t\left[\ln\left(\frac{M(T)}{M(t)}\right) - \ln\left(\frac{\Pi(T)}{\Pi(t)}\right)\right]\right\}. \end{aligned}$$

Let $\phi_1 = \phi'\rho\phi$ and $\rho = \begin{bmatrix} 1 & \rho_{mr} & \rho_{m\pi} \\ \rho_{mr} & 1 & \rho_{r\pi} \\ \rho_{m\pi} & \rho_{r\pi} & 1 \end{bmatrix}$. Then

$$\begin{aligned} \ln\left(\frac{M(T)}{M(t)}\right) &= \int_t^T \left(-r(s) - \frac{1}{2}\phi_1\right) ds + \int_t^T \phi' dZ \\ \ln\left(\frac{\Pi(T)}{\Pi(t)}\right) &= \int_t^T \left(-\hat{\pi}(s) - \frac{1}{2}\xi_\pi^2\right) ds + \int_t^T \xi_\pi d\hat{z}_\pi \end{aligned}$$

Define $D(t,T) = \frac{1}{\kappa}(1 - e^{\kappa(t-T)})$ and $C(t,T) = \frac{1}{\alpha}(1 - e^{\alpha(t-T)})$. Then,

$$\begin{aligned} E_t\left[\ln\left(\frac{M(T)}{M(t)}\right)\right] &= -\bar{r}(T-t) + (\bar{r} - r(t))\, D(t,T) - \frac{1}{2}\phi_1(T-t) \\ Var_t\left[\ln\left(\frac{M(T)}{M(t)}\right)\right] &= -\frac{\sigma_r^2}{2\kappa^3}\left[2\kappa\left(D(t,T) - (T-t)\right) + \kappa^2 D^2(t,T)\right] \\ &\quad +\phi_1(T-t) - \frac{2\sigma_r}{\kappa}\left(\phi_m\rho_{mr} + \phi_r + \phi_\pi\rho_{\pi r}\right)\left[(T-t) - D(t,T)\right] \end{aligned}$$

and

$$\begin{aligned} E_t\left[\ln\left(\frac{\Pi(T)}{\Pi(t)}\right)\right] &= \pi(T-t) - (\bar{\pi} - \hat{\pi}(t))\, C(t,T) - \frac{1}{2}\xi_\pi^2(T-t) \\ Var_t\left[\ln\left(\frac{\Pi(T)}{\Pi(t)}\right)\right] &= -\frac{\hat{\sigma}_\pi^2}{2\alpha^3}\left[2\alpha\left(C(t,T) - (T-t)\right) + \alpha^2 C^2(t,T)\right] \\ &\quad +\xi_\pi^2(T-t) - \frac{2\hat{\sigma}_\pi}{\alpha}\xi_\pi\left[(T-t) - D(t,T)\right] \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
CV &= Cov_t\left[\ln\left(\frac{M(T)}{M(t)}\right), \ln\left(\frac{\Pi(T)}{\Pi(t)}\right)\right] \\
&= -\frac{\sigma_r\hat{\sigma}_\pi\rho_{r\pi}}{\alpha\kappa}\left[(T-t) - D(t,T) - C(t,T) + \frac{1-e^{(\alpha+\kappa)(t-T)}}{\alpha+\kappa}\right] \\
&\quad -\frac{\sigma_r}{\kappa}\xi_\pi\rho_{r\pi}\left[(T-t) - D(t,T)\right] \\
&\quad +\frac{\hat{\sigma}_\pi}{\alpha}\left(\phi_m\rho_{m\pi} + \phi_\pi + \phi_r\rho_{\pi r}\right)\left[(T-t) - C(t,T)\right] \\
&\quad +\left(\phi_m\rho_{m\pi} + \phi_r\rho_{\pi r} + \phi_\pi\right)\xi_\pi(T-t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
B(t,T) &= \exp\left\{A(t,T) - D(t,T)r(t) - C(t,T)\hat{\pi}(t)\right\}, \text{ where} \\
A(t,T) &= \left[D(t,T) - (T-t)\right]\bar{r}^* + \left[C(t,T) - (T-t)\right]\bar{\pi}^* \\
&\quad -\frac{\sigma_r^2}{4\kappa^3}\left[2\kappa\left(D(t,T) - (T-t)\right) + \kappa^2 D^2(t,T)\right] \\
&\quad -\frac{\hat{\sigma}_\pi^2}{4\alpha^3}\left[2\alpha\left(C(t,T) - (T-t)\right) + \alpha^2 C^2(t,T)\right] \\
&\quad +\frac{\sigma_r\hat{\sigma}_\pi\rho_{r\pi}}{\alpha\kappa}\left[(T-t) - D(t,T) - C(t,T) + \frac{1-e^{(\alpha+\kappa)(t-T)}}{\alpha+\kappa}\right] \\
&\quad +\xi_\pi\lambda_\pi(T-t) \\
\bar{r}^* &= \bar{r} - \lambda_r\frac{\sigma_r}{\kappa},\ \bar{\pi}^* = \bar{\pi} - \lambda_\pi\frac{\hat{\sigma}_\pi}{\alpha} \\
\lambda_m &= \xi_\pi\rho_{m\pi} - \left(\phi_m + \phi_r\rho_{mr} + \phi_\pi\rho_{m\pi}\right) \\
\lambda_r &= \xi_\pi\rho_{r\pi} - \left(\phi_m\rho_{mr} + \phi_r + \phi_\pi\rho_{r\pi}\right) \\
\lambda_\pi &= \xi_r\rho_{r\pi} - \left(\phi_m\rho_{m\pi} + \phi_r\rho_{r\pi} + \phi_\pi\right)
\end{aligned}
$$

Using Itô's lemma, we obtain

$$
\begin{aligned}
\frac{dB(t,T)}{dB(t,T)} &= \left[r(t) + \hat{\pi}(t) - D\sigma_r\lambda_r - C\hat{\sigma}_\pi\lambda_\pi\right]dt - D\sigma_r dz_r - C\hat{\sigma}_\pi d\hat{z}_\pi \\
&= R(t)dt - D\sigma_r\left(dz_r + \lambda_r dt\right) - C\hat{\sigma}_\pi\left(d\hat{z}_\pi + \lambda_\pi dt\right) \\
&= R(t)dt - \kappa^{-1}(1-e^{\kappa(t-T)})\sigma_r\left(dz_r + \lambda_r dt\right) \\
&\quad -\alpha^{-1}(1-e^{\alpha(t-T)})\hat{\sigma}_\pi\left(d\hat{z}_\pi + \lambda_\pi dt\right)
\end{aligned}
$$

where $R(t) = r(t) + \hat{\pi}(t) - \xi_\pi \lambda_\pi$.

We assume there exists two bonds with different maturities. The first bond, $B(t, T_1)$, with short maturity T_1:

$$\frac{dB(t,T_1)}{dB(t,T_1)} = R(t)dt + \sigma_{r1}(t)\left(dz_r + \lambda_r dt\right) + \sigma_{\pi 1}(t)\left(d\hat{z}_\pi + \lambda_\pi dt\right)$$

where $\sigma_{r1}(t) = -\kappa^{-1}(1 - e^{\kappa(t-T_1)})\sigma_r$, $\sigma_{\pi 1}(t) = -\alpha^{-1}(1 - e^{\alpha(t-T_1)})\hat{\sigma}_\pi$.

The price process of the bond with the longer maturity T_2, $B(t, T_2)$, can also be expresses as:

$$\frac{dB(t,T_2)}{dB(t,T_2)} = R(t)dt + \sigma_{r2}(t)\left(dz_r + \lambda_r dt\right) + \sigma_{\pi 2}(t)\left(d\hat{z}_\pi + \lambda_\pi dt\right)$$

where $\sigma_{r2}(t) = -\kappa^{-1}(1 - e^{\kappa(t-T_2)})\sigma_r$, $\sigma_{\pi 2}(t) = -\alpha^{-1}(1 - e^{\alpha(t-T_2)})\hat{\sigma}_\pi$.

Then, the trading strategy resulting in $B_K(t)$ is a linear combination of investing a proportion $\left(\frac{\sigma_B^K \sigma_{\pi 2} - \sigma_{r2}\sigma_\pi^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right)$ in $B(t, T_1)$, a proportion $\left(\frac{\sigma_\pi^K \sigma_{r1} - \sigma_{\pi 1}\sigma_B^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right)$ in $B(t, T_2)$, and a proportion $\left(1 - \left(\frac{\sigma_B^K \sigma_{\pi 2} - \sigma_{r2}\sigma_\pi^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right) - \left(\frac{\sigma_\pi^K \sigma_{r1} - \sigma_{\pi 1}\sigma_B^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right)\right)$ in cash. Therefore,

$$\begin{aligned}
\frac{dB_K(t)}{B_K(t)} &= R(t)dt + \sigma_B^K(dz_r + \lambda_r dt) + \sigma_\pi^K(d\hat{z}_\pi + \lambda_\pi dt) \\
&= \frac{\sigma_B^K \sigma_{\pi 2} - \sigma_{r2}\sigma_\pi^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}} \frac{dB(t,T_1)}{dB(t,T_1)} \\
&\quad + \frac{\sigma_\pi^K \sigma_{r1} - \sigma_{\pi 1}\sigma_B^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}} \frac{dB(t,T_2)}{dB(t,T_2)} \\
&\quad + \left(1 - \left(\frac{\sigma_B^K \sigma_{\pi 2} - \sigma_{r2}\sigma_\pi^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right) - \left(\frac{\sigma_\pi^K \sigma_{r1} - \sigma_{\pi 1}\sigma_B^K}{\sigma_{r1}\sigma_{\pi 2} - \sigma_{r2}\sigma_{\pi 1}}\right)\right) \frac{dS_0(t)}{S_0(t)}
\end{aligned}$$