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Blow-up positive solutions of semilinear wave equations in one space dimension

$\Box\, u - u^p = 0$

with non-positive energy

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Abstract In this paper we treat the blow-up set and the blow-up rate of positive solutions of semilinear wave equations

$$\Box u - u^p = 0$$

with initial and boundary values problem in $1-$space dimension.

1. Introduction

Consider the initial value problem for the semilinear wave equation of the type

$$u_{tt} - \triangle u := \Box u = -g(u) = 0 \quad \text{in} \quad [0, T) \times \mathbb{R}, \tag{1}$$

$$u\,(0, \cdot) = u_0, \qquad u_t\,(0, \cdot) = u_1, \tag{2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a real valued function, the initial data are given sufficiently smooth functions and $u_t = \frac{\partial u}{\partial t}$, Δ is the Laplace operator. The linear case $g(u) = mu$, where m is aconstant, corresponds to the classical Klein Gordon equation in relativistic particle physics; the constant m is interpreted as the mass and is assumed to be nonnegative generally. To model also nonlinear phenomena like quantization, in the 1950s equations of (1) type with nonlinearities like

$$g(u) = mu + u^3, \qquad m \geq 0,$$

were proposed as models in relativistic quantum mechanics with local interaction; see for instance Schi [28] and Segal [29]. Solutions could be considerd as real or complex valued functions. In the latter case it was assumed that the nonlinearity commutes with the phase; that is,

$$g(e^{i\varphi}u) = e^{i\varphi}g(u) \quad \text{for all} \quad \varphi \in \mathbb{R}$$

and that $g(0) = 0$. In this case, g may be expressed $g(u) = uf\left(|u|^2\right)$, which gives the study of equation (1) [3]. Here, for simplicity, we confine ourselves to the study of real-valued solutions of equation (1). In spinor fields u, the scalar equation (1) also was considered in space dimensions $n \geq 3$; see [29].

Various other models involving nonlinearities g depending also on u_t and ∇u, have been studied. The " σ-model" involves an equation of type (1) for vector-valued functions subject to a certain (nonlinear) constraint. In this case

$$g(u) = u\left(|u_t|^2 - |\nabla u|^2\right), u = (u_1, ..., u_n),$$

and u is assumed to satisfy the condition $|u|^2 = u_1^2 + \cdots + u_n^2 = 1$; for some results on this problem see Shatah [30] and the references.

We restrict our study to nonlinearities depending only on u. The abovementioned examples suggest we assume that $g(0) = 0$ and that g satisfies the following for all $u \in \mathbb{R}$

(3) $$g(u) \leq C(1 + |u|^{p-2})|u| \quad \text{for some } p \geq 2,\ C \in \mathbb{R}.$$

Following Strauss [13,Theorem 3.1], we assume that g satisfies the conditions

(4) $$G(u) \geq C|u|^2 \quad \text{for some } C \in \mathbb{R},$$

and

(5) $$G(u)/g(u) \to \infty \quad \text{as} \quad |u| \to \infty, G(u) = \int_0^u g(v)dv.$$

(4) and (5) include the linear case (with no sign condition) or, more generally, the case of local Lipschitz nonlinearities. In the super-linear case; that is, if $|g(u)|/|u| \to \infty$ as $|u| \to \infty$, the conditions (4), (5) should be regarded as a coerciveness condition. In fact, in this case finite propagation speed ≤ 1 and conservation of energy imply locally uniform a priori bounds in L^2 for solutions of (1) in terms of the initial data.

Contrastly, in the noncoercive case it is easy to construct solutions of (1) with smooth initial data that blow up in finite time; for instance, for any $\alpha > 0$ the function $u(t,x) = (1-t)^{-1/m}$ solves the equation

$$\Box u = -\alpha(1+\alpha)u|u|^{2m}, m \in \mathbb{N}$$

and blows up at $t = 1$. Modifying the initial data off $\{x; |x| \leq 2\}$, say, we even posses a singular solution with C^∞-data having compact support. (See John [23] for a blow-up result for a similar equation.) Thus, conditions like (3) – (5) seem natural if we are interested in global solutions.

The class (3) – (5) includes the following special cases

(6) $$g(u) = mu|u|^{q-2} + u|u|^{p-2}, \quad m \geq 0, 2 \leq q < p.$$

For nonlinearities of this kind the answer to the existence problem for (1), (2) in a striking way depends on the space dimension n and on the exponent p. In particular, in the physically interesting case $n = 3$, global existence for $p < 6$ can be established, while the same question for $p > 6$ so far has eluded all research attempts.

In fact, the apparent existence of a " critical power" for (1) and recent advances on elliptic problems involving critical nonlinearities prompted the interest in the u^5-Klein Gordon equation. " Critical powers" very often come into play in nonlinear problems through Sobolev embedding. In particular, $p = 6$ is the critical power for the Sobolev embedding $W^{1,2}_{loc}(\mathbb{R}^3) \hookrightarrow L^p_{loc}(\mathbb{R}^3)$. (In n-dimension the critical power for this embedding is $p = \frac{2n}{n-2}$.) Moreover, they very often arise naturally from the requirements of scale invariance, that is, whenever " intrinsic" notions are involved. An example of such a problem is the Yamabe problem concerning the existence of conformal metrics with constant scalar curvature on a given (compact) Riemannian manifold. Through the work of Trudinger, Aubin, and—finally— Schoen this problem has been completely solved and it has become apparent that at the critical power properties like " compactness of the solution set" depend crucially on global aspects of the problem; in this case, on the topological and differentiable structure of the manifold. See Lee and Parker [24] for a survey of the Yamabe problem.

Incidentally, for nonlinear wave equations (or nonlinear Schrödinger equations $iu_t - \triangle u + u|u|^{p-2} = 0$) there appear to be many " critical powers," depending on what aspect of the problem we consider: global existence, scattering theory, . . . ; see Strauss [13, p. 14f.]. As regards global existence, it remains to be seen whether the critical power represents only a technical barrier or, in fact, defines the dividing line between qualitatively different regimes of behavior of (1), (2). We conclude this introduction with a short overview of the existence results in the case of a pure power

$$\Box u + u|u|^{p-2} = 0, \qquad p > 2.$$

The sub-critical case. For $n = 3, p < 6$ global existence and regularity was established by Jörgens [3] in 1961. Jörgens also was able to show local (small time) existence of regular solutions to (7), (2) for arbitrarily large p. Moreover, he was able to reduce the problem of existence of global, regular solutions to (1) to (local) estimates of the L^∞-norms of solutions. These results were generalized to higher dimensions; however, such extensions have been very hard to obtain. While Jörgens' work relies on the classical representation formula for the 3-dimensional wave equation, this method fails in higher dimensions $n > 3$. The fundamental solution to the wave equation no longer is positive; moreover, it carries derivatives transverse to the wave cone. Nevertheless, at least for $n \leq 9$, the existence results of Pecher [26], Brenner-von Wahl [19] now cover the full sub-critical range $p < \frac{2n}{n-2}$. Regular solutions are unique.

Global weak solutions. On the other hand, by a suitable approximation and using energy estimates, for all $p > 2, n \geq 3$ it is possible to construct global weak solutions, satisfying (7) in a distributional sense; see Segal [29], Lions [25]. In this case, it even suces to assume that the initial data $u_0, u_1 \in L^2_{loc}(\mathbb{R}^n)$ with $u_0 \in L^p_{loc}(\mathbb{R}^n)$ and distributional derivative $\nabla u_0 \in L^2_{loc}(\mathbb{R}^n)$. Energy estimates immediately give uniqueness of weak solutions in case $p \leq \frac{2n}{n-2} - \frac{2}{n-2}$; see Browder [1]. However, this range is well below the critical Sobolev exponent $p = \frac{2n}{n-2}$. In order to improve the range of admissable exponents, more sophisticated tools were developed, based, in particular, on the $L^p - L^q$-estimates for the wave operator by Strichartz [31]; see also Brenner [18]. In their simplest version, these estimates allow to prove uniqueness of solutions to $(7),(2)$ for $p \leq \frac{2(n+1)}{n-1}$, the Sobolev exponent in $(n+1)$ space dimensions. In fact, uniqueness can be established for $p < \frac{2n}{n-2}$; see Ginibre-Velo [20]. In this case, moreover, the unique solution can be shown to be " strong," that is, to possess second derivatives in L^2 and to satisfy the energy identity [20].

The critical case. In dimension $n = 3$, global existence of C^2-solutions in the critical case $p = 6$ was first obtained by Rauch [27], assuming the initial energy

$$E\left(u(0)\right) = \int_{\mathbb{R}^3} \left(\frac{|u_1|^2 + |u_0|^2}{2} + \frac{|u|^6}{6} \right) dx$$

to be small. In 1987, also for " large" data global C^2-solutions were shown to exist by this author [32] in the radially symmetric case $u_0(x) = u_0\left(|x|\right), u_1(x) = u_1\left(|x|\right)$. Finally, Grillakis [22] in 1989 was able to remove the latter symmetry assumption, yielding the following result:

Theorem: For any $u_0 \in C^3(\mathbb{R}^3), u_1 \in C^2(\mathbb{R}^3)$ there exists a unique solution $u \in C^2\left(\mathbb{R}^3 \times [0,\infty)\right)$ to the Cauchy problem

$$\Box u + u^5 = 0, \tag{8}$$

$$u\left|_{t-0}\right. = u_0, u_t\left|_{t=0}\right. = u_1. \tag{9}$$

Related partial regularity results independently have been obtained by Kapitanskii [34] in 1989. Uniqueness holds among C^2-solutions. The proof procedes via a priori estimates. The classical representation formula crucially enters. It seems unlikely that regularity or uniqueness of weak solutions to $(8),(9)$ can be established in a similar way. Research on the critical case in higher dimensions is in progress; however, to this moment the results on this subject still seem incomplete. Advances in these questions may require eliminating the use of the wave kernel.

The super-critical case. We observe that for suciently small initial data the existence of global regular solutions, for instance, to the equation

$$\Box u + u^5 + u|u|^{p-2} = 0 \quad in \quad [0,\infty) \times \mathbb{R}^3,$$

for any $p > 2$ can be deduced as a corollary to Rauch' s result. Various qualitative properties of solutions in the super-critical case have been studied [8] [33]. Other open problems concern scattering theory, involving, in particular, decay estimates for solutions of (1) (see Ginibre-Velo [21]), or existence and regularity results for initial-boundary value problems.

In this paper, in another topic, we want to estimate the life-span and later seek for the blow-up set of positive solutions for the 1-dimensional semilinear wave equation

$$\Box\ u = u^p \text{ in } [0,T) \times \mathbb{R} \tag{0.1}$$

with initial values $u\left(0,x\right) = u_0\left(x\right) \in H^2\left(\mathbb{R}\right) \cap H_0^1\left(\mathbb{R}\right)$ and $\dot{u}\left(0,x\right) = u_1\left(x\right) \in H_0^1\left(\mathbb{R}\right)$, where $p > 1$,that is, the superlinear case. We will use the following notations:

$$\cdot := \frac{\partial}{\partial t},\ Du := (\dot{u}, u_x)\,,\ \Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2},$$

$$a\,(t) := \int_{\mathbb{R}} u^2\,(t,x)\,dx,\ E\,(t) := \int_{\mathbb{R}} \left(|Du|^2 - \frac{2}{p+1} u^{p+1} \right)(t,x)\,dx.$$

For a Banach space X and $0 < T \leq \infty$ we set

$$C^k\,(0,T,X) = \text{Space of } C^k - \text{ functions} : [0,T) \to X,$$
$$H1 := C^1\left(0,T,H_0^1\,(\mathbb{R})\right)\ \cap\ C^2\left(0,T,L^2\,(\mathbb{R})\right).$$

Jörgens [3] published the first exist Theorem for global solutions to the wave equation of the form

$$(*) \qquad \Box\ u + f\,(u) = 0 \text{ in } [0,T) \times \Omega,$$

for $\Omega = \mathbb{R}^n, n = 3$ and $f\,(u) = g\left(u^2\right) \cdot u$, his result can be applied to the equation $\Box u + u^3 = 0$; and Browder [1] generalized Jörgens's result to $n > 2$. For local Lipschitz f , Li [10] proved the nonexistence of global Solution of the initial-boundary value problem of semilinear wave equation $(*)$ in bounded domain $\Omega \subset \mathbb{R}^n$ under the assumption

$$\bar{E}\,(0) = \|Du\|_2^2\,(0) + 2\int_{\Omega} f\,(u)\,(0,x)\,dx \leq 0,$$

$$\eta f\,(\eta) - 2\,(1+2\alpha)\int_0^{\eta} f\,(r)\,dr \leq \lambda_1 \alpha \eta^2 \quad \forall \eta \in R$$

where $\alpha > 0$, $\lambda_1 := \sup\left\{\|u\|_2 / \|\nabla u\|_2 : u \in H_0^1\,(\Omega)\right\}$ and $a'\,(0) > 0$. There we have a rough estimate for the life-span

$$T \leq \beta_2 := 2\left[1 - \left(1 - k_2 a\,(0)^{-\alpha}\right)^{1/2}\right] / k_1 k_2,$$

where $k_1 := \alpha a\,(0)^{-\alpha-1}\sqrt{a'\,(0)^2 - 4E\,(0)\,a\,(0)}, k_2 := \left(-4\alpha^2 E\,(0)\,/k_1^2\right)^{\alpha/1+2\alpha}$.

For $n = 3$ and $f(u) = -u^3$, there exist global solutions of (SL) for small initial data [8]; but if $E(0) < 0$ and $a'(0) > 0$ then the solutions are only local, i.e. $T < \infty$ [11].

John [4] showed the nonexistence of solutions of the initial-boundary value problem for the wave equation $\Box u = A|u|^p$, $A > 0$,

$$1 < p < 1 + \sqrt{2},\ \ \Omega = \mathbb{R}^3.$$

This problem was considered by Glassey [2] in two dimensional case $n = 2$; for $n > 3$ Sideris [15] showed the nonexistence of global solutions under the conditions

$$\|u_0\|_1 > 0 \ \text{ and } \ \|u_1\|_1 > 0.$$

According to this result Strauss [13, p.27] guessed that the solutions for the above mentioned wave equation are global for $p \geq p_0(n) = \lambda$ which is the positive root of the quadratic equation

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0$$

and $\Omega = \mathbb{R}^n$. Further literature about blow up one can see [4], [5], [6], [12] and [13] and their reference.

In this paper we treat the blow-up set and the blow-up rate of the solution to the equation (0.1) with non-positive energy.

2. Definition and Fundamental Lemmas

There are many definitions of the weak solutions of the initial-boundary problems of the wave equation, we use here as following.

Definition 2.1: *For $p > 1$, $u \in H1$ is called a positive weakly solution of equation* (0.1), *if*

$$\int_0^t \int_{\mathbb{R}} \begin{pmatrix} \dot{u}(r,x)\,\dot{\varphi}(r,x) - \nabla u(r,x) \cdot \nabla \varphi(r,x) \\ +u^p(r,x)\,\varphi(r,x) \end{pmatrix} dxdr = 0 \ \forall \varphi \in H1$$

and

$$\int_0^t \int_{\mathbb{R}} u(r,x)\,\psi(r,x)\,dxdr \geq 0$$

for each positive $\psi \in C_0^\infty([0,T) \times \mathbb{R})$.

Remark 2.2: 1) This definition 2.1 is resulted from the multiplication with φ to the equation (0.1) and integration in $\mathbb{R}$ from 0 to t.

2) From the local Lipschitz functions u^p, $p \geq 1$, the initial-boundary value problem (0.1) possesses a unique solution in $H1$ [9]. Hereto we use the notations:

$$\frac{1}{C} := \eta_1 = \sup \left\{ \|u\|_2 \ / \ \|Du\|_2 : u \in H_0^1(\mathbb{R}) \right\},$$

$$\lambda_q \ = \sup \left\{ \|u\|_q \ / \ \|Du\|_2 : u \in H_0^1(\mathbb{R}) \ \cap \ L_q(\mathbb{R}) \right\},$$

for $q \geq 1$.

In this paper we need the following lemmas

Lemma 2.3: *Suppose that $u \in H1$ is a weakly positive solution of* (SL) with $E(0) = 0$ *for* $p \geq 1$, $a(0) > 0$, *then we have*:

(i) $a \in C^2(0,T)$ *and* $E(t) = E(0) \quad \forall t \in [0,T)$.

(ii) $a'(t) > 0 \quad \forall t \in [0,T)$, *provided* $a'(0) > 0$.

(iii) $a'(t) > 0 \quad \forall t \in (0,T)$, *if* $a'(0) = 0$.

(iv) *For* $a'(0) < 0$, *there exists a constant* $t_0 > 0$ *with*

$$a'(t) > 0 = a'(t_0) \quad \forall t > t_0.$$

Lemma 2.4: *Suppose that* u *is a positive weakly solution in* $H1$ *of equation (0.1) with* $u(0,\cdot) = 0 = \dot{u}(0,\cdot)$ in $L^2(\mathbb{R})$. *For* $p \geq 1$, *we have* $u \equiv 0$ *in* $H1$.

3. Estimates for the Life-Span of the Solutions of (0.1) under Null-Energy

In this section we focus on the case that $E(0) = 0$, $p \geq 1$ and divide it into two parts

(i) $a(0) > 0, a'(0) \geq 0$

(ii) $a(0) > 0, a'(0) < 0$

3.1. Estimates for the Life-span of the Solutions of (0.1) under $a'(0) \geq 0$.

Theorem 3.1.1: *Suppose that* $u \in H1$ *is a positive weakly solution of equation (0.1) with* $a'(0) \geq 0$ *and* $E(0) = 0$. *Then the Life-span of* u *is finite, further*

$$T \leq \alpha_1 := k_2^{-1} \sin^{-1}\left(\frac{k_2}{k_1 a^{\frac{p-1}{4}}(0)}\right) \tag{3.1.1}$$

with

$$k_1 := \frac{p-1}{4} a^{-\frac{p-1}{4}}(0) \sqrt{a'(0)\, a^{-2}(0) + 4C^2},$$

$$k_2 := \frac{p-1}{2} C.$$

If $T = \alpha_1$, *then* $a(t) \to \infty, t \to T$.

Furthermore, we have also the estimate for $a(t)$*:*

$$a(t) \geq \left(\frac{k_2}{k_1}\right)^{\frac{4}{p-1}} \left(\sin\left(k_2\alpha_1 \;-\; k_2 t\right)\right)^{-\frac{4}{p-1}} \quad \forall t \in [0, T). \tag{3.1.2}$$

This means that the blow-up rate of u *is* $\frac{4}{p-1}$ *in the sin-growth.*

Remark 3.1.2. 1) The Theorem 3.1.1 is a extension of my own Satz 2 in [9]. And the local existence and uniqueness of solutions of equation (0.1) in $H1$ are known [10].

2) For special cases:

i) For $n = 2$, $p > 1$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded by $T \leq \alpha_1$.

ii) For $n = 3$, $p = 2$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded

$$T \leq \alpha_2 := 2C^{-1} \sin^{-1} \left(2C \left(a'(0)^2 a(0)^{-2} + 4C^2 \right)^{-\frac{1}{2}} \right).$$

If $T = \alpha_2$, then $a(t) \to \infty, t \to T$.

iii) For $n = 3$, $p = 3, E(0) = 0$ the life-span of the positive solution $u \in H1$ of equation (0.1) is bounded

$$T \leq \alpha_3 := C^{-1} \sin^{-1} \left(2C \left(a'(0)^2 a(0)^{-2} + 4C^2 \right)^{-\frac{1}{2}} \right).$$

If $T = \alpha_3$, then $a(t) \to \infty, t \to T$.

iv) For $a'(0) = 0$, we have $\alpha_1 = \frac{\pi}{p-1} C$.

v) For $|\Omega| \to \infty$, we have also $\alpha_1 \to \dfrac{1}{p-1} \dfrac{a(0)}{a'(0)}$.

As $|\Omega| \to 0$, then $\alpha_1 \to \dfrac{2}{p-1} \sin^{-1} \left(\dfrac{1}{4C} \right)$.

3.2. **Estimates for the Life-span of the Solutions of equation (0.1) under $a'(0) < 0$.**

Theorem 3.2.1: *Suppose that $u \in H1$ is a positive weakly solution of the initial-boundary value problem equation (0.1) with $a(0) > 0$, $E(0) = 0$ and $a'(0) < 0$. Then the life span of u is bounded:*

$$T \leq \alpha_5 := \frac{\pi}{(p-1)C} - \frac{a'(0)}{p-1} \left(\frac{2\lambda_{p+1}^{p+1}}{p-1} \right)^{\frac{2}{p-1}}.$$

If $T = \alpha_5$, then

$$a(t) \to \infty, T \to \alpha_5.$$

Further, we have the estimate for the blow-up rate of $a(t)$ in the neighborhood of α_5:

$$a(t) \geq a(t_0) \left(\sin \left(\frac{(p-1)C}{2} (\alpha_5 - t) \right) \right)^{-\frac{4}{p-1}} \quad \forall t \in [t_0, T), t_0 \leq t_1$$

with $t_1 := \dfrac{-1}{p-1} \left(\dfrac{p+1}{2\lambda_{p+1}^{p+1}} \right)^{-\frac{2}{p-1}} a'(0).$

Theorem 3.2.2: *Suppose that u is a positive weakly solution of equation (0.1) with $a(0) > 0$, $E(0) = 0$, and* *(i)* $-\frac{1}{2}r_1 a(0) < a'(0) < 0$

(ii) $\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)} \leq e^{2r_1 t_1}$,

where $r_1 := \sqrt{2(p-1)}C$. *Then the life-span of* u *is bounded:*

$$T \leq \alpha_6 := \frac{\pi}{(p-1)C} + \frac{1}{2r_1}\ln\left(\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)}\right) \leq \alpha_5.$$

And there is a constant $t_4 > 0$ *with*

(iii) $t_4 \leq t_3 := \frac{1}{2r_1}\ln\left(\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)}\right)$

(iv) $a(t) \geq a(t_4)\left(\sin\left(\frac{p-1}{2}C(\alpha_6 - t)\right)\right)^{-\frac{4}{p-1}}$.

Remark 3.2.3: In Theorem 3.2.1 we have no restriction (i) or (ii) under Theorem 3.2.2. It seems that Theorem 3.2.1 is better as Theorem 3.2.2, yet under the suppositions (i) , (ii) Theorem 3.2.2 is better then Theorem 3.2.1.

4. Estimates for the Life-Span of the Solutions of equation (0.1) under Negativ-Energy

In this chapter we suppose the energy $E(0)$ is negative and consider the following cases:

(i) $a(0) > 0, a'(0) > 0$ (ii) $a(0) > 0, a'(0) = 0$ (iii) $a(0) > 0, a'(0) < 0$.

4.1. **Fundamental Lemmas.** In this section we use the following lemmas and

those argumentations of proof to lemmas are not true for positive energy, so under positive energy we need to seek another method to show the results.

Lemma 4.1: *Suppose that* $u \in H1$ *is a positive weakly solution of equation (0.1) with* $a(0) > 0$ *and* $E(0) < 0$. *Then*

(i) for $a'(0) \geq 0$, *we have* $a'(t) > 0 \ \forall t > 0$.

(ii) for $a'(0) < 0$, *there exists a constant* $t_5 > 0$ *with* $a'(t) > 0 \ \forall t > t_5$, $a'(t_5) = 0$ *and*

$$t_5 \leq t_6 := \frac{-a'(0)}{(p-1)(\delta^2 - E)},$$

where δ *is the positive root of the equation*

$$\frac{2}{p+1}\lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E(0) = 0.$$

4.2. **Estimates for the Life-Span of the Solutions of equation (0.1) under** $E(0) < 0$**,** $a'(0) \geq 0$**.** Theorem 4.2: *Suppose that* $u \in H1$ *is a positive weakly solution of equation (0.1) with* $E(0) < 0$*, and* $a'(0) \geq 0$*. Then the life-span of* u *is bounded:*

$$T \leq \alpha_5 := k_0^{-1} k_2^{-1} \cdot \sin^{-1}\left(k_2 a(0)^{-\frac{p-1}{4}}\right) \tag{4.2.1}$$

where

$$k_0 := \frac{p-1}{2} a(0)^{-\frac{p+1}{4}} \sqrt{\frac{1}{4} a(0)^{-1} a'(0)^2 + \frac{p-1}{p+1}\left(\delta^2 - \frac{p+1}{p-1} E(0)\right)},$$

$$k_2 := \left(\frac{k_1}{k_0}\right)^{\frac{p-1}{p+1}}, k_1 := \frac{p-1}{2} \sqrt{\frac{\delta^2 - \frac{p-1}{p+1} E(0)}{p+1}}.$$

Further we have

$$a(t) \geq k_2^{\frac{4}{p-1}} \left(\sin\left(k_0 k_2 \left(\alpha_5 - t \right)\right)\right)^{-\frac{4}{p-1}} \quad \forall t \in [0, T]. \tag{4.2.2}$$

Remark4.2: 1) We can good estimate the rate of the singularity of $a(t)$ and the life-span of u, but we can not get them contemporaneously:

$$T \leq \alpha_6 := k_0^{-1} k_2^{-1} \frac{\tan^{-1}\left(k_2 a(0)^{-\frac{p-1}{4}}\right)}{\sqrt{1 - k_2^2 a(0)^{-\frac{p-1}{2}}}} \tag{4.2.3}$$

(4.2.4)

$$a(t) \geq k_2^{\frac{4}{p-1}} \left\{\tan\left\{\tan^{-1}\left[\left(k_2 a(0)^{-\frac{p-1}{4}}\right) - k_0 k_2 \sqrt{1 - k_2^2 a(0)^{-\frac{p-1}{2}}}\, t\right]\right\}\right\}^{-\frac{4}{p-1}}$$

for each $t \in [0, T]$.

2) For $k_2 \cdot a(0)^{-\frac{p-1}{4}} = 1$, that is,

$$4(2-p)\left(\delta^2 - E(0)\right) a(0) = (p+1) a'(0)^2 \quad n \geq 4, \tag{4.2.5}$$

then we can get a better estimate for the life-span of u:

$$T \leq k_0^{-1} k_2^{-1} \frac{p-1}{2(p+1)} \sqrt{\pi} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{p+1}\right)}{\Gamma\left(\frac{p}{p+1}\right)} := \alpha_7. \tag{4.2.6}$$

Blow-up set of the solution

According to the above results concerning blow-up solution, we want to seek the (set of) blow-up point(s) and the blow-up rate and blow-up constant of the solution for the semilinear wave equation $\Box u = u^p$ with smooth initial values, for instance, u_0, u_1 are both in $C_0^\infty(\mathbb{R})$ and we consider the sets

$$S := \left\{ (t_0, x_0) \in \mathbb{R}^2 \;\middle|\; u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (t_0, x_0) \right\},$$

$$S_{T^*} := \left\{ x_0 \in \mathbb{R} \;\middle|\; u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (T^*, x_0) \right\},$$

$$S_{T^*,\ L^q} := \left\{ x_0 \in \mathbb{R} \;\middle|\; \begin{array}{c} \lim_{t \to T^*} \left(\int_{B_r(x_0)} |u|^q (t,x)\, dx \right)^{-1} = 0 \\ \lim_{t \to T^*} \left(\int_{\mathbb{R} - B_r(x_0)} |u|^q (t,x)\, dx \right)^{-1} > 0 \end{array} \quad \text{for each } r > 0 \right\},$$

where $B_r(x_0) = \left\{ x \in \mathbb{R} : |x - x_0| \leq r \right\}$. We call S, S_{T^*} and $S_{T^*,\ L^q}$ the blow-up set, blow-up set at time T^* and the blow-up set in the sense of L^q of u. The problems are:

(1) What are the sets S, S_{T^*} and $S_{T^*,\ L^q}$?

(2) How large are these sets?

(3) What are the blow-up rate of u in the neighborhoods of S, S_{T^*} and $S_{T^*,\ L^q}$?

(4) What are the blow-up constants of u in the neighborhoods of S, S_{T^*} and $S_{T^*,\ L^q}$?

To study the above hard problems we concentrate on the properties later.

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