行政院國家科學委員會專題研究計畫 成果報告

貝氏 Edgeworth expansion 及其應用
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 99-2118-M-004-001-
執 行 期 間 ： 99 年 08 月 01 日至 101 年 01 月 31 日
執 行 單 位 ： 國立政治大學統計學系

計 畫 主 持 人 ： 翁久幸

計畫參與人員： 碩士班研究生-兼任助理人員：張孫浩
博士班研究生-兼任助理人員：蘇健郎
博士班研究生-兼任助理人員：許正宏

報 告 附 件 ： 出席國際會議研究心得報告及發表論文

公 開 資 訊 ： 本計畫可公開查詢

中 華 民 國 101 年 01 月 04 日

中文摘要： Edgeworth expansion 是一個關於機率分配的近似展開式.其主要之精神在於看出一機率分配的動差(cumulants)可以用來描述與近似該機率分配.這樣一個展開式的想法可以追溯到一百多年前(Chebyshev 1890 and Edgeworth 1896, 1905). 而由於它能幫助探索當代統計方法,因此過去幾十年再度引起相當的興趣.關於這個主題的文獻回顧 可以參考 Hall (The Bootstrap and the Edgeworth expansion 1992).

除了統計學,Edgeworth expansion 也被應用在其他領域；比方天文物理 (Blinnikov and Moessner (1998) 比較 Gram-Charlier, Gauss-Hermite 與 Edgeworth expansions 一天文物理問題之表現), 和財務數學(Filho and Rosenfeld (2004) 藉由 Edgeworth expansion 討論選擇權定價問題).

另一方面,Edgeworth expansion 可以視為中央極限定理的精緻化. 中央極限定理在貝氏統計中所對應的理論是關於後驗分配的常態性,也稱為 Bernstein-von Mises theorem. 而且不論是中央極限定理或是後驗分配的常態性,兩者都有若干高次漸近結果. 然而貝氏 Edgeworth expansion 卻未曾被發展出來.

本研究計畫利用 Stein ‘s Identity 與 Hermite 多項式,在適當條件下推展出個別參數的後驗分配的貝氏 Edgeworth expansion.

中文關鍵詞： 貝氏, Edgeworth 展開, Hermite 多項式

英文摘要： The Edgeworth expansion, named in honor of F. Y. Edgeworth (1845-1926),is a series that approximates a probability distribution in terms of its cumulants. It is over a century old and in the past decades it has received a revival of interest due to its usefulness for exploring properties of contemporary statistical methods； see Hall (1992) for an excellent review of this subject.

The Edgeworth expansion has been applied to other areas as well； for example, Blinnikov and Moessner (1998) compared Gram-Charlier, Gauss-Hermite and Edgeworth expansions in astrophysics problems, and Filho and Rosenfeld (2004) considered the option pricing problem using Edgeworth expansion, among others.

The Bayesian counterpart of the central limit theorem is the Bernstein-von Mises theorem, which states the asymptotic normality of the posterior. There have been extensive studies on refinements of the Bernstein-von Mises theorem. These refinements are Bayesian counterpart of the higher order asymptotic normality.
However, an Edgeworth expansion for posterior distributions has not been developed yet.

The present project showed that by Stein ‘s Identity and some properties of Hermite polynomials, one can obtain an expansion of the marginal posterior distribution that is valid under mild conditions and resembles the Edgeworth expansion.

英文關鍵詞： Bayesian, Edgeworth expansion, Hermite polynomials

行政院國家科學委員會補助專題研究計畫

成果報告

（計畫名稱）貝氏 Edgeworth expansion 及其應用

計畫類別： 個別型計畫
計畫編號：NSC 99-2118-M-004-001-
執行期間： 2010 年 8 月 1 日至 2012 年 1 月 31 日

執行機構及系所：國立政治大學統計學系

計畫主持人： 翁久幸
共同主持人：
計畫參與人員：

成果報告類型(依經費核定清單規定繳交)： 完整報告

本計畫除繳交成果報告外，另須繳交以下出國心得報告：

出席國際學術會議心得報告

The Final Report

The Edgeworth expansion, named after F. Y. Edgeworth (1845-1926), is an expansion that approximates a probability distribution in terms of its cumulants. It is over a century old and it provides an improvement to the central limit theorem. In the past decades it has received a revival of interest in statistics; for example, see Hall [5] on how Edgeworth expansion and bootstrap methods can help explain each other. The Edgeworth expansion has been applied to other areas as well; for example, Blinnikov and Moessner [2] compared Gram-Charlier, Gauss-Hermite and Edgeworth expansions in problems of astrophysics, and Filho and Rosenfeld [4] considered the problem of testing option pricing with Edgeworth expansion, among others. Actually, Blinnikov and Moessner [2] gave a simple algorithm to calculate higher-order terms of Edgeworth expansion, and they obtained the cumulants up to 10th order in the application to peculiar velocities from cosmic strings.

Wallace [15, Section 3] and Blinnikov and Moessner [2] provided reviews on early developments of the series. Let F be the distribution to be approximated and $\{\kappa_r\}$ its cumulants; let γ_r be the cumulants of a standard normal distribution and D the differential operator representing differentiation with respect to x; let Φ and ϕ be the cdf and pdf of a standard normal variable. Chebyshev and Charlier considered the identity

$$F(x) = \exp \sum_{r=1}^{\infty} (\kappa_r - \gamma_r) \frac{(-D)^r}{r!} \Phi(x)$$

and proceeded by expanding and collecting terms according to the order of the derivatives. The resulting expansion is commonly known as the Gram-Charlier series (of type A) and it turned out to be identical with the expansion of F in Hermite orthogonal functions; or equivalently, for a pdf $p(x)$,

$$p(x) \sim \sum_{k=0}^{\infty} c_k q_k(x) \phi(x), \tag{1}$$

where q_k are Hermite polynomials and, by the orthogonal property below,

$$c_k = \frac{1}{k!} \int_{-\infty}^{\infty} p(x) q_k(x) dx. \tag{2}$$

Blinnikov and Moessner [2, Section 4] also showed that the Gram-Charlier series (1) is just a Fourier expansion of $p(x)/\phi(x)$ in Hermite polynomials. Note that the sample size plays no role in this expansion, and it is known that this expansion has poor convergence properties (see Cramér [3]). Edgeworth considered the standardized sum of n independent and identically distributed random variables, and developed a similar expansion. Actually, the

Edgeworth series can be obtained by collecting terms in the Gram-Charlier series according to powers of n.

The most basic result of Edgeworth expansion is for independent and identically distributed random variables $X_1, ..., X_n$ with mean θ_0 and finite variance σ^2. Let $\hat{\theta}_n$ be the sample mean of X_i's. Under regularity conditions, the distribution function of $Y \equiv n^{1/2}(\hat{\theta}_n - \theta_0)$ may be expanded as

$$P(\frac{n^{1/2}(\hat{\theta}_n - \theta_0)}{\sigma} \leq x) = \Phi(x) + n^{-1/2}p_1(x)\phi(x) + \cdots + n^{-j/2}p_j(x)\phi(x) + \cdots . \quad (3)$$

Formula (3) is termed an Edgeworth expansion. The functions p_j are polynomials with coefficients depending on cumulants of $\hat{\theta}_n - \theta_0$. In particular, p_j is a polynomial of degree at most $3j - 1$ and is an odd or even function according to whether j is even or odd.

Many researchers have derived Edgeworth expansions in non-iid contexts; for example, Bickel and Ghosh [1] considered the signed-root transformation, and Jing and Wang [6] obtained expansions for U-statistics. There are also studies from Bayesian perspective. Let g be a smooth function of the parameter θ. The usual approach to asymptotic posterior expansions starts from writing the posterior mean of $g(\theta)$ as a ratio of two integrals,

$$E_\xi[g(\theta)|x_t] = \frac{\int g(\theta)\exp(\ell_t(\theta))\xi(\theta)d\theta}{\int \exp(\ell_t(\theta))\xi(\theta)d\theta},$$

where ℓ_t is the loglikelihood function and ξ the prior density, next takes a Taylor series expansion at the maximum likelihood estimator and develops expansions on both the numerator and denominator, and then obtains an approximation of the posterior mean by formal division of the two series. Johnson [7, 8] provides a careful account of this approach. There are other work that apply Laplace method to both numerator and denominator and then take the ratio; see, for example, Lindley [9, 10], Mosteller and Wallace [11], Tierney and Kadane [14], and references therein. However, these asymptotic expansions for posterior distributions are not in terms of the cumulants or moments.

Recently Weng [16] and Weng and Tsai [18] applied a version of Stein's Identity, established by Woodroofe [19, 20] for integrable expansions for posterior distributions, to asymptotic posterior normality; and Weng and Lin [17] applied it for Bayesian online ranking. The idea of this identity originated from the famous Stein's lemma [12, 13], but the latter considers the expectations of normal distributions, while the former the expectations of distributions which are "nearly" normal (to be defined later). The application of this identity to posterior normality starts by writing the posterior density of a normalized maximum likelihood estimator Z_t in a form close to normal, next applies Stein's Identity to obtain an expansion for posterior expectations of $h(Z_t)$, and then analyzes the remainder term in the expansion. The present paper takes one step further to show that by repeatedly employing Stein's Identity, together with some properties of Hermite polynomials, one can expand the marginal posterior distribution in the form of (1); then, we proceed to obtain the orders of the c_k terms (2) and form an asymptotic series. Note that our expansion resembles the classic Edgeworth expansion in that both are directly connected to the cumulants or moments, and both can be viewed as an expansion of the probability distribution in Hermite orthogonal functions together with rearrangement of terms in powers of the sample size. These two properties are lost in existing posterior expansions in the literature. The advantage of expression a distribution in terms of the moments is that the information about the distribution can be efficiently stored.

This work was published in Bayesian Analysis, Vol. 5, 741-764, 2010.

References

[1] P. Bickel and J. K. Ghosh. A decomposition for the likelihood ratio statistic and the bartlett correction–a bayesian argument. *Ann. Statist.*, 18:1070–1090, 1990.

[2] S. Blinnikov and R. Moessner. Expansions for nearly Gaussian distributions. *Astron. Astrophys. Suppl. Ser.*, 130:193–205, 1998.

[3] H. Cramér. *Mathematical Methods of Statistics.* Princeton University Press, Princeton, 1957.

[4] R. G. B. Filho and R. Rosenfeld. Testing option pricing with the edgeworth expansion. *Physica A*, 344:484–490, 2004.

[5] P. Hall. *The Bootstrap and Edgeworth Expansion.* Springer, New York, 1992.

[6] B.-Y. Jing and Q. Wang. Edgeworth expansion for U-statistics under minimal conditions. *Ann. Statist.*, 31(4):1376–1391, 2003.

[7] R. Johnson. An asymptotic expansion for posterior distributions. *Ann. Math. Statist.*, 38:1899–1906, 1967.

[8] R. Johnson. Asymptotic expansions associated with posterior distributions. *Ann. Math. Statist.*, 41:851–864, 1970.

[9] D. V. Lindley. The use of prior probability distributions in statistical inference and decisions. *Proc. 4th. Berkeley Symp.*, 1:453–468, 1961.

[10] D. V. Lindley. Approximate bayesian methods. In J. M. Bernardo, M. H. DeGroot, D. V. Lindley, and A. F. M. S. (Eds.), editors, *Bayesian Statistics*. University Press, 1980.

[11] F. Mosteller and D. L. Wallace. *Inference and Disputed Authorship: The Federalist Papers*. Addison-Wesley, Reading, Mass., 1964.

[12] C. Stein. Estimation of the mean of a multivariate normal distribution. *Ann. Statist.*, 9:1135–1151, 1981.

[13] C. Stein. *Approximate Computation of Expectations*. IMS, Hayward, Calif, 1987.

[14] L. Tierney and J. B. Kadane. Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, 81:82–86, 1986.

[15] D. L. Wallace. Asymptotic approximations to distributions. *Annals of Mathematical Statistics*, 29:635–654, 1958.

[16] R. C. Weng. On Stein's identity for posterior normality. *Statistica Sinica*, 13:495–506, 2003.

[17] R. C. Weng and C.-J. Lin. A Bayesian approximation method for online ranking. *Accepted by Journal of Machine Learning Research*, 2010.

[18] R. C. Weng and W.-C. Tsai. Asymptotic posterior normality for multiparameter problems. *Journal of Statistical Planning and Inference*, 138:4068–4080, 2008.

[19] M. Woodroofe. Very weak expansions for sequentially designed experiments: linear models. *The Annals of Statistics*, 17:1087–1102, 1989.

[20] M. Woodroofe. Integrable expansions for posterior distributions for one-parameter exponential families. *Statistica Sinica*, 2:91–111, 1992.

Report on attending International Conference on Advances in Probability and Statistics (ICAPS) - Theory and Applications. December 28 - 31, 2011 The Chinese University of Hong Kong, Hong Kong SAR, China

The ICAPS 2011 was a celebration of N. Balakrishnan's 30 years of Contributions to Statistics. I had an oral presentation on the topic "A Bayesian rating system using W-Stein's Identity", which was in the session of "Bayesian method and probability models with applications" on Thursday December 29.

In this talk I presented online ranking algorithms and compared with other systems such as TrueSkill and Glicko. After the talk, I met people who also worked on ranking problems and had some discussions. I also attended several other oral presentations; for example, "Applying sensitivity analysis on probabilistic graphical models" by Hei Chan, The Institute of Statistical Mathematics, Japan, "Coin tossing, Fibonacci numbers, and Indian music" by M. B. Rao, University of Cincinnati, USA, and the session on "Statistical methods in assessment of agreement", the session on "Multivariate distribution theory", among others.

By attending the conference, I met some researchers and also knew more about old friends' current research. This inspired me and encouraged me to keep on moving.

Abstract:

For Internet games, large online ranking systems are much needed. We propose a Bayesian approximation method, based on a variant of Stein's identity (1981), to obtain online ranking algorithms with simple analytic update rules. Experiments on game data show that the accuracy of our approach is competitive with state of the art systems such as TrueSkill, but the running time as well as the code are much shorter. We also compare our method with Glicko rating system, designed for rating chess players.

Note: The conference program is available on `http://faculty.smu.edu/ngh/icaps2011.html`

國科會補助計畫衍生研發成果推廣資料表

日期:2012/01/04

國科會補助計畫	計畫名稱: 貝氏Edgeworth expansion及其應用
	計畫主持人: 翁久幸
	計畫編號: 99-2118-M-004-001- 學門領域: 數理統計

無研發成果推廣資料

99 年度專題研究計畫研究成果彙整表

計畫主持人：翁久幸			計畫編號：99-2118-M-004-001-				
計畫名稱：貝氏 Edgeworth expansion 及其應用							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	1	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

　　□實驗失敗

　　□因故實驗中斷

　　□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

Edgeworth expansions have applications in many areas；

and Bayesian methods are getting popular in recent decades.

We expect that Bayesian Edgeworth expansions have potential in many applications.