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# 由 GL_2 與 GSp_4 所衍生的志村多樣體及其幾何(第 2 年)
# 研究成果報告(完整版)

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中 文 摘 要 ： 我們研究某些志村曲線的特殊纖維之幾何,並且給了大域至局部 Jacquet-Langlands 相容理論的直接證明。

中文關鍵詞： 志村曲線,消沒圈層,加羅瓦表示,p 進位單值化理論

英 文 摘 要 ： We study the geometry of the special fibers of certain Shimura curves over a totally real field and give a direct
proof of global-to-local Jacquet-Langlands compatibility.

英文關鍵詞： Shimura curves； Vanishing cycles； Galois representations； Cerednik-Drinfeld uniformizations

# RAMAKRISHNA-KHARE SYSTEMS AND MODULARITY LIFTING THEOREMS IN HIGHER WEIGHTS

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Abstract. Following the ideas of Khare and Ramakrishna-Khare, we give a different approach to prove the modularity lifting theorem in higher weights without using Taylor-Wiles systems.

## 1. Introduction

The focus of this article is to give a different approach of proving modularity lifting theorems of Galois representations in Wiles [27] and Taylor-Wiles [26]:

**Theorem.** *Let $\rho : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathcal{O})$ be an odd, continuous, absolutely irreducible, $p$-adic Galois representation which is ramified at finitely many primes, and de Rham at $p$ with Hodge-Tate weights $(k-1, 0)$ with $k \geq 2$. Assume that the reduction modulo $p$ of $\rho$ is modular. Then $\rho$ is isomorphic to an integral model of a $p$-adic representation $\rho_f$ arising from a newform $f$.*

This is sometimes described as "$R = \widehat{\mathbb{T}}$"-theorems, where $R$ is the universal deformation ring of the reduction $\overline{\rho}$ of $\rho$ and $\widehat{\mathbb{T}}$ is a certain localized Hecke algebra.

In this article, we generalize the approach introduced by Khare and Ramakrishna from weight 2 [15, 14] to higher weights $k < p$; we prove:

**Theorem.** *Let $N$ be a square-free positive integer, let $p > 5$ be a prime not dividing $N$. Let $f \in S_k(\widehat{\Gamma}_0(N))$ be a cusp newform. Let $2 \leq k < p$ be an integer. Let $\overline{\rho} = \overline{\rho}_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{F})$ be the $\bmod p$ Galois representation attached to $f$. Assume that $\overline{\rho}$ is irreducible, minimally ramified at primes at primes dividing $Np$, and that $\overline{\rho}\,|_{I_p} = \begin{pmatrix} \overline{\chi}_p^{k-1} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$, where $\overline{\chi}_p$ is the $\bmod p$ cyclotomic character. Then the universal deformation ring $R$ associated to $\overline{\rho}$ is canonically isomorphic to $\widehat{\mathbb{T}}_{\emptyset} (\simeq \widehat{\mathbb{T}})$.*

The essential point of this generalization is to replace the use of Jacobians of Shimura curves with the cohomology of certain sheaves over the curves in question.

The paper is organized as follows. We first review the basic properties of Shimura curves, and then take up the study of the bad reduction of the Shimura curves at a prime $r$ dividing the level in question by the Tate-Oort theory. Along this line, we obtain an explicit description of the special fiber as the union of exactly two irreducible components

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of multiplicity 1 and $r-1$ respectively. These two components cross transversally at the supersingular points and nowhere else. This description enables us calculate the vanishing cycles (Proposition 4.5) and the cohomology of Shimura curves (Proposition 4.7).

Suppose that $\overline{\rho}$ is an odd, continuous, absolutely irreducible Galois representation with values in $\mathbf{GL}_2(\mathbb{F})$ where $\mathbb{F}$ is a finite field of characteristic $p$. Under certain hypotheses, we impose the deformation conditions on such $\overline{\rho}$, and show the existence of deformation ring $R_Q$ (Proposition 3.3) which is universal for the deformations of $\overline{\rho}$ unramified outside $S \cup Q$ and minimally ramified on $S$, where $S$ is the set of primes at which $\overline{\rho}$ is ramified and $Q$ is a set of auxiliary primes. According to Khare and Ramakrishna, we define for each $\alpha \subseteq Q$ a quotient $R_Q^\alpha$ of $R_Q$ which is universal for the deformations $\rho$ of $\overline{\rho}$ such that, for $q \in \alpha$, the local representation $\rho\mid_{G_q}$ is special, i.e., it is of the form $\begin{pmatrix} \chi_p & * \\ 0 & 1 \end{pmatrix}$ where $\chi_p$ is the $p$-adic cyclotomic character. With the result at hand, we shall identify the tangent space of $R_Q^\alpha$ with a suitable Selmer group.

From now on, we suppose that $\overline{\rho}$ is modular with weight $k < p$. The first step (Proposition 5.1) is to show that there exists a set $Q$ of auxiliary primes such that $R_Q^Q$ is isomorphic to $W(\mathbb{F})$ and that the corresponding deformation $\rho_Q^Q$ is modular. The existence of $Q$ is due to Khare and Ramakrishna [15]. The proof of this isomorphism and the modularity of $\rho_Q^Q$ use the fact that the tangent space of $R_Q^Q$ is trivial and the work of Diamond-Taylor [9]: there is a $p$-adic lifting of $\overline{\rho}$ that arises from the $Q$-new quotient of $\widehat{\mathbb{T}}_Q$.

The second step (Theorem 5.2) is to show that the deformation $\rho_Q$ parametrized by $R_Q$ is modular. As in the founder article of Wiles, we introduce a localized Hecke algebra $\widehat{\mathbb{T}}_Q$ parametrizing a modular deformation of geometric origin and show that the canonical homomorphism $R_Q \to \widehat{\mathbb{T}}_Q$ is actual an isomorphism. In the proof of Theorem 5.2, we use a variant of Wiles' numerical isomorphism criterion refined by Lenstra (Proposition 6.1). Thus, we are quickly reduced to study how a certain congruence module grows as one relaxes conditions of newness at primes in $Q$. Let $\pi : \widehat{\mathbb{T}}_Q \to \widehat{\mathbb{T}}_Q^Q \simeq R_Q^Q \simeq W(\mathbb{F})$ be the canonical homomorphism resulting from the first step, $\phi : R_Q \to \widehat{\mathbb{T}}_Q$, $\Phi = \ker(\pi\phi)/\ker(\pi\phi)^2$ and $\eta = \pi(\mathrm{Ann}_{\mathbb{T}}(\ker(\pi)))$. The criterion consists with verifying the equality $|W(\mathbb{F})/\eta| = |\Phi|$.

The verification of this equality is the main part of this paper. First, we need a sophisticated calculations of Galois cohomology and identify $\Phi$ to certain Selmer groups; we thus obtain an upper bound for $|\Phi|$. Then in the proof of theorem in weight 2, Khare used a result of Ribet-Takahashi [25] which generalized a calculation of Ribet [24] in his work on Serre's conjecture. The idea of Ribet is to compare two Shimura curves such that two prime numbers $q$ and $q'$ dividing the discriminant for one and dividing the level for the other. Hence in weight 2 we could compare the Jacobians of corresponding Shimura curves. For most of our work, we extend the level-lowering part by replacing Ribet's method via Jacobians of certain Shimura curves with arguments using vanishing cycles on those curves (Proposition 7.6). This requires a study of Boutot-Carayol's version of Čerednik-Drinfel′d uniformization of Shimura curves [2]. With these results, the lifting of isomorphism $R_Q^Q = \widehat{\mathbb{T}}_Q^Q$ to $R_Q = \widehat{\mathbb{T}}_Q$ is carried out by applying the level-lowering (Proposition 8.2), and the numerical isomorphism criterion alluded to above.

The third and final step (Theorem 5.3) is to get rid of the set of auxiliary primes $Q$ and yields that the ramified minimal universal deformation $\rho_\emptyset$ is modular. More precisely, we use the local-to-global principle of Böckle [1] to show the canonical morphism $R_\emptyset \to \widehat{\mathbb{T}}_\emptyset$ is an actual isomorphism.

The ideas developed here can be applied to the case over totally real fields, and this will be in our forthcoming work.

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## 2. Background on Shimura curves

We first give a brief review of the basic properties of Shimura curves, following Buzzard [3]. We then take up the study of the reduction modulo a prime of the Shimura curves. As a preparation for our later work, we shall also present the Hecke correspondences.

2.1. **Model of Shimura curves.** Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$; let $S$ be the set of places where it ramifies. Let $D = \prod_{\ell \in S} \ell$, and let $M$ be a square-free integer prime to $D$. Let $\mathcal{O}_B$ be a maximal order of $B$.

For all $\ell \notin S$, we choose an isomorphism $\phi_\ell : B \otimes \mathbb{Q}_\ell = B_\ell \xrightarrow{\sim} \mathbf{M}_2(\mathbb{Q}_\ell)$ such that $\phi_\ell(\mathcal{O}_{B_\ell}) = \mathbf{M}_2(\mathbb{Z}_\ell)$. Let $u_M : (\mathcal{O}_B \otimes \widehat{\mathbb{Z}})^\times = \widehat{\mathcal{O}}_B^\times \to \mathbf{GL}_2(\mathbb{Z}/M\mathbb{Z})$ induced by $\phi_\ell$. We let

- $\widehat{\Gamma}_0^D(M)$ denote the preimage of $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}/M\mathbb{Z}) \middle| c = 0 \right\}$ under $u_M$;
- $\widehat{\Gamma}_1^D(M)$ denote the preimage of $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}/M\mathbb{Z}) \middle| c = 0, d = 1 \right\}$ under $u_M$.

We define an *Eichler order of level $M$* as follow:

$$R_{M,D} := \left\{ x \in \mathcal{O}_B \middle| \phi_\ell(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\ell} \text{ for all } \ell \mid M \right\}.$$

We write $\widehat{R}_{M,D} = R_{M,D} \otimes \widehat{\mathbb{Z}} = \prod_\ell (R_{M,D} \otimes \mathbb{Z}_\ell)$. We see that $\widehat{R}_{M,D}^\times = \widehat{\Gamma}_0^D(M)$.

Let $\Gamma$ be an open compact subgroup of $\widehat{B}^\times$. The *Shimura curve* $X^D(\Gamma)$ is defined by:

$$X^D(\Gamma)(\mathbb{C}) := B^\times \backslash B_{\mathbb{A}}^\times / \mathbb{R}^\times \Gamma \cdot C_{\mathrm{i}}$$

where $C_{\mathrm{i}}$ is the stabilizer of $\sqrt{-1}$ in $(B \otimes \mathbb{R})^\times$. In particular, if $\Gamma = \widehat{R}_{M,D}^\times$, we will denote the corresponding Shimura curve by $X^D(M)(\mathbb{C})$.

Consider a torsion-free subgroup $\Gamma$ of $\widehat{\mathcal{O}}_B^\times$ of level $M$. Let $r$ be a prime, not dividing $MD$. Let $\Gamma_0 = \Gamma \cap \widehat{\Gamma}_0^D(r)$ and $\Gamma_1 = \Gamma \cap \widehat{\Gamma}_1^D(r)$. We study here the reduction modulo $r$ of the Shimura curves $X^D(\Gamma_i)$ for $i = 0, 1$.

We fixed an isomorphism $\phi_r : \mathcal{O}_B \otimes \mathbb{Z}_r \simeq \mathbf{M}_2(\mathbb{Z}_r)$. Let $e$ be the idempotent in $\mathcal{O}_B/r\mathcal{O}_B$ corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by $\phi_r$. Following Buzzard [3], we define a $\Gamma_0(r)$-structure, (resp. $\Gamma_1(r)$-structure, on a false elliptic curve $A$ as a finite flat group scheme $K_1$ of rank $r$ inside $(1-e)A[r]$, resp. a (Drinfel′d) generator of this subgroup. For rigidification, we also introduce a $\Gamma$-level $\overline{\nu}$ structure on $A$ (that is, a full level structure $\nu$ of level $N$ taken modulo $\Gamma$).

By the Corollary 4.2 of [3], the moduli problem on $\mathbb{Z}_r$-schemes $S \mapsto \{$isomorphism classes of $(A, \iota, \overline{\nu}, K_1)_{/S}\}$ is representable by a proper $\mathbb{Z}_r$-scheme which we denote by $X^D(\Gamma_0)$. Let us recall that this moduli problem is isomorphic to the problem $(A, \iota, \overline{\nu}, C)$ where $C$ is an isotropic subgroup of $A[r]$ of order $r^2$. (See the paragraph after Definition 3.1 in [3].) There is a universal triple $(A^u, \iota^u, K_1^u)$ defined over $X^D(\Gamma_0)$. Recall the Theorem 4.7 of [3]:

**Proposition 2.1.** (i) *The scheme $X^D(\Gamma_0)$ is proper over $\mathbb{Z}_r$.*

(ii) *It is semistable over $\mathbb{Z}_r$, i.e. regular, and smooth away from the supersingular points in characteristic $r$, with strictly henselian local ring at such a geometric point $\mathbb{Z}_r^{\mathrm{ur}}[\![X,Y]\!]/(XY-r)$; moreover, there are exactly two smooth irreducible components, $X^m$ and $X^e$, in the special fiber; they can be described as the Zariski closure of the locus $X^{m,0}$ where $K_1$ is of multiplicative type, resp. of $X^{e,0}$ where $K_1$ is étale.*

(iii) *The map $\pi : X^D(\Gamma_0) \to X^D(\Gamma)$ forgetting the $\Gamma_0$-structure is finite and flat.*

We consider the moduli problem of $\mathbb{Q}_r$-schemes $S \mapsto \{$isomorphism classes of $(A, \iota, P)_{/S}\}$ where $P$ is a generator of $K_1$; it is representable over $\mathbb{Q}_r$ by the curve $X^D(\Gamma_1)_{\mathbb{Q}_r}$. Let $X^D(\Gamma_1)$ be the normalization of $X^D(\Gamma_0)$ in $X^D(\Gamma_1)_{\mathbb{Q}_r}$.

Following [11] Proposition 3.3.6, we shall use Tate-Oort theory in order to prove:

**Proposition 2.2.** (i) *The model $X^D(\Gamma_1)$ of $X^D(\Gamma_1)_{\mathbb{Q}_r}$ is regular and flat over $\mathbb{Z}_r$.*

(ii) *The map $\pi_{10} : X^D(\Gamma_1) \to X^D(\Gamma_0)$ is finite flat; the special fiber of $X^D(\Gamma_1)$ is a divisor with normal crossings, with exactly two irreducible components $Y^e = \pi_{10}^{-1}(X^e)$ and $Y^m = \pi_{10}^{-1}(X^m)$ with multiplicity $1$ and $r-1$ respectively, whose underlying reduced subschemes are smooth.*

(iii) *The two components cross (transversally) at the supersingular points and nowhere else.*

*Proof.* Let us consider the finite group scheme $C = K_1^u$ of rank $r$ over the $\mathbb{Z}_r$-scheme $X^D(\Gamma_0)$ and $C^+$ the complement of the zero section in $C$.

If $s \in X^{e,0}$, then $C^+$ is étale over $X^D(\Gamma_0)$ in a neighbourhood of $s$ by Proposition 2.1. If $s \in X^{m,0}$, then $C^+$ is of multiplicative type, hence is isomorphic to $\mu_r$ on an étale neighbourhood of $s$. Hence, the statement about multiplicities is obvious, as the map $\pi_{10}$ is an isomorphism. Let $s \in X^m \cap X^e$. By Proposition 2.1, the completed local ring $\widehat{\mathcal{O}}_{X^D(\Gamma_0),s}$ at $s$ is isomorphic to $R = \mathbb{Z}_r[\![u,v]\!]/(uv-r)$ in such a way that

(♠) the completion at $s$ of $X^e$ (resp. $X^m$) has equation $v = 0$ (resp. $u = 0$).

Tate-Oort theory [19] classifies finite flat group schemes of rank $r$ over $R$; in particular the pull-back $C_R$ of $C$ over Spf $R$ is isomorphic to $G_R(x,y)$, the Tate-Oort group scheme of rank $r$ over $R$ for some parameters $x, y \in R$, where, in the notations of Tate-Oort, $a = \nu x$, $b = y$ and $w_r = \nu r = ab$, where $w_r \in \mathbb{Z}_r$ is an explicit Gauss sum. Note that $G_R(x,y) \simeq G_R(x',y')$ if and only if $x'x^{-1}$ is the $(r-1)^{\text{st}}$ power of a unit in $R$. As in [11] Corollary 3.3.5, we deduce from (♠) that $x = \alpha u$ and $y = \alpha^{-1}v$ for some $\alpha \in R^\times$.

Therefore, $C_R \simeq G_R(u,v)$ if and only if the unit $\alpha$ is a $(r-1)^{\text{st}}$ power in $R$. Now the extension $R[(\alpha)^{\frac{1}{r-1}}]$ is finite étale over $R$. Since the problem is local in the étale topology, we may assume $C_R \simeq G_R(u,v)$.

Now we recall the Tate-Oort equations for $G_R(u,v)$ over $R$. Put $X_1 = \nu u$, $X_2 = v$, then $G_R(u,v) = \mathrm{Spf}(R[Y][\![X_1,X_2]\!]/(X_1X_2 - w_r, Y^r - X_1Y))$. (See p.13 of Tate-Oort [19]).

Now the factorization $Y^rX_1Y = Y(Y^{r-1} - X_1)$ provides an embedding of the algebra of formal functions on $G_R(u,v)$ into the ring $R_0 \times R^*$, where

$$R_0 = \mathbb{Z}_r[\![X_1,X_2]\!]/(X_1X_2 - w_r); \quad R^* = \mathbb{Z}_r[\![X_2,Y]\!]/(Y^{r-1}X_2 - w_r).$$

Dually, we have a surjective morphism

$$\mathrm{Spf}(R_0) \sqcup \mathrm{Spf}(R^*) \to G_R(u,v).$$

The image of $\mathrm{Spf}(R_0)$ corresponds to the zero section, while the image of $\mathrm{Spf}(R^*)$ is the scheme-theoretic closure of $C^+$. It follows that $\mathrm{Spf}(R^*)$ is a local model for $C^+$ over a neighbourhood of the singular point $s$. Obviously, $\mathrm{Spf}(R^*)$ has the properties required. Moreover $\mathrm{Spf}(R^*)$ is normal, and it follows that $\mathrm{Spf}(R^*)$ is a formal local model for $X^D(\Gamma_1)$. □

2.2. **Local Hecke correspondences.** The curve $X^D(\Gamma_1)$ is a fine moduli space for triples $x = (A, \overline{\nu}, P)$, where, if $D > 1$, $A$ is a false elliptic curve with a level $\Gamma_1$ structure $\overline{\nu}$ and a $\mathcal{O}_B$-stable group scheme $K_1$ of rank $r$ inside $(1-e)A[r]$ and a Drinfel′d generator $P$ of $K_1$, and, if $D = 1$, $A$ is a generalized elliptic curve and a generator $P$ of a cyclic subgroup $K_1$ of order $r$. There is a universal triple $(A^u, \iota^u, K_1^u)$ defined over $X^D(\Gamma_1)$; let $f : A^u \to X^D(\Gamma_1)$.

Similarly, $X^D(\Gamma_1 \cap \widehat{\Gamma}_0^D(\ell))$ classifies $(x, C)$ where $x$ is a triple as above and $C$ is an isotropic subgroup of order $\ell^2$ in $A[\ell]$ as noticed before Proposition 2.1.

For $\ell \nmid DMrp$, we have two degeneracy maps $\alpha_\ell$ and $\beta_\ell$ from $X^D(\Gamma_1 \cap \widehat{\Gamma}_0^D(\ell))$ to $X^D(\Gamma_1)$.

$$\begin{tikzcd} & X^D(\Gamma_1 \cap \widehat{\Gamma}_0^D(\ell)) \arrow[dl, "\alpha_\ell"'] \arrow[dr, "\beta_\ell"] & \\ X^D(\Gamma_1) & & X^D(\Gamma_1) \end{tikzcd}$$

They are defined by $\alpha_\ell((x,C)) = x$ and $\beta_\ell((x,C)) = (\phi_* x)$ where $\phi : A \to A/C$ denotes the quotient map and $\phi_* x = (A/C, \phi_*\overline{\nu}, \phi_* P)$.

Recall that if we are given a lisse sheaf $\mathscr{F}$ on $X^D(\Gamma_1)$ with a morphism $A_\ell : \beta_\ell^*\mathscr{F} \to \alpha_\ell^*\mathscr{F}$ over $X^D(\Gamma_1 \cap \widehat{\Gamma}_0^D(\ell))$, we can define Hecke correspondence acting on the pair

$(X^D(\Gamma_1), \mathscr{F})$. By contravariant functoriality, it induces an endomorphism $T_\ell$ of $\mathrm{H}^\bullet(X^D(\Gamma_1), \mathscr{F})$ given by $T_\ell = \alpha_{\ell,*} \circ A_{\ell,*} \circ \beta_\ell^*$.

In our situation, we take $\mathscr{F} = \mathrm{Sym}^{k-2}\,\mathrm{R}f_*\mathbb{Z}_p$. $\Gamma_1$ acts on $\mathrm{Sym}^{k-2}\,\mathbb{Z}_p^2$ from the left by its $p$-component. We recall that the lisse sheaf associated to the corresponding representation of the fundamental group of $X^D(\Gamma_1)$ is $\mathrm{Sym}^{k-2}\,\mathrm{R}f_*\mathbb{Z}_p$. Let us consider the morphism $A_\ell : \beta_\ell^*\mathscr{F} \to \alpha_\ell^*\mathscr{F}$ induced by the left action of $(1, \ldots, 1, \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}, 1, \ldots, 1)$ on $\mathrm{Sym}^{k-2}\,\mathbb{Z}_p^2$. Note that this action being through the $p$-component is trivial if $\ell \neq p$. We define the $\ell^{\text{th}}$ Hecke correspondence $T_\ell$ as $t_\ell(A_\ell)$.

For a false elliptic curve $A$, $(\mathbb{Z}/\ell\mathbb{Z})^\times$ acts on $A[\ell]$ by multiplication. We thus have an action of $(\mathbb{Z}/\ell\mathbb{Z})^\times$ on $\Gamma_1$-level structures on $A$. We let $\langle a\rangle(A, K_1, P) = (A, K_1, aP)$ as an endomorphism of $X^D(\Gamma_1)$.

These operators all commute with each other. We let $\mathbb{T}(\Gamma_1)$ denote the $\mathbb{Z}$-algebra generated by $T_\ell$ for all $\ell \nmid MDp$ and the diamond operators.

## 3. Deformation rings and Hecke rings

Let $\mathbb{F}$ be a finite field of characteristic $p > 5$, and let $W = W(\mathbb{F})$ be the ring of Witt vectors with coefficients in $\mathbb{F}$, and let $\mathcal{O}$ be a totally ramified extension of $W$ (hence its residue field is $\mathbb{F}$). Consider the continuous absolutely irreducible mod$p$ Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{F})$. We write $S$ to be the set of primes containing $p$, $\infty$ and the primes at which $\overline{\rho}$ is ramified, and $S' = S \setminus \{p\}$. Let $\mathrm{Ad}^0$ be the set of all trace zero two-by-two matrices over $\mathbb{F}$ with Galois action through $\overline{\rho}$ and by conjugation.

Suppose that $\overline{\rho}$ is *modular* and satisfies the following conditions:

- The Serre weight $k := k(\overline{\rho})$ of $\overline{\rho}$ is greater than 2 and strictly less than $p$.
- $\det(\overline{\rho}) = \overline{\chi}_p$ the mod$p$ cyclotomic character.
- $\mathrm{Ad}^0$ is absolutely irreducible.
- $\overline{\rho}$ is *semistable* at every primes in $S$.
- Moreover, $\overline{\rho}$ is *crystalline* and *ordinary* at $p$.

Note that this implies the order of $\mathrm{im}(\overline{\rho})$ is divisible by $p$ which gives the properties of $\overline{\rho}$ required in [23].

*Remark* 3.1. If $p > k > 2$, the condition being ordinary implies that being crystalline by Perrin-Riou and Fontaine [20].

3.1. **Deformation rings.** We briefly recall the existence of certain deformation rings parametrizing liftings of $\overline{\rho}$ with given local conditions, referring [15] for the details.

Let $Q$ be a finite set of primes disjoint from $S$ such that for all $q \in Q, q \not\equiv \pm 1 \pmod p$ and $\overline{\rho}(\mathrm{Frob}_q)$ has eigenvalues with ratio $q$. Consider the following covariant deformation functor $\mathscr{D}_Q$ from the category of complete noetherian local $W$-algebras to the category of sets:

$$\begin{aligned} \mathscr{D}_Q : \underline{\mathbf{CNL}_W} &\rightsquigarrow \underline{\mathbf{Sets}} \\ (A, \varphi) &\rightsquigarrow \{\rho : G_{\mathbf{Q}} \to \mathbf{GL}_2(A) | \rho \bmod \mathfrak{m}_A = \overline{\rho}\}/\sim, \end{aligned}$$

such that

**(DC1)** $\det\rho = \widetilde{\gamma}\chi_p^{k-1}$, where $\widetilde{\gamma}$ is the Teichmüller lifting of $S'$-ramified character $\gamma$ and $\chi_p$ is the $p$-adic cyclotomic character;

**(DC2)** $\rho$ is unramified outside $S \cup Q$;

**(DC3)** $\rho|_{I_\ell} = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ for all $\ell \in S'$;

**(DC4)** for $\ell = p$, $\rho$ is ordinary and cristalline at $p$,

where $\rho_1 \sim \rho_2$ if and only if there exists $M \in \ker(\mathbf{GL}_2(A) \to \mathbf{GL}_2(\mathbb{F}))$ such that $\rho_1 = M^{-1}\rho_2 M$.

**Definition 3.2.** For any representation satisfying conditions **(DC1)** – **(DC4)**, we will call it *minimally $S$-ramified.*

Since $\mathrm{Ad}^0$ is absolutely irreducible, then by the Schlessinger's criterion it is easy to see that:

**Proposition 3.3.** *The functor $\mathscr{D}_Q$ is pro-representable. We denote its universal couple by $(R_Q, \rho_Q)$.*

*Remark* 3.4. There is *no* condition at any primes $q \in Q$.

More generally for any subset $\alpha \subseteq Q$, we now consider the closed subfunctor of $\mathscr{D}_Q$:

$$\mathscr{D}_Q^\alpha(A) = \{\rho : G_\mathbb{Q} \to \mathbf{GL}_2(A) | \rho \bmod \mathfrak{m}_A = \overline{\rho}\}/\sim$$

such that the conditions **(DC1)** – **(DC4)** hold and moreover

**(DC5)** $\rho|_{G_q} \sim \begin{pmatrix} \chi_p & * \\ 0 & 1 \end{pmatrix}$ for any $q \in \alpha$. (2.5)

Since the functor $\mathscr{D}_Q^\alpha$ is relatively representable, hence we have the following:

**Proposition 3.5.** *The functor $\mathscr{D}_Q^\alpha$ is pro-representable. We denote the corresponding universal couple by $(R_Q^\alpha, \rho_Q^\alpha)$.*

*Remark* 3.6. There is a sequence of natural surjections of local $W$-algebras $R_Q \twoheadrightarrow R_Q^\alpha \twoheadrightarrow R_Q^Q$. If $Q = \emptyset$, we denote the corresponding universal couple by $(R_\emptyset, \rho_\emptyset)$, and call $R_\emptyset$ the *minimal deformation ring.*

*Remark* 3.7. We could also consider similar deformation problems $\mathscr{D}_{Q,\mathcal{O}}^\alpha$ over $\mathcal{O}$ instead of $W$. One checks easily that $R_Q^\alpha \otimes_W \mathcal{O}$ is the universal deformation ring of $\mathscr{D}_{Q,\mathcal{O}}^\alpha$.

3.2. **The local conditions.** Let $G_{S\cup Q}$ be the Galois group of the maximal extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ which is unramified outside $S \cup Q$. We introduce local conditions in order to define the Selmer group.

- For $v \in S'$, we let

$$\mathcal{L}_v = \mathrm{H}^1_{\mathrm{nr}}(G_v, \mathrm{Ad}^0) := \ker\left(\mathrm{H}^1(G_v, \mathrm{Ad}^0) \to \mathrm{H}^1(I_v, \mathrm{Ad}^0)\right).$$

- For $v = p$, we define

$$\mathcal{L}_p = \ker\left(\mathrm{H}^1(G_p, \mathrm{Ad}^0) \to \mathrm{H}^1(I_p, \mathrm{Ad}^0 / Z)\right),$$

  where $Z$ consists of $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$.

- For $v \in Q$, $\mathcal{L}_v$ is spanned by the 1-cocycles class given by

$$g(\sigma_v) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ g(\tau_v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

modulo 1-coboundaries, where $\sigma_v$ and $\tau_v$ generate the tame quotient of $G_v$ and satisfy $\sigma_v \tau_v \sigma_v^{-1} = \tau_v^{p_v}$.

Let $\mathcal{L}$ be the collection of these local conditions, and define the *Selmer group* to be

$$\mathrm{H}^1_{\mathcal{L}}(G_{S\cup Q}, \mathrm{Ad}^0) := \ker\left(\mathrm{H}^1(G_{S\cup Q}, \mathrm{Ad}^0) \to \bigoplus_{v \in S\cup Q} \mathrm{H}^1(G_v, \mathrm{Ad}^0)/\mathcal{L}_v\right).$$

The importance of Selmer group stems from the fact that it is canonically isomorphic to the reduced tangent space of the deformation problem. The proof of the following proposition is routine (See [17]).

**Proposition 3.8.** *Let $\mathfrak{m}_Q$ be the maximal ideal of $R_Q^Q$. We have an isomorphism of $\mathbb{F}$-vector space:*

$$\mathrm{H}^1_{\mathcal{L}}(G_{S\cup Q}, \mathrm{Ad}^0) \simeq \mathrm{Hom}(\mathfrak{m}_Q/(p, \mathfrak{m}_Q^2), \mathbb{F}).$$

3.3. **Hecke rings and modular Galois representations.** Let $N \geq 1$ be a square-free integer. We let $\Gamma = \widehat{\Gamma}_0(N)$ be the subgroup of $\mathbf{GL}_2(\widehat{\mathbb{Z}})$ consisting of elements $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$ with determinant 1. Let $f \in S_k(\widehat{\Gamma}_0(N))$ be a cusp eigen newform. We fix a prime $p > 5$ not dividing $N$. Let $K$ be the $p$-adic field generated by the Fourier coefficients of $f$, let $\mathcal{O}$ be the ring of integers of $K$, and let $\mathbb{F}$ be its residue field. Let $\mathbb{T}$ be the $\mathbb{Z}$-algebra generated by Hecke operators $T_\ell$ for all $\ell \nmid N$ acting on $S_k(\widehat{\Gamma}_0(N))$. We consider the attached character $\lambda_f : \mathbb{T} \to K$ and assume that $f$ is *ordinary*; that is, $\lambda_f(T_p) \in \mathcal{O}_K^\times$. We are interested in the Galois representation $\rho_f : G_\mathbb{Q} \to \mathbf{GL}_2(K)$ attached to $f$. Let $\mathfrak{p} = \ker(\lambda_f)$ and let $\mathfrak{m}_f$ be the unique maximal ideal of $\mathbb{T}$ containing $\mathfrak{p} + p\mathbb{T}$.

Suppose that

- **(H1)** $2 \leq k < p$;
- **(H2)** $\mathfrak{m}_f \subset \mathbb{T}$ is non-Eisenstein;
- **(H3)** the residual representation $\overline{\rho}_f : G_\mathbb{Q} \to \mathbf{GL}_2(\mathbb{F})$ is minimally ramified at primes dividing $Np$;
- **(H4)** $\overline{\rho}_f \mid_{I_p} = \begin{pmatrix} \overline{\chi}_p^{k-1} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$.

Consider a finite set of primes $Q = \{q_2, \dots, q_{2m}\}$ of odd cardinality such that $\overline{\rho}_f$ is unramified on $Q$, $q_i \not\equiv \pm 1 \bmod p$ for $q_i \in Q$, and such that $\mathrm{Tr}(\overline{\rho}_f(\mathrm{Frob}_{q_i})) = \pm(q_i + 1)$ for $q_i \in Q$. Let $D = \prod_{q \in Q} q$. We write $\widetilde{N} = pND$, and fix a prime $q_1$ dividing $pN$.

For $0 \leq s \leq m$, let $Q_s = \{q_1, q_2, \dots, q_{2m-2s}\}$ and let $B_s$ be the indefinite quaternion algebra ramified on $Q_s := \{q_1, \dots, q_{2m-2s}\}$. Let $D_s = q_1 \cdots q_{2m-2s}$ and $M_s = \widetilde{N}_s/D_s$ and for $0 \leq s \leq m$. Choose an Eichler order $R_{M_s,D_s}$ of level $M_s$ in $B_s$. Denote the corresponding Shimura curve by $X^{D_s}(M_s)$.

Let $A$ be a $\mathbb{Z}_p$-algebra, and let $\Lambda_k(A) = \mathrm{Sym}^{k-2}\,\mathbb{Z}_p^2 \otimes_{\mathbb{Z}_p} A$. Notice that if $k < p$ and $A$ is a $\mathbb{Z}_p$-flat algebra, $\mathrm{H}^1(X^{D_s}(M_s), \Lambda_k(A))$ is a torsion-free $A$-module. If $A = \mathbb{Z}_p$, we simply write it $\Lambda_k$ or $\Lambda$. The $\ell^{\text{th}}$ Hecke correspondence $T_\ell$ (defined in §2.2) defines an endomorphism, still denoted by $T_\ell$, of $\mathrm{H}^1(X^{D_s}(M_s), \Lambda_k(A))$ for all $\ell \nmid \widetilde{N}$. We define the *Hecke algebra* $\mathbb{T}_Q^{D_s}$ to be the $A$-algebra generated by these endomorphisms $T_\ell$ for all primes $\ell \nmid \widetilde{N}$. For $A = W$, we drop the $A$ in the notation and we simply write $\mathbb{T}_Q^{D_s}$.

We also have the *minimal Hecke algebra* $\mathbb{T}_\emptyset$ generated over $W$ by Hecke operators $T_\ell$ on the corresponding modular curve $X(\widehat{\Gamma}_0(Np))$ for all primes $\ell$ such that $(\ell, \widetilde{N}) = 1$.

For any $0 \le s \le m$, we have obvious $W$-algebra homomorphisms

$$\mathbb{T}_Q^{D_s} \to \mathbb{T}_\emptyset \to \mathbb{T}.$$

Note that they are surjective. We let $\mathfrak{m}_Q$ be the preimage of $\mathfrak{m}_f$ under the map $\mathbb{T}_Q^{D_s} \to \mathbb{T}$, and denote the completion of the Hecke algebra $\mathbb{T}_Q^{D_s}$ at $\mathfrak{m}_Q$ by $\widehat{\mathbb{T}}_Q^{D_s}$. Note that for any $0 \le s \le m$, the Hecke algebra $\mathbb{T}_Q^{D_s}$ is finite flat over $W$.

**Lemma 3.9.** (i) *We have Galois representations*

$$\rho_{Q,\mathrm{mod}}^{D_s} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\widehat{\mathbb{T}}_Q^{D_s}) \quad \textit{(resp. } \rho_{\emptyset,\mathrm{mod}} : G_{\mathbb{Q}} \to \mathbf{GL}_2(\widehat{\mathbb{T}}_{\mathfrak{m}_f})\textit{)}$$

*which are unramified outside $S \cup Q$ such that for $\ell \notin S \cup Q$,*

$$\mathrm{Tr}\,\rho_{Q,\mathrm{mod}}^{D_s}(\mathrm{Frob}_\ell) = T_\ell \qquad \textit{and} \qquad \mathrm{Tr}\,\rho_{\emptyset,\mathrm{mod}}(\mathrm{Frob}_\ell) = T_\ell.$$

*They arise by uniquely determined specializations of the universal representations*

$$\rho_Q^{D_s} : G_{\mathbb{Q}} \to \mathbf{GL}_2(R_Q^{D_s}) \quad \textit{(resp. } \rho_\emptyset : G_{\mathbb{Q}} \to \mathbf{GL}_2(R_\emptyset)\textit{)}.$$

(ii) *They satisfy*

$$\rho_{Q,\mathrm{mod}}^{D_s} \in \mathscr{D}_{Q,\mathcal{O}}^{D_s}(\widehat{\mathbb{T}}_Q^{D_s}) \quad \textit{(resp. } \rho_{\emptyset,\mathrm{mod}} \in \mathscr{D}_{\mathcal{O}}(\widehat{\mathbb{T}}_{\mathfrak{m}_f})\textit{)}.$$

(iii) *The local $\mathcal{O}$-algebra homomorphisms defined by the universal property*

$$R_Q^{D_s} \otimes_W \mathcal{O} \to \widehat{\mathbb{T}}_Q^{D_s} \quad \textit{(resp. } R_\emptyset \otimes_W \mathcal{O} \to \widehat{\mathbb{T}}_{\mathfrak{m}_f}\textit{)}$$

*are surjective.*

*Proof.* By the irreducibility of the residual representations, we can apply the theorem of Carayol [7] and Nyssen [18] to construct a representation using pseudo-representations on $\widehat{\mathbb{T}}_Q^{D_s}$ and $\widehat{\mathbb{T}}_Q^{D_s} \otimes \mathbb{Q}$. By definition of the representation, it satisfies the local conditions; also by [7], the same holds for its integral structure.

According to Carayol [6], the representations $\rho_{Q,\mathrm{mod}}^{D_s}\big|\, G_q$ and $\rho_{\emptyset,\mathrm{mod}}\big|\, G_q$ are of the form

$$\pm \begin{pmatrix} \chi_p & * \\ 0 & 1 \end{pmatrix}.$$

Hence, the existence of the specialization map follows from the universal property of deformation ring $R_Q^{D_s}$. ∎

3.4. **The main theorem.** Our main theorem is:

**Theorem 3.10.** *Let $N$ be a square-free positive integer, let $p > 5$ be a prime not dividing $N$. Let $f \in S_k(\widehat{\Gamma}_0(N))$ be a cusp newform. Let $2 \leq k < p$ be an integer. Let $\overline{\rho} = \overline{\rho}_f : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{F})$ be the* mod$p$ *Galois representation attached to $f$. Assume it satisfies conditions* **(H1)** – **(H4)**. *Then the universal deformation ring $R$ associated to $\overline{\rho}$ is canonically isomorphic to $\widehat{\mathbb{T}}_{\emptyset}(\simeq \widehat{\mathbb{T}})$.*

*Remark* 3.11. The analogue result has been proven by Taylor-Wiles in weight 2 case using the Taylor-Wiles systems method; it has been generalized by Ramakrishna [22] following a similar method. In weight 2, it has been reproved by Khare; for higher weights, we will prove the theorem stated above by following the ideas of Khare.

## 4. Review of vanishing cycles

In this section, we assume that $X$ is a proper semistable curve over $S = \operatorname{Spec} \mathbb{Z}_q$ such that the irreducible components of the special fiber are isomorphic to $\mathbb{P}^1$. Let $\overline{\mathbb{Q}}_q$ an algebraic closure of $\mathbb{Q}_q$. We denote by $\overline{\mathbb{Z}}_q$ the normalization of $\mathbb{Z}_q$ in $\overline{\mathbb{Q}}_q$. Let $I_q = \mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q^{\mathrm{nr}})$ be the inertia subgroup at $q$. Let $\overline{S} = \operatorname{Spec} \overline{\mathbb{Z}}_q$, and let $\overline{\eta}$ (resp. $\overline{s}$) the generic point of $\overline{S}$ (resp. closed point of $\overline{S}$). We consider the commutative diagram

$$\begin{array}{ccccc} X_{\overline{s}} & \xrightarrow{i} & X & \xleftarrow{j} & X_{\overline{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{s} & \longrightarrow & \overline{S} & \longleftarrow & \overline{\eta} \end{array}$$

Here, we still denote by $X/\overline{S}$ the base change to $\overline{S}$ of $X$. We denote by $\mathcal{G}$ the dual graph of $X_{\overline{s}}$, and by $\Sigma_i$ the $i$-simplices of $\mathcal{G}$ for $i = 0$, 1. We assume that a neighbourhood of each point $x \in \Sigma_1$ in $X$ is locally $S$-isomorphic to the subscheme of $\mathbb{A}^2_S = \mathbb{Z}_q[t_1, t_2]$ with $t_1 t_2 = q \neq 0$.

In this section we follow the presentation of Illusie [12]. We also borrow from Jarvis [13] and Rajaei [21]. §3.1 – §3.3 provide a general framework which we apply to our situation in §3.4. We thus obtain the crucial proposition 3.3.

Let $Y = X_{\overline{s}}$. For each $x \in \Sigma_1$, let $Y_{(x)}$ denote the henselization of $Y$ at $x$, and let $C_x$ denote the set of the two branches of $Y$ at $x$. As in Illusie [12] §1.1, define $\mathbb{Z}(x)$ and $\mathbb{Z}'(x)$ according to the following two dual exact sequences:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{(a)}} \mathbb{Z}^{C_x} \longrightarrow \mathbb{Z}(x) \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}'(x) \longrightarrow \mathbb{Z}^{C_x} \xrightarrow{\text{(b)}} \mathbb{Z} \longrightarrow 0$$

where (a) is the diagonal map, and (b) is the sum. Choosing an ordering for $C_x$ for each $x \in \Sigma_1$ defines a base $\delta'_x = (1, -1)$ for $\mathbb{Z}'(x)$ and the dual base for $\mathbb{Z}(x)$ will be denoted by $\delta_x$. Let

$$\mathscr{F} = \mathbb{Z}_p, \quad \mathscr{F}(x) = \mathbb{Z}(x) \otimes \mathscr{F}, \quad \mathscr{F}'(x) = \mathbb{Z}'(x) \otimes \mathscr{F}.$$

4.1. **Vanishing cycles.** We have the following well-known results of vanishing cycles, for which we refer to SGA7 [10, 8], especially to the Exposés I, XIII, XIV, XV by Deligne.

One has the $I_q$-equivariant *Leray spectral sequence*

$$\mathrm{H}^m(X, \mathrm{R}^n j_* \Lambda) \Longrightarrow \mathrm{H}^{m+n}(X_{\overline{\eta}}, \Lambda).$$

By the Proper Base Change theorem, the morphism of functors $i^*$ induces an isomorphism

$$\mathrm{H}^m(X, \mathrm{R}^n j_* \Lambda) \simeq \mathrm{H}^m(X_{\overline{s}}, i^* \mathrm{R}^n j_* \Lambda).$$

Let $\mathrm{R}^n\Psi(\Lambda) := i^*\mathrm{R}^n j_*\Lambda$ be the $n^{\text{th}}$-*sheaf of vanishing cycles.* Let $\mathrm{R}\Phi(\Lambda) = \mathrm{Cone}(\Lambda \to \mathrm{R}\Psi(\Lambda))$ be the *sheaf of nearby cycles.* By [10], Exposés XV, §3.1.2, the sheaves $\mathrm{R}^n\Phi(\Lambda) = 0$ for $n \neq 1$, $\mathrm{R}^0\Psi(\Lambda) = \Lambda$, $\tau_{\geq 1}\ \mathrm{R}^\bullet\Psi(\Lambda) = \mathrm{R}^\bullet\Phi(\Lambda)$, and $\mathrm{R}^1\Phi(\Lambda)$ is a sheaf concentrated on the set $\Sigma_1$ of singular points of $X_{\overline{s}}$.

Using these information, we obtain the following *exact sequence of specialization*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(X_{\overline{s}}, \Lambda)(1) &\xrightarrow{\mathrm{sp}} \mathrm{H}^1(X_{\overline{\eta}}, \Lambda)(1) \xrightarrow{\beta} \bigoplus_{x\in\Sigma_1} (\mathrm{R}^1\Phi(\Lambda))_x(1) \\ &\xrightarrow{d_2} \mathrm{H}^2(X_{\overline{s}}, \Lambda)(1) \xrightarrow{\mathrm{sp}(1)} \mathrm{H}^2(X_{\overline{\eta}}, \Lambda)(1) \longrightarrow 0. \end{aligned} \tag{4.1}$$

Define $\mathbb{X}(\Lambda)$ to be:

$$\mathbb{X}(\Lambda) := \ker\left(\bigoplus_{x\in\Sigma_1} (\mathrm{R}^1\Phi(\Lambda))_x(1) \to \ker(\mathrm{sp}(1))\right).$$

For $x \in \Sigma_1$, $\mathrm{H}^i_x(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda)) = 0$ for $n \neq 1, 2$, and for $n = 2$ we have the trace isomorphism:

$$\mathrm{Tr} : \mathrm{H}^2_x(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda)) \xrightarrow{\simeq} \Lambda(-1)$$

whereas for $n = 1$ we have

$$\mathrm{H}^1_x(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda)) \xrightarrow{\simeq} \Lambda(x).$$

So for any singular points $x \in \Sigma_1$, we get a *vanishing cycle* $\delta_x \in \mathrm{H}^1_x(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda))$. Similarly, we have

$$\Lambda'(x) \xrightarrow{\simeq} \mathrm{R}^1\Phi(\Lambda)_x(1),$$

and $\delta'_x \in \mathrm{R}^1\Phi(\Lambda_x)(1)$. These cycles are dual to each other with respect to the canonical pairing on $\mathrm{H}^1(X_{\overline{\eta}}, \Lambda)$ to $\Lambda(-1)$. That is, this pairing

$$\begin{aligned} \mathrm{R}^1\Phi(\Lambda)_x \times \mathrm{H}^1_x(X_{\overline{s},(x)}, \mathrm{R}\Psi(\Lambda)) &\to \Lambda(-1) \\ (a, b) &\mapsto \mathrm{Tr}(ab) \end{aligned}$$

is perfect between free rank one modules.

Recall that $Y = X_{\overline{s}}$. Under the trace mapping, the free $\Lambda$-modules $\mathrm{H}^1(Y, \mathrm{R}\Psi(\Lambda))(1)$ and $\mathrm{H}^2(Y, \mathrm{R}\Psi(\Lambda))$ are respectively dual to $\mathrm{H}^1(Y, \mathrm{R}\Psi(\Lambda))$, and $\mathrm{H}^0(\widetilde{Y}, \Lambda)$ ($\widetilde{Y}$ being the normalization of $Y$). The exact sequence dual to the specialization sequence would be

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(\widetilde{Y}, \mathrm{R}\Psi(\Lambda)) &\longrightarrow \mathrm{H}^0(\widetilde{Y}, \Lambda) \longrightarrow \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \\ &\xrightarrow{\beta'} \mathrm{H}^1(Y, \mathrm{R}\Psi(\Lambda)) \longrightarrow \mathrm{H}^1(Y, \Lambda) \longrightarrow 0, \end{aligned} \tag{4.2}$$

which is called the *exact sequence of cospecialization*. We define

$$\check{\mathbb{X}}(\Lambda) = \mathrm{im}(\beta').$$

4.2. **The variation morphism.** Let $t_p : I_q \to \Lambda(1)$ be the $p$-component of the canonical homomorphism $I_q \to \prod_{q' \neq q} \mathbb{Z}_{q'}(1)$. For $x \in \Sigma_1$ and $\sigma \in I_q$, the *variation morphism at $x$* is defined by (see [8], Exposé XV, §3.3.5):

$$\begin{array}{rcl} \mathrm{Var}(\sigma)_x : \mathrm{R}^1\Phi(\Lambda)_x & \to & \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \\ a & \mapsto & -e_x t_p(\sigma)(a\delta_x)\delta_x \end{array}$$

where $(a\delta_x) \in \mathbb{Z}_p(-1)$ is the coordinate of $a$ with respect to $\delta'_x$. Notice that here $e_x = 1$ because we assume that $X/S$ is semistable. The *monodromy logarithm at $x$* is defined to be

$$N_x : \mathrm{R}^1\Phi(\Lambda)_x(1) \to \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda))$$

$$N_x(t_p(\sigma)a) = \mathrm{Var}(\sigma)_x(a), \quad \text{for } a \in \mathrm{R}^1\Phi(\Lambda)_x \text{ and } \sigma \in I_q.$$

We have the following commutative diagram:

$$\begin{tikzcd} \mathrm{R}^1\Phi(\Lambda)_x(1) \arrow[r, "\simeq"] \arrow[d, "N_x"'] & \Lambda'(x) \arrow[d] & \delta'_x \arrow[d, mapsto] \\ \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \arrow[r, "\simeq"] & \Lambda(x) & -\delta_x. \end{tikzcd}$$

This, together with the factorization of $\sigma - 1 : \mathrm{H}^1(X_{\overline{\eta}}, \Lambda) \to \mathrm{H}^1(X_{\overline{\eta}}, \Lambda)$ as sum of $\mathrm{Var}(\sigma)_x(a)$ ([8], Exposé XIII, §2.4.6) implies that there is a unique homomorphism

$$N : \mathrm{H}^1(X, \Lambda)(1) \to \mathrm{H}^1(X, \Lambda),$$

$$N(t_p(\sigma)a) = (\sigma - 1)(a), \quad \text{for } a \in \mathrm{H}^1(X_{\overline{\eta}}, \Lambda) \text{ and } \sigma \in I_q$$

such that $N$ makes the following diagram commutate

$$\begin{tikzcd} \mathrm{H}^1(X_{\overline{\eta}}, \Lambda)(1) = \mathrm{H}^1(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda))(1) \arrow[r, "\beta"] \arrow[d, "N"'] & \bigoplus_{x \in \Sigma_1} \mathrm{R}^1\Phi(\Lambda)_x(1) \arrow[d, "\bigoplus N_x"] \\ \mathrm{H}^1(X_{\overline{\eta}}, \Lambda) = \mathrm{H}^1(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda)) & \bigoplus_{x \in \Sigma_1} \mathrm{H}^1_x(X_{\overline{s}}, \mathrm{R}\Psi(\Lambda)). \arrow[l, tail, "\beta'"] \end{tikzcd}$$

4.3. **Monodromy pairing.** If $C$ denotes the set of irreducible components of $Y = X_{\overline{s}}$, the summation map $\mathbb{Z}^C \to \mathbb{Z}$ has the image of $\bigoplus_{x \in \Sigma_1} \mathbb{Z}'(x) \to \mathbb{Z}^C$ as its kernel, so we can define a $\mathbb{Z}$-module $M$ via the following exact sequence

$$0 \to M \to \bigoplus_{x \in \Sigma_1} \mathbb{Z}'(x) \to \mathbb{Z}^C \to \mathbb{Z} \to 0,$$

and dually we have

$$0 \to \mathbb{Z} \to \mathbb{Z}^C \to \bigoplus_{x \in \Sigma_1} \mathbb{Z}(x) \to \check{M} \to 0,$$

where $\mathbb{Z}^C \to \mathbb{Z}(x)$ is the composition of the projections $\mathbb{Z}^C \to \mathbb{Z}^{C_x}$ and $\mathbb{Z}^{C_x} \to \mathbb{Z}(x)$. We define

$$u : M \otimes_{\mathbb{Z}} M \to \mathbb{Z}$$

by the composite homomorphism:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "u_*"'] & \bigoplus_{x\in\Sigma_1} \mathbb{Z}'(x) \arrow[d] & \delta'_x \arrow[d, mapsto] \\ \check{M} & \bigoplus_{x\in\Sigma_1} \mathbb{Z}(x) \arrow[l] & -\delta_x \end{tikzcd}$$

In fact, $u$ is the symmetric bilinear form induced on $M$ by the quadratic form $-\sum \delta_x \otimes \delta_x$ on $\bigoplus \mathbb{Z}'(x)$. So in particular $u_*$ is *injective* and $u_* \otimes \mathbb{Q}$ is an isomorphism.

We identify $\mathrm{H}^2(X_{\overline{s}}, \Lambda)(1)$ with $\Lambda^C$ via the trace isomorphism, and using the isomorphism between $\mathrm{R}^1\Phi(\Lambda)_x(1)$ and $\Lambda'(x)$ for $x \in \Sigma_1$, we see that

$$M \otimes \Lambda = \mathrm{im}\left(\mathrm{H}^1(X_{\overline{\eta}}, \Lambda)(1) \to \bigoplus_{x\in\Sigma_1} \mathrm{R}^1\Phi(\Lambda)_x(1)\right)$$

and

$$\check{M} \otimes \Lambda = \mathrm{coker}\left(\mathrm{H}^0(\widetilde{Y}, \Lambda) \to \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda))\right).$$

Define $c$ and $c'$ by the commutative diagram:

$$\begin{tikzcd} \mathrm{H}^1(X_{\overline{\eta}}, \Lambda)(1) \arrow[r, two heads, "c"] \arrow[d, "N"'] \arrow[rr, bend left, "\beta"] & M \otimes \Lambda \arrow[r, hook] \arrow[d, "u_*\otimes\Lambda"'] & \bigoplus_{x\in\Sigma_1} \mathrm{R}^1\Phi(\Lambda)_x(1) \arrow[d, "\bigoplus N_x"] \\ \mathrm{H}^1(X_{\overline{\eta}}, \Lambda) & \check{M} \otimes \Lambda \arrow[l, hook', "c'"] & \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \arrow[l, two heads] \arrow[ll, bend left, "\beta'"] \end{tikzcd}$$

in which the horizontal composite maps are the maps $\beta$ and $\beta'$.

4.4. **Application I.** Using the formalism of vanishing cycles, we have an injective map

$$\lambda'_q : \mathbb{X}(\Lambda) \to \check{\mathbb{X}}(\Lambda),$$

and we deduce the following commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(X_{\overline{\eta}}, \Lambda)(1) \arrow[r, two heads, "c"] \arrow[d, "N"'] \arrow[rr, bend left, "\beta"] & \mathbb{X}(\Lambda) \arrow[r, hook] \arrow[d, "\lambda'_q"'] & \bigoplus_{x\in\Sigma_1} \mathrm{R}^1\Phi(\Lambda)_x(1) \arrow[d, "\bigoplus N_x"] \\ \mathrm{H}^1(X_{\overline{\eta}}, \Lambda) & \check{\mathbb{X}}(\Lambda) \arrow[l, hook', "c'"] & \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \arrow[l, two heads] \arrow[ll, bend left, "\beta'"] \end{tikzcd}$$

Taking cohomology with supports at $x$ in the specialization exact sequence (4.1), we induce the following exact sequence

$$0 = \mathrm{H}^0_x(Y, \mathrm{R}\Phi(\Lambda)) \to \mathrm{H}^1_x(Y, \Lambda) \xrightarrow{\beta_x} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \to \mathrm{H}^1_x(Y, \mathrm{R}\Phi(\Lambda)) \xrightarrow{\gamma} \mathrm{H}^2_x(Y, \Lambda),$$

and define the map $\beta_x : \mathrm{H}^1_x(Y,\Lambda) \to \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda))$. By (4.1), we may regard $\mathrm{H}^1(Y,\Lambda)$ as a subset of $\mathrm{H}^1(X_{\overline{\eta}},\Lambda)$. The morphism $\gamma$ is injective because the composite

$$\mathrm{R}^1\Phi(\Lambda)_x \xrightarrow{\simeq} \mathrm{H}^1_x(Y, \mathrm{R}\Phi(\Lambda)) \to \mathrm{H}^2_x(Y,\Lambda)$$

is injective (see Illusie [12], Lemme 1.5(b)). Hence, $\beta_x$ is an isomorphism. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y,\Lambda) \arrow[r, "\bigoplus \beta_x"] \arrow[d] & \bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \arrow[d, "\beta'"] \\ \mathrm{H}^1(Y,\Lambda) \arrow[r, hook, "\mathrm{sp}"'] & \mathrm{H}^1(Y, \mathrm{R}\Psi(\Lambda)) \end{tikzcd},$$

and this shows the image of $\beta'$ lands in this subspace $\mathrm{H}^1(Y,\Lambda)$.

**Lemma 4.1.** *We have that* $\mathrm{im}(\beta') \subset \mathrm{H}^1(X_{\overline{s}},\Lambda)$.

Let $\mu$ be the normalization map of $\mu : \widetilde{Y} \to Y$ over $\mathbb{Z}_q$. Define the sheaf $\mathscr{G}$ on $Y$ by the exact sequence of sheaves:

$$0 \to \Lambda \to \mu_*\mu^*\Lambda \to \mathscr{G} \to 0. \tag{4.3}$$

Note that $\mathscr{G}$ is a sheaf supported on $\Sigma_1$. All its positive cohomology groups vanish. Hence we get the following exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(Y,\Lambda) &\to \mathrm{H}^0(Y,\mu_*\mu^*\Lambda) \xrightarrow{\theta} \mathrm{H}^0(Y,\mathscr{G}) \xrightarrow{\tau} \mathrm{H}^1(Y,\Lambda) \\ &\to \mathrm{H}^1(Y,\mu_*\mu^*\Lambda) \to 0 \to \mathrm{H}^2(Y,\Lambda) \to \mathrm{H}^2(Y,\mu_*\mu^*\Lambda) \to 0. \end{aligned} \tag{4.4}$$

We define

$$\check{\mathbb{Y}}_q(\Lambda) := \mathrm{H}^0(Y,\mathscr{G})/\operatorname{im}(\theta),$$

and

$$\mathbb{Y}_q(\Lambda) := \ker\left(\bigoplus_{x\in\Sigma_1} (\mathrm{R}\Phi(\Lambda))_x(1) \to \ker(\mathrm{sp})(1)\right).$$

By the monodromy pairing, we have

$$\lambda_q : \mathbb{Y}_q(\Lambda) \to \check{\mathbb{Y}}_q(\Lambda).$$

Since all irreducible components of $X$ over $k$ are rational curves, we have $\mathrm{H}^1(\widetilde{Y},\mu^*\Lambda) = 0$; hence $\mathrm{H}^1(Y,\mu_*\mu^*\Lambda) = 0$. Therefore, we get

$$\mathbb{Y}_q(\Lambda) \simeq \mathrm{H}^1(Y,\Lambda).$$

Let $\widetilde{\Sigma}_1 = \mu^{-1}(\Sigma_1)$. We may take the cohomology with supports in $\Sigma_1$ corresponding to (4.3) on $Y$:

$$\cdots \to \mathrm{H}^0_{\Sigma_1}(Y,\mu_*\mu^*\Lambda) \to \mathrm{H}^0_{\Sigma_1}(Y,\mathscr{G}) \to \mathrm{H}^1_{\Sigma_1}(Y,\Lambda) \to \mathrm{H}^1_{\Sigma_1}(Y,\mu_*\mu^*\Lambda) \to \cdots.$$

**Lemma 4.2.** *We have the following canonical isomorphism induced by* $\beta'$*:*

$$\bigoplus_{x\in\Sigma_1} \mathrm{H}^1_x(Y,\mathrm{R}\Psi(\Lambda)) \xrightarrow{\simeq} \mathrm{H}^0(Y,\mathscr{G}).$$

*Proof.* Note that the normalization map $\mu$ is finite. It follows that $\mathrm{H}^i_{\Sigma_1}(Y, \mu_*\mu^*\Lambda) \simeq \mathrm{H}^i_{\widetilde{\Sigma}_1}(\widetilde{Y}, \mu^*\Lambda)$.

Since $\mathscr{F}$ is locally constant and each component of $\widetilde{Y}$ is smooth over $\overline{s}$, it follows that $\mathrm{H}^i_{\widetilde{\Sigma}_1}(\widetilde{Y}, \mu^*\Lambda) = 0$ for $i = 0, 1$. Therefore, in the above exact sequence we have $\mathrm{H}^0_{\Sigma_1}(Y, \mathscr{G}) \xrightarrow{\simeq} \mathrm{H}^1_{\Sigma_1}(Y, \Lambda)$. Since each of the maps $\beta_x$ are isomorphisms, we obtain

$$\mathrm{H}^1_{\Sigma_1}(Y, \Lambda) \simeq \bigoplus_{x \in \Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)).$$

As $\mathscr{G}$ is supported on $\Sigma_1$, we see that $\mathrm{H}^0_{\Sigma_1}(Y, \mathscr{G}) \simeq \mathrm{H}^0(Y, \mathscr{G})$. Hence, we get

$$\bigoplus_{x \in \Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \simeq \mathrm{H}^1_{\Sigma_1}(Y, \Lambda) \simeq \mathrm{H}^0_{\Sigma_1}(Y, \mathscr{G}) \simeq \mathrm{H}^0(Y, \mathscr{G}).$$

□

Recall that the map $\theta$ is given in the exact sequence (4.4).

**Proposition 4.3.** $\check{\mathbb{X}}(\Lambda) \simeq \mathrm{H}^0(Y, \mathscr{G})/\theta(\mathrm{H}^0(Y, \mu_*\mu^*\Lambda))$.

*Proof.* By Lemma 4.2, we have identified $\bigoplus_{x \in \Sigma_1} \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda))$ with $\mathrm{H}^0(Y, \mathscr{G})$. The rest is to show that the image of $\mathrm{H}^0(Y, \mathscr{G})$ in $\mathrm{H}^1(Y, \Lambda)$ is the same as the image of $\beta'$. Also note that we have proved that the image of $\beta'$ lies in $\mathrm{H}^1(Y, \Lambda)$.

By Illusie [12] §1.5, we have the following "diagramme des 9" over $Y_{(x)}$ for the inclusions $i_x : \{x\} \hookrightarrow Y_{(x)}$ and $j_x : U_x = Y_{(x)} \setminus \{x\} \hookrightarrow Y_{(x)}$

$$\begin{tikzcd}
i_{x,*}\mathrm{R}i_x^!\Lambda \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & \mathrm{R}j_{x,*}j_x^*\Lambda \arrow[d, equal] \\
\mu_*\mu^* i_{x,*}\mathrm{R}i_x^!\Lambda \arrow[r] \arrow[d] & \mu_*\mu^*\Lambda \arrow[r] \arrow[d] & \mathrm{R}j_{x,*}j_x^*\Lambda \arrow[d] \\
\mathscr{G} \arrow[r, equal] & \mathscr{G} \arrow[r] & 0
\end{tikzcd}$$

So by [12], Lemme 5.4, the composite

$$\gamma_x : \mathrm{H}^0(Y_{(x)}, \mu_*\mu^*\Lambda) \to \mathrm{H}^0(Y_{(x)}, \mathrm{R}j_*j^*\Lambda) \to \mathrm{H}^1(Y_{(x)}, i_{x,*}\mathrm{R}i_x^!\Lambda) \simeq \mathrm{H}^1_x(Y, \Lambda)$$

is negative of the composite map

$$\gamma'_x : \mathrm{H}^0(Y_{(x)}, \mu_*\mu^*\Lambda) \to \mathrm{H}^0(Y_{(x)}, \mathscr{G}) \to \mathrm{H}^1(Y_{(x)}, i_{x,*}\mathrm{R}i_x^!\Lambda) \simeq \mathrm{H}^1_x(Y, \Lambda).$$

In particular, the image of $\partial_x : \mathrm{H}^0(U_x, \Lambda) \simeq \mathrm{H}^0(Y_{(x)}, \mathrm{R}j_{x,*}j_x^*\Lambda) \to \mathrm{H}^1_x(Y, \Lambda)$ is the same as that of $\tau_x : \mathrm{H}^0(Y_{(x)}, \mathscr{G}) \to \mathrm{H}^1(Y_{(x)}, i_{x,*}\mathrm{R}i^!\Lambda) \simeq \mathrm{H}^1_x(Y, \Lambda)$. But we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(U_x, \mathrm{R}\Psi(\Lambda)) \arrow[r, "\partial'_x"] & \mathrm{H}^1_x(Y, \mathrm{R}\Psi(\Lambda)) \arrow[d, "\beta_x^{-1}"] \\
\mathrm{H}^0(U_x, \Lambda) \arrow[u, "\simeq"] \arrow[r, "\partial_x"'] & \mathrm{H}^1_x(Y, \Lambda)
\end{tikzcd}$$

which implies that the image of $\beta_x^{-1}$ is also the same as that of $\partial_x$. Thus, $\beta_x^{-1}$ and $\tau_x$ have the same image.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0_{\Sigma_1}(Y,\mathscr{G}) \arrow[r] \arrow[d, "\tau"'] & \bigoplus_{x\in\Sigma_1}\mathrm{H}^0_x(Y,\mathscr{G}) \arrow[r, "\simeq", "(1)"'] \arrow[d, "\bigoplus\tau_x"] & \bigoplus_{x\in\Sigma_1}\mathrm{H}^1_x(Y,\mathrm{R}\Psi(\Lambda)) \arrow[r] \arrow[d, "\bigoplus\beta_x^{-1}"'] & \mathrm{H}^1_{\Sigma_1}(Y,\mathrm{R}\Psi(\Lambda)) \arrow[d, "\beta'"] \\
\mathrm{H}^1(Y,\Lambda) & \bigoplus_{x\in\Sigma_1}\mathrm{H}^1_x(Y,\Lambda) \arrow[l, "\sigma"] & \bigoplus_{x\in\Sigma_1}\mathrm{H}^1_x(Y,\Lambda) \arrow[r, "\sigma"'] & \mathrm{H}^1(Y,\Lambda),
\end{tikzcd}$$

where (1) is given by Lemma 4.2. Thus we have $\mathrm{im}(\beta')=\mathrm{im}(\tau)$ and this is the same as $\mathrm{coker}(\theta)$. $\square$

*Remark* 4.4. Note that $\mathbb{X}(\Lambda)\subset\bigoplus_{x\in\Sigma_1}(\mathrm{R}^1\Phi(\Lambda)_x)(1)$, so it is torsion-free. Since $\Lambda$ is irreducible for $k-2<p$, we can also see that $\check{\mathbb{X}}(\Lambda)$ has no $p$-torsion.

4.5. **Application II: Descent.** For our another application to descent from an auxiliary level group $\widehat{\Gamma}^D_1(r)$ to a level prime to $r$, we shall need Langlands-Deligne-Carayol theorem on the compatibility between local and global Langlands correspondance for $B^\times$. For this purpose, we will consider a regular scheme $X=X^D(\Gamma_1)$ (cf. §2, Prop. 2.2) flat of finite type over $\mathbb{Z}_r$ with smooth generic fiber $X_\eta$ and special fiber $X_s$ in this section.

4.5.1. *Calculation of vanishing cycles and monodromy.* Let $\Lambda$ be a $\mathbb{Z}/p^m$-module or a lisse sheaf of $\mathbb{Q}_p$-vector space over $X$. The special fiber is assumed to be étale locally one of the following types:

**(1)** $X_s$ *is smooth.* This corresponds to a neighbourhood of a point in $Y^m$ not in $Y^e$. Then the regularity of $X$ implies $X$ is smooth. In this case, $\mathrm{R}^q\Psi\Lambda=0$ for $q>0$, and $\mathrm{R}^q\Psi\Lambda=\Lambda$.

**(2)** $X$ *is of the form* $\mathrm{Spec}\,\mathbb{Z}_r[X,Y^{\pm1}]/(X^{r-1}Y-r)$. This corresponds to a neighbourhood of a point in $Y^e$ not in $Y^m$. Let $X'=\mathrm{Spec}\,\mathbb{Z}_r[x,y^{\pm1}]/(x^{r-1}-r)$, and define a map $\pi: X'\to X$ via the embedding

$$\begin{array}{rcl}\mathbb{Z}_r[X,Y^{\pm1}]/(X^{r-1}Y-r) & \to & \mathbb{Z}_r[x,y^{\pm1}]/(x^{r-1}-r)\\ (X,Y) & \mapsto & (xy^{-1},y^{r-1})\end{array}$$

The morphism $\pi$ is étale with Galois group $(\mathbb{Z}/r\mathbb{Z})^\times\simeq\mu_{r-1}$ where $\zeta\in\mu_{r-1}$ acts by multiplying both $x$ and $y$ by $\zeta$. The special fiber $X_s$ is a non-reduced divisor with multiplicity $r-1$; the associated reduced divisor $X_{s,\mathrm{red}}$ is defined by $X=0$ and is smooth; it is isomorphic to $\mathrm{Spec}\,\mathbb{F}_r[Y^{\pm1}]$ which we view as $(\mathbb{G}_m)_{\mathbb{F}_r}$.

We first compute the vanishing cycle $\mathrm{R}^q\Psi\Lambda$ in the étale neighbourhood of $X'$ of $X$. Let $\mathcal{O}=\mathbb{Z}_r[\sqrt[r-1]{r}]=\mathbb{Z}_r[x]/(x^{r-1}-r)$. Then we write

$$X'=\mathrm{Spec}\,\mathcal{O}\times_{\mathrm{Spec}\,\mathbb{Z}_r}Y \tag{4.5}$$

where $Y=\mathrm{Spec}\,\mathbb{Z}_r[y^{\pm1}]$. The second factor is smooth over $\mathbb{Z}_r$, hence $(\mathrm{R}^q\Psi\Lambda)_{X'}$ is the pullback from $\mathrm{R}^q\Psi$ for the finite flat morphism $\mathrm{Spec}\,\mathcal{O}\to\mathrm{Spec}\,\mathbb{Z}_r$. The morphism is of relative dimension zero, hence $\mathrm{R}^q\Psi=0$ for $q>0$. Similarly, $(\mathrm{R}^0\Psi\Lambda)_{\mathrm{Spec}\,\mathcal{O}}$ is the pull-back of $\mathrm{Spec}\,\mathcal{O}\to\mathrm{Spec}\,\mathbb{Z}_r$ and $(\mathrm{R}^0\Psi)_{\mathrm{Spec}\,\mathbb{Z}_r}=\Lambda$; hence $(\mathrm{R}^0\Psi\Lambda)_{\mathrm{Spec}\,\mathcal{O}}\simeq\Lambda^{r-1}$ as $\mathbb{Z}/p^m$-modules or $\mathbb{Q}_p$-vector space. Since the inertia group of $\mathcal{O}$ over $\mathbb{Z}_r$ acting on $(\mathrm{R}^0\Psi\Lambda)_{\mathrm{Spec}\,\mathcal{O}}$ by $\mu_{r-1}$ transitively, we thus have $(\mathrm{R}^0\Psi\Lambda)_{\mathrm{Spec}\,\mathcal{O}}$ is the group algebra $\Lambda[\mu_{r-1}]$.

It follows that $\mathrm{R}^q\Psi\Lambda = 0$ for $q > 0$, and that $\mathrm{R}^0\Psi\Lambda$ is a lisse $p$-adic sheaf of rank $r-1$ on $X_{s,\mathrm{red}}$ that becomes constant over $X'_{s,\mathrm{red}}$. Moreover, since $\mathrm{Gal}(X'/X)$ acts as inertia group on the first factor of (4.5), one sees that the canonical action of $\mathrm{Gal}(X'/X)$ on $(\mathrm{R}^0\Psi\Lambda)_{X'_{s,\mathrm{red}}}$ identifies the latter with the group algebra $\Lambda[\mathrm{Gal}(X'/X)]$. It follows that

$$\mathrm{R}^0\Psi\Lambda \xrightarrow{\simeq} \pi_*\Lambda.$$

The inertia group $\mu_{r-1}$ acts on $\mathrm{R}^0\Psi$, and we have seen that the lift of this action to $X'_{s,\mathrm{red}}$ coincides with the action of $\mathrm{Gal}(X'/X)$. We write

$$\mathrm{R}^0\Psi\Lambda = \bigoplus_{\chi} \mathrm{R}^0\Psi\Lambda[\chi],$$

the decomposition with respect to characters of the inertia group. We write $L = \mathrm{R}^0\Psi$, and $L[\chi]$ for the rank one local system $\mathrm{R}^0\Psi\Lambda[\chi]$. Let $\chi_0$ denote the trivial character. Consider the embedding

$$i : X_{s,\mathrm{red}} = (\mathbb{G}_m)_{\mathbb{F}_r} \hookrightarrow \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{F}_r[Y]$$

as the complement of the origin $Y = 0$. The morphism $\pi$ is totally ramified along $Y = 0$. It follows that

$$\mathrm{R}^0 i_* L[\chi] = i_! L[\chi]; \quad \mathrm{R}^q i_* L[\chi] = 0,\ q > 0 \quad (\chi \neq \chi_0); \tag{4.6}$$

$$\mathrm{R}^0 i_* L[\chi_0] = \Lambda; \quad \mathrm{R}^q i_* L[\chi_0] = 0,\ q > 0.$$

More generally, suppose $X = \mathrm{Spec}\,R[X, Y^{\pm 1}]/(X^{r-1}Y - r)$, where $R$ is a smooth $\mathbb{Z}_r$-algebra of finite type. Then $X$ is the fiber product

$$X = \mathrm{Spec}\,R \times_{\mathrm{Spec}\,\mathbb{Z}_r} \mathrm{Spec}\,\mathbb{Z}_r[X, Y^{\pm 1}]/(X^{r-1}Y - r),$$

where the first factor is smooth. We define

$$i_X = 1 \times i : X_{s,\mathrm{red}} = \mathrm{Spec}\,R \times_{\mathrm{Spec}\,\mathbb{Z}_r} \mathrm{Spec}(\mathbb{G}_m)_{\mathbb{F}_r} \to \mathrm{Spec}\,R \times_{\mathrm{Spec}\,\mathbb{Z}_r} \mathrm{Spec}\,\mathbb{F}_r[Y].$$

Let $X_2$ denote the second factor above and let $\mathrm{pr}_2$ denote the projection $X \to X_2$. We see that

$$\mathrm{R}^q\Psi\Lambda = \mathrm{pr}_2^*\,\mathrm{R}^q\Psi_{X_2}\Lambda,$$

where $\mathrm{R}\Psi_{X_2}$ denotes the vanishing cycle sheaves for the map from $X_2$ to $\mathrm{Spec}\,\mathbb{Z}_r$. In particular, $\mathrm{R}^q\Psi\Lambda = 0$ for $q > 0$, while $\mathrm{R}^q\Psi\Lambda$ breaks up under the action of the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_r/\mathbb{Q}_r)$ as the sum of rank one local system $L[\chi]$: $L[\chi_0] = \Lambda$, whereas $L[\chi]$ for nontrivial $\chi$ satisfies the analogue of (4.6):

$$\mathrm{R}^0 i_{X,*} L[\chi] = i_{X,!} L[\chi]; \quad \mathrm{R}^q i_{X,*} L[\chi] = 0,\ q > 0 \quad (\chi \neq \chi_0).$$

**(3)** *$X$ is of the form* $\mathrm{Spec}\,R[X, Y]/(X^{r-1}Y - r)$*, where $R$ is a smooth $\mathbb{Z}_r$-algebra of finite type.* This corresponds to a neighbourhood of a point in $Y^e \cap Y^m$. We will calculate the stalks of $\mathrm{R}^q\Psi\Lambda$ at a geometric point $\overline{x}$ of the singular locus $X_{\mathrm{sing}}$ of the special fiber defined by $X = Y = 0$. In this case, we simply have $\mathrm{R}^q\Psi\Lambda = 0$,

$q > 1$; $(\mathrm{R}^0\Psi\Lambda)_{\overline{x}} = \Lambda$, $(\mathrm{R}^1\Psi\Lambda)_{\overline{x}} = \Lambda(-1)$, with trivial action of the inertia group on $\Lambda$, $(-1)$ denoting Tate twist.

We write $Y^a = Y^m \cap Y^e$. Let $i_m : Y^m \to X^D(\Gamma_1)$, $i_e : Y^e \to X^D(\Gamma_1)$, and $i_a : Y^a \to X^D(\Gamma_1)$ be the natural maps. Let $Y^e_0$ denote the complement of $Y^a$ in $Y^e$, and let $j_e : (Y^e_0)_{\mathrm{red}} \to (Y^e)_{\mathrm{red}}$ be the open immersion. Then the vanishing cycle sheaves $\mathrm{R}^q\Psi\Lambda$ are calculated as follows:

**Proposition 4.5.** *Let $I$ denote the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_r/\mathbb{Q}_r)$. Then the action of $I$ on $\mathrm{R}\Psi^q\Lambda$ factors through the map to $(\mathbb{Z}/r\mathbb{Z})^\times$ (which we identify to $\mu_{r-1}(\mathbb{Z}_r)$ by Teichmüller lifting) given by the action on $\mathbb{Q}[\zeta_r]$. For a character $\chi$ of $\mu_{r-1}$, let $[\chi]$ denote the $\chi$-isotypic component, and let $\chi_0$ denote the trivial character. Then*

(i) $\mathrm{R}^0\Psi\Lambda[\chi_0] = \Lambda$.
(ii) $\mathrm{R}^1\Psi\Lambda = \Lambda[\chi_0]$ *is a rank one local system supported on $Y^a$, locally isomorphic at any point of $Y^a$ to $\Lambda(-1)$.*
(iii) *For $\chi \neq \chi_0$, $\mathrm{R}^0\Psi\Lambda$ is the extension by zero of a rank one lisse sheaf $L[\chi]$ supported on $Y^m_0$. Moreover, the natural map $i_{m,!}\mathrm{R}^0\Psi\Lambda[\chi] \to \mathrm{R}i_{m,*}\Psi^0\Lambda[\chi]$ is a quasi-isomorphism.*
(iv) $\mathrm{R}^q\Psi\Lambda = 0$ *for $q > 1$.*

*Proof.* Everything follows from the cases **(1)**–**(3)** discussed above except the global triviality of $\mathrm{R}^0\Psi\Lambda[\chi_0]$. But there is always an injection $\Lambda \to \mathrm{R}^0\Psi\Lambda[\chi_0]$, so (i) follows from the fact that all stalks of $\mathrm{R}^0\Psi\Lambda[\chi_0]$ are one-dimensional. ☐

Since the tame vanishing cycle sheaves are concentrated in two degrees, the vanishing cycle spectral sequence degenerates into a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{H}^i(X^D(\Gamma_1)_{\overline{s}}, \mathrm{R}^0\Psi\Lambda) &\to \mathrm{H}^i(X^D(\Gamma_1)_{\overline{\eta}}, \Lambda)\\ &\to \mathrm{H}^{i-1}(X^D(\Gamma_1)_{\overline{s}}, \mathrm{R}^1\Psi\Lambda) \to \mathrm{H}^{i+1}(X^D(\Gamma_1)_{\overline{s}}, \mathrm{R}^0\Psi\Lambda) \to \cdots .\end{aligned}$$

Using Proposition 4.5 (ii), we rewrite this

$$\cdots \to \mathrm{H}^i(X^D(\Gamma_1)_{\overline{s}}, \mathrm{R}^0\Psi\Lambda) \to \mathrm{H}^i(X^D(\Gamma_1)_{\overline{\eta}}, \Lambda) \to \mathrm{H}^{i-1}(Y^a_{\mathrm{red}}, \mathrm{R}^1\Psi\Lambda[\chi_0]) \to \cdots . \tag{4.7}$$

We deduce from (i) and (iii) of Proposition 4.5 that the first term in turn is calculated by a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{H}^{i-1}(Y^a_{\mathrm{red}}, \Lambda) &\to \mathrm{H}^i(X^D(\Gamma_1)_{\overline{\eta}}, \mathrm{R}^0\Psi\Lambda)\\ &\to \mathrm{H}^i(Y^m, \Lambda) \oplus \mathrm{H}^i((Y^e)_{\mathrm{red}}, \Lambda) \oplus \bigoplus_{\chi\neq\chi_0} \mathrm{H}^i_c(Y^e_0, L[\chi]) \to \cdots .\end{aligned} \tag{4.8}$$

Here and in (4.7) , we have replaced $Y^e$ and $Y^a$ by the associated reduced schemes, since the étale cohomology is insensitive to nilpotents.

The diamond operators act $X^D(\Gamma_1)_{\overline{\eta}}$ as well as on $Y^e$ and $Y^m$, and thus induce compatible actions on the spaces in the exact sequence (4.7) and (4.8). These are determined as follows:

**Lemma 4.6.** *The diamond operators $\langle a\rangle$ act on the outer terms of the exact sequence* (4.8) *as follows: The action acts via $\chi$ on $L[\chi]$, and acts trivially on $\mathrm{H}^\bullet((Y^e)_{\mathrm{red}}, \Lambda)$, on $\mathrm{H}^\bullet(Y^a, \Lambda)$, and on $\mathrm{H}^{i-1}((Y^a)_{\mathrm{red}}, \mathrm{R}^1\Psi\Lambda[\chi_0])$.*

*Proof.* The diamond operators acts trivially on $(Y^e)_{\mathrm{red}}$ and $(Y^a)_{\mathrm{red}}$, so it suffices to determine their action on $\mathrm{R}^0\Psi\Lambda$ and $\mathrm{R}^1\Psi\Lambda$.

For $\mathrm{R}^0\Psi\Lambda$, by the discussion in **(2)**, we see it suffices to determine the action of the diamond operators on $\mathrm{H}^0(\mathrm{Spec}(\overline{\mathbb{Q}}_r \otimes_{\mathbb{Q}_r} \mathbb{Q}_r[\zeta_r]), \Lambda)$, via the identification of $\mathbb{Q}_r[\zeta_r]$ with the generic fiber of $\mu_{r-1}$ and the latter with $C_R$ in **(1)**–**(2)**. But the diamond operators on $\mu_{r-1}$ are tautologically given by the cyclotomic character.

For the action on $\mathrm{R}^1\Psi\Lambda$, this is again local. But locally the discussion in **(3)** shows that $\mathrm{R}^1\Psi\Lambda$ is a constant sheaf, so the triviality of the action of the diamond operators is clear. ☐

**Proposition 4.7.** *Suppose $\chi \neq \chi_0$, and denote by $^{\langle\ \rangle = \chi}$ the $\chi$-isotypic component for the action of the diamond operators. Then for any $i$, there is a canonical isomorphism of* $\mathrm{Gal}(\overline{\eta}/\eta)$*-modules*

$$\mathrm{H}^i(Y^m, \Lambda)^{\langle\ \rangle = \chi} \oplus \mathrm{H}^i_c((Y^e_0)_{\mathrm{red}}, L[\chi]) \xrightarrow{\simeq} \mathrm{H}^i(X^D(\Gamma_1)_{\overline{\eta}}, \Lambda)^{\langle\ \rangle = \chi}.$$

*Proof.* Indeed, in (4.8), the diamond operators act trivially on the term $\mathrm{H}^i((Y^a)_{\mathrm{red}}, \Lambda)$ and coincide with inertia on $L[\chi]$, inducing an isomorphism

$$\mathrm{H}^i(X^D(\Gamma_1)_{\overline{s}}, \mathrm{R}^0\Psi\Lambda)^{\langle\ \rangle = \chi} \xrightarrow{\simeq} \mathrm{H}^i(Y^m, \Lambda)^{\langle\ \rangle = \chi} \oplus \mathrm{H}^i_c(Y^e_0, L[\chi]).$$

Similarly, the diamond operators act trivially on the $\mathrm{H}^{i-1}((Y^a)_{\mathrm{red}}, \mathrm{R}^1\Psi\Lambda[\chi_0])$ term in (4.8). ☐

4.5.2. *Descent.* Let us consider the following condition on a prime $r$:

**UR($r$)**: $r$ does not divide $\widetilde{N}$, $r \not\equiv 1 \bmod p$, and the ratio of the eigenvalues of $\rho(\mathrm{Frob}_r)$ is not congruent to 1 or $r^{\pm 1}$ mod $p$.

The existence of a prime $r$ satisfying **UR($r$)** follows from Čebotarev density theorem.

We fix such a prime $r$ in the sequel. For any $0 \leq s \leq m$, let $\Gamma_1(M_s; r) = \widehat{R}^\times_{M_s, D_s} \cap \widehat{\Gamma}^{D_s}_1(r) \subset \widehat{\mathcal{O}}_{B_s}$ and $\Gamma_0(M_s; r) = \widehat{R}^\times_{M_s, D_s} \cap \widehat{\Gamma}^{D_s}_0(r) \subset \widehat{\mathcal{O}}_{B_s}$. Denote the corresponding Shimura curves by $X^{D_s}(M_s; r)$. Let $A$ be a $\mathbb{Z}_p$-algebra. Applying the Jacquet-Langlands correspondence, we denote by $\mathbb{T}^{rQ_s}_Q$ the *Hecke algebra* generated by Hecke correspondences $T_\ell$ on $\mathrm{H}^1(X^{D_s}(M_s; r), \Lambda_k(A))$ for all $\ell \nmid \widetilde{N}$ for primes $\ell \nmid \widetilde{N}$. Note that if $s = m$, we also have the *minimal Hecke algebra* $\mathbb{T}_\emptyset$ generated by Hecke operators $T_\ell$ on the corresponding moduli curve $X(\widehat{\Gamma}_0(Np) \cap \widehat{\Gamma}_1(r))$ for primes $\ell$ such that $(\ell, \widetilde{N}) = 1$.

For each $0 \leq s \leq m$, we set $\Gamma_1 = \widehat{R}^\times_{M_s, D_s} \cap \widehat{\Gamma}^{D_s}_1(r)$ and $\Gamma_0 = \widehat{R}^\times_{M_s, D_s} \cap \widehat{\Gamma}^{D_s}_0(r)$, and let $X^{D_s}(\Gamma_1)$ and $X^{D_s}(\Gamma_0)$ be the corresponding Shimura curves. Let $\mathfrak{m}_{(r)} = \mathfrak{m} + (T_r - \alpha_r - \beta_r, rS_r - \alpha_r\beta_r)$. If $\pi'$ occurs in $\mathrm{H}^1(X^{D_s}(\Gamma_1), \Lambda)^{T_r \equiv \alpha_r}_{\mathfrak{m}_{(r)}}$ is special, it occurs in $\mathrm{H}^1(X^{D_s}(\Gamma_0), \Lambda)^{T_r \equiv \alpha_r}_{\mathfrak{m}_{(r)}}$. By the weight monodromy conjecture for curves, the eigenvalues of $\mathrm{Frob}_r$ on $\rho_{\pi'} \subset \mathrm{H}^1(X^{D_s}(\Gamma_0), \Lambda)^{T_r \equiv \alpha_r}_{\mathfrak{m}_{(r)}}$ are of the form $\alpha'_r, r\alpha'_r$. (See also Carayol [6].) However, we have

$$\rho_\pi(\mathrm{Frob}_r) \sim \begin{pmatrix} \alpha_r & 0 \\ 0 & \beta_r \end{pmatrix}$$

which implies that $\alpha_r/\beta_r \equiv r^{\pm 1}$ modulo $p$, and deduces a contradiction. Hence, $\pi'_r$ belongs to the ramified principal series and there exists a non trivial character $\chi$ of

$(\mathbb{Z}/r\mathbb{Z})^\times$ such that the diamond operator act on $\pi'_r$ by $\chi$. In particular, $\pi'$ occurs in $\mathrm{H}^1(X^{D_s}(\Gamma_1)_{\overline{\eta}}, \Lambda)_{\mathfrak{m}_{(r)}}^{\langle\ \rangle=\chi}$. Then by Proposition 4.7, we see that

$$\rho_{\pi'}\ |_{I_r} \sim \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix}.$$

Since $r \not\equiv 1 \bmod p$, this implies that $\overline{\rho}_{\pi'} = \overline{\rho}$ is also ramified at $r$ which is a contradiction. Therefore, $\rho_{\pi'}$ is unramified.

*Remark* 4.8. Since the associated Galois representations of the Hecke algebras $\widehat{\mathbb{T}}_Q^{rD_s}$ are unramified at $r$, we still have the surjective specialization maps of local $W$-algebras $R_Q^{D_s} \to \widehat{\mathbb{T}}_Q^{rD_s}$ for $0 \le s \le m$. Hence we will ignore the auxiliary prime $r$ in the sequel to reduce our notations.

## 5. Isomorphisms between deformation rings and Hecke rings

By the Proposition 21 of [15], there exists a finite set of primes $Q = \{q_2, \ldots, q_{2m}\}$ of odd cardinality such that for each $q \in Q$, $q \not\equiv \pm 1 \bmod p$, $\mathrm{Tr}\,\overline{\rho}(\mathrm{Frob}_q) = \pm(q+1)$, and such that

$$\mathrm{H}^1_{\mathcal{L}}(G_{S\cup Q}, \mathrm{Ad}^0) = 0.$$

Then by Proposition 3.8, $\mathfrak{m}_Q = pR_Q^D$; therefore, $R_Q^D/pR_Q^D = \mathbb{F} = W/pW$ and by Nakayama's lemma the structure morphism $W \to R_Q^D$ is surjective. Since $k \le p-1$, by Diamond-Taylor [9], there is a $p$-adic modular lifting of $\overline{\rho}$ which arises from a specialization of the universal representation $\rho_Q^Q : G_{\mathbb{Q}} \to \mathbf{GL}_2(R_Q^D)$ which gives a surjection from $R_Q^D$ to $W$. Hence the structure morphism is also injective.

By Lemma 3.9, we have a surjective $W$-morphism $R_Q^D \twoheadrightarrow \widehat{\mathbb{T}}_Q^D$, and since the algebra $\widehat{\mathbb{T}}_Q^D$ is finite flat over $W$, thus $\psi : R_Q^D \to \widehat{\mathbb{T}}_Q^D$ is injective. Hence we deduce the following:

**Proposition 5.1.** *Let $2 \le k < p$. Then for $D = \prod_{q\in Q} q$ we have isomorphisms of local $W$-algebras*

$$W \simeq R_Q^D \xrightarrow{\sim} \widehat{\mathbb{T}}_Q^D.$$

*We call such set $Q$ a* RK system *or a RK set.*

A proof of the following theorem which deduce it from Proposition 5.1, will be given in the Section 8.2.

**Theorem 5.2.** *If $Q$ is a RK system, we have an isomorphism $R_Q \simeq \widehat{\mathbb{T}}_Q$ of complete intersection rings.*

An argument of Böckle implies that:

**Theorem 5.3.**

$$R_\emptyset \simeq \widehat{\mathbb{T}}_\emptyset$$

*is an isomorphism of complete intersection rings.*

## 6. A numerical inequality

As in Wiles' method, the proof of modularity is based on a numerical inequality relating the length of a congruence module to the cardinality of a Selmer group. We recall a numerical criterion due to Lenstra which refines a result of Wiles.

**Proposition 6.1** (Numerical criterion)**.** *Let $R, T \in \underline{\mathbf{CNL}}_W$. Suppose that $T$ is finite flat as $W$-module and $\phi : R \to T$ is a surjective local $W$-algebra homomorphism. Let $\pi : T \to W$ be a homomorphism of local $W$-algebras, and set $\Phi(R) = \ker(\pi\phi)/\ker(\pi\phi)^2$ and $\eta_T = \pi(\mathrm{Ann}_T(\ker(\pi)))$. Then we have the following:*

(i)
$$|W/\eta_T| \leq |\Phi(R)| .$$

(ii) *Assume that $\eta_T$ is not zero. Then the following are equivalent:*
- *The equality $|W/\eta_T| = |\Phi(R)|$ is satisfied.*
- *The rings $R$ and $T$ are complete intersections and $\phi$ is an isomorphism.*

We let $\phi : R_Q^\alpha \to \widehat{\mathbb{T}}_Q^\alpha$ and $\pi : \widehat{\mathbb{T}}_Q^\alpha \to \widehat{\mathbb{T}}_Q^D$. For any prime $q \in Q$, let $t_q$ generate the unique $\mathbb{Z}_p$ quotient of $I_q$. Then $\rho_Q^{D_s}(t_q)$ is of the form

$$\begin{pmatrix} 1 & x_q \\ 0 & 1 \end{pmatrix}$$

for some $x_q \in W \setminus \{0\}$, and $(x_q)$ does not depend on the choice of an integral model for $\rho_Q^D$. Note that $x_q \neq 0$. Indeed, if $x_q = 0$ then $\rho_Q^D$ is unramified at $q$, and as $R_Q^D \simeq \widehat{\mathbb{T}}_Q^D$ we have $\mathrm{Tr}\, \rho_Q^D(\mathrm{Frob}_q) = \pm(q+1)$ which contradicts the Ramanujan bound.

**Lemma 6.2.** *Let $q \in Q$ and let $n$ be the largest number such that $\rho_Q^D\,|_{G_q}$ is unramified. Then we have:*

(i) *Let $Z \subset \mathrm{Ad}^0(\rho_Q^D)$be the upper triangular one dimensional subspace of nilpotent elements which are preserved under the conjugation by $\rho_Q^D\,|_{G_q}$. The sequence*
$$0 \to \mathrm{Ad}^0(\rho_Q^D) \otimes p^{-n}\mathbb{Z}/\mathbb{Z} \to \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q} \to Z \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$
*is exact.*

(ii) *The inflation map*
$$\mathrm{H}^1\left(G_q/I_q, \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q}\right) \xrightarrow{\sim} \mathrm{H}^1(G_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$
*is an isomorphism.*

(iii) $\mathrm{H}^1(G_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ *is isomorphic to $W/p^n$.*

(iv) *For all $q \in Q$, $\mathcal{L}_q$ is trivial.*

*Proof.* (i) The ramification first occurs mod$p^{n+1}$. The exactness of the sequence follows.

(ii) Using the inflation-restriction sequence
$$\begin{aligned} 0 \;\to\; & \mathrm{H}^1\left(G_q/I_q, \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q}\right) \to \mathrm{H}^1(G_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ \to\; & \mathrm{H}^1(I_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p), \end{aligned}$$

it is enough to show that $\mathrm{H}^1(I_q, \mathrm{Ad}^0(\rho_Q^D)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$ is trivial. Let $M = \mathrm{Ad}^0(\rho_Q^D)\otimes \mathbb{Q}_p/\mathbb{Z}_p$. The exact sequence

$$0 \to P_q \to I_q \to I_q/P_q \simeq \prod_{\ell\neq q} \mathbb{Z}_\ell(1) \to 0$$

induces

$$0 \to \mathrm{H}^1(I_q/P_q, M^{P_q}) \to \mathrm{H}^1(I_q, M) \to \mathrm{H}^1(P_q, M).$$

Note that $P_q$ acts trivially on $M$ because $\rho_Q^D$ is tame at $q$, hence $\mathrm{H}^1(P_q, M) = \mathrm{Hom}(P_q, M) = 0$.

(iii) By (i), we have the exact sequence of $G_q/I_q$-modules

$$0 \to \mathrm{Ad}^0(\rho_Q^D) \otimes p^{-n}\mathbb{Z}/\mathbb{Z} \to \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q} \to Z \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Since $q \not\equiv \pm 1 \bmod p$, these cohomology groups $\mathrm{H}^i(G_q/I_q, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ are trivial for $i = 0, 1$. Therefore,

$$\mathrm{H}^1(G_q/I_q, \mathrm{Ad}^0(\rho_Q^D) \otimes p^{-n}\mathbb{Z}/\mathbb{Z}) \simeq \mathrm{H}^1\left(G_q/I_q, \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q}\right).$$

We have the following decomposition as $G_q/I_q$-modules

$$\mathrm{Ad}^0(\rho_Q^D) \otimes p^{-n}\mathbb{Z}/\mathbb{Z} \simeq W(-1)/p^m \oplus W/p^n \oplus W(-1)/p^n$$

Denote $W/p^n$ by $M_1$, $W(-1)/p^n \oplus W(-1)/p^n$ by $M_2$ and let $M = M_1 \oplus M_2$. Since $q \not\equiv \pm 1 \bmod p$, we see that $\mathrm{H}^0(G_q/I_q, M_2) = 0$ and that

$$\mathrm{H}^1(G_q/I_q, M_1) \hookrightarrow \mathrm{H}^1(G_q/I_q, M).$$

In fact, $\mathrm{H}^1(G_q/I_q, M_1)$ and $\mathrm{H}^1(G_q/I_q, M)$ are isomorphic because their dimensions are equal. The group $G_q/I_q$ acts trivially on $M_1$, so

$$\mathrm{H}^1(G_q/I_q, M_1) = \mathrm{Hom}(G_q/I_q, M_1) = W/p^n;$$

therefore by (ii),

$$\begin{aligned} \mathrm{H}^1(G_q/I_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p) &\xrightarrow{\sim} \mathrm{H}^1\left(G_q/I_q, \left(\mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^{I_q}\right) \\ &= W/p^n. \end{aligned}$$

(iv) We have a short exact sequence:

$$0 \to Z \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \left(\mathrm{Ad}^0(\rho_Q^D)/Z\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Taking the $G_q$-cohomology, we get a long exact sequence

$$\begin{aligned} 0 \;\to\;& W/p^m \to W \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(G_q, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ \to\;& \mathrm{H}^1(G_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \mathrm{H}^1(G_q, \left(\mathrm{Ad}^0(\rho_Q^D)/Z\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \end{aligned}$$

We claim that $W \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(G_q, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is a surjection; indeed, by the inflation-restriction we know $\mathrm{H}^1(G_q, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is isomorphic to $\mathrm{H}^1(I_q, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_q/I_q}$ and the later is just $W \otimes \mathbb{Q}_p/\mathbb{Z}_p$. By this claim, we have

$$0 \to \mathrm{H}^1(G_q, \mathrm{Ad}^0(\rho_Q^D) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \mathrm{H}^1(G_q, \left(\mathrm{Ad}^0(\rho_Q^D)/Z\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

which implies $\mathcal{L}_q = 0$.

□

By identifying $\Phi(R_Q^\alpha)$ to the Selmer group, we thus obtain an upper bound for $\Phi(R_Q^\alpha)$:

**Proposition 6.3.** *For any subset $\alpha \subset Q$, we have*

$$\left|\Phi(R_Q^\alpha)\right| \leq \prod_{q\in Q\setminus\alpha} |W/(x_q)|\,.$$

*Proof.* Let us abbreviate as above $M = \mathrm{Ad}^0(\rho_Q^D)\otimes \mathbb{Q}_p/\mathbb{Z}_p$. We may identify $\Phi(R_Q)$ to the Selmer group. That is

$$\mathrm{Hom}_W(\Phi(R_Q), \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathrm{H}^1_{\mathcal{L}}(G_{S\cup Q}, M).$$

Since $R_Q^D$ is isomorphic to $W$, the kernel of this map

$$\mathrm{H}^1(G_{S\cup Q}, M) \to \bigoplus_{q\in S\cup Q} \frac{\mathrm{H}^1(G_q, M)}{\mathcal{L}_q}$$

is trivial. Hence,

$$\mathrm{Hom}_W(\Phi(R_Q), \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow \bigoplus_{q\in Q} \frac{\mathrm{H}^1(G_q, M)}{\mathcal{L}_q}$$

By the previous lemma, we see

$$\mathrm{Hom}_W(\Phi(R_Q), \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow \bigoplus_{q\in Q} W/p^{m_q},$$

and

$$\left|\Phi(R_Q^D)\right| \leq \prod_{q\in Q} |W/(x_q)|\,.$$

Moreover, since we have a surjective morphism $R_D^\alpha \twoheadrightarrow R_Q^D$ for any subset $\alpha \subset Q$, $\Phi(R_Q^\alpha)$ is isomorphic to a subgroup of $\prod_{q\in Q\setminus\alpha} W/(x_q)$; hence the result is derived. $\square$

## 7. Ribet's short exact sequence

By the cohomological formalism of vanishing cycles (as opposed to Néron model theorem), we can establish a higher weight version of character groups and Ribet's short exact sequence for the curves.

7.1. **Residual characteristic divides the level.** We suppose that the prime $q$ does not divide the discriminant $D$ of the indefinite quaternion algebra $B$ over $\mathbb{Q}$. Let $\Gamma = \Gamma_q\Gamma^q = \widehat{\Gamma}_0(qM)\cap\widehat{\Gamma}_1^D(r)$ be a sufficiently small, open compact subgroup of $\widehat{B}^\times$. We have defined the Shimura curves $X^D(qM;r)$ associated to $\Gamma$. Let $V_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}_q) \middle| c \equiv 0 \bmod q \right\}$.

Let $\mu$ be the normalization map for the special fiber $Y$ of $X = X^D(qM;r)$ over $\mathbb{Z}_q$. Recall that we have a natural morphism $\theta : \mathrm{H}^0(Y, \mu_*\mu^*\Lambda) \to \mathrm{H}^0(Y, \mathscr{G})$. Let $\mathbb{T}_{qM;r}$ denote the Hecke algebra generated over $\mathbb{Z}_q$ by the endomorphisms $T_\ell$ ($\ell \nmid Mqr$) of $\mathrm{H}^1(X^D(qM : r), \Lambda)$. We will write $\mathbb{X}_q(qM;r)$ resp. $\check{\mathbb{X}}_q(qM;r)$ instead of $\mathbb{X}(\mathscr{F})$ resp. $\check{\mathbb{X}}(\mathscr{F})$ in order to emphasis the level structure.

**Proposition 7.1.** *If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbb{T}_{qM;r}$, then we have:*

(i) $\mathrm{im}(\theta)_\mathfrak{m} = 0$, *and we have a canonical isomorphism* $\check{\mathbb{X}}_q(qM;r)_\mathfrak{m} \simeq \left(\bigoplus_{x\in\Sigma_1}\mathscr{G}_x\right)_\mathfrak{m}$.

(ii) *In the exact sequence* (4.1)*, after localization at* $\mathfrak{m}$ *the map* $\bigoplus_{x\in\Sigma_1}\left(\mathrm{R}^1\Phi(\Lambda)\right)_x \to \ker(\mathrm{sp})$ *is the zero map, i.e.* $\mathbb{X}_q(qM;r)_{\mathfrak{m}} \simeq \left(\bigoplus_{x\in\Sigma_1}\left(\mathrm{R}^1\Phi(\Lambda)\right)_x\right)_{\mathfrak{m}}$.

(iii) *From (i) and (ii), we deduce*

$$\check{\mathbb{X}}_q(qM;r)_{\mathfrak{m}} \simeq \left(\bigoplus_{x\in\Sigma_1}\mathscr{G}_x\right)_{\mathfrak{m}} \overset{(1)}{\simeq} \left(\bigoplus_{x\in\Sigma_1}\left(\mathrm{R}^1\Phi(\Lambda)\right)_x\right)_{\mathfrak{m}} \simeq \mathbb{X}_q(qM;r)_{\mathfrak{m}},$$

*where* (1) *is induced from the monodromy logarithm* $N_x$ *at each* $x$*.*

*Proof.* (i) We first study $\mathrm{im}(\theta)$. Note that the normalization map $\mu : X^D(qM;r)\otimes\overline{\mathbb{F}}_q \sqcup X^D(qM;r)\otimes\overline{\mathbb{F}}_q \to X^D(qM;r)\otimes\overline{\mathbb{F}}_q$ is a finite morphism. We thus have an isomorphism

$$\mathrm{H}^0(X^D(qM;r)\otimes\overline{\mathbb{F}}_q, \mu_*\mu^*\mathscr{F}) \simeq \mathrm{H}^0(X^D(M;r)\otimes\overline{\mathbb{F}}_q,\mathscr{F})^2.$$

In Carayol [5] §2, we have that

$$\begin{aligned}\pi_0(X^D(M;r)\otimes\overline{\mathbb{F}}_q) &\simeq \mathbb{Q}_+^\times\backslash(\mathbb{A}^\infty)^\times/(\mathrm{Nm}(\Gamma(M)\cap\Gamma_1(r))\times\mathbb{Z}_q^\times)\\ &= (\mathbb{Z}_{(q)})_+^\times\backslash(\mathbb{A}^{q\infty})^\times/\,\mathrm{Nm}(\Gamma(M)\cap\Gamma_1(r)),\end{aligned}$$

where we write $\mathbb{Z}_{(q)}$ for $\mathbb{Q}\cap\mathbb{Z}_q$. Then $G(\mathbb{A}^{q\infty})$ acts on the set of components through via reduced norm (see [5] §1.3). But the maximal ideals in $\mathbb{T}_{qq'M}$ lying in the support of $\check{\mathbb{X}}$ must correspond to the one-dimensional automorphic representations, as cuspidal representations on quaternion groups admit infinite-dimensional components at almost every place, and thus do not factor through the norm (see [6] §4.4). However, the automorphic representations in $\mathrm{im}(\theta)$ factor through the norm. By Lemma 4.2 and Proposition 4.3, we deduce $\check{\mathbb{X}}_q(qM;r)_{\mathfrak{m}} \simeq \left(\bigoplus_{x\in\Sigma_1}\mathscr{G}_x\right)_{\mathfrak{m}}$.

(ii) Since $\mathscr{G}$ concentrates at points, its cohomology groups vanish in degree greater that one. Hence, we also have an isomorphism induced from the normalization map $\mu$:

$$\mathrm{H}^2(X^D(qM;r)\otimes\overline{\mathbb{F}}_q,\mathscr{F}) \simeq \mathrm{H}^2(X^D(qM;r)\otimes\overline{\mathbb{F}}_q,\mu_*\mu^*\mathscr{F}).$$

On the other hand, we may regard the latter group as $\mathrm{H}^2(X^D(M;r)\otimes\overline{\mathbb{F}}_q,\mathscr{F})^2$, and this is Poincaré dual to the group $\mathrm{H}^0(X^D(M;r)\otimes\overline{\mathbb{F}}_q,\check{\mathscr{F}}(1))^2$. Using similar analysis in (i), the second point follows as before.

(iii) Notice that the homomorphisms $N_x$ are isomorphisms for any regular model for $X$ over $\mathbb{Z}_q$. The third point (iii) follows from this together with the first and the second assertions.

□

7.2. **Čerednik-Drinfel'd uniformization theorem.** Throughout this section, we fix a prime $q$. Let $DM$ be a square-free integer. Suppose $D$ is a product of an even number of primes, and that $q \mid D$. Let $B$ be an *indefinite* quaternion algebra over $\mathbb{Q}$ with discriminant $D$.

Let $B'$ be the *definite* quaternion algebra ramified precisely at the primes dividing $D' = D/q$, and the archimedean place. Let $G'$ be the group scheme over $\mathbb{Z}$ associated to $B'^\times$. We fix an isomorphism $\phi_q : B'\otimes_{\mathbb{Q}}\mathbb{Q}_q \xrightarrow{\simeq} \mathbf{M}_2(\mathbb{Q}_q)$.

The $q$-adic upper half-plane is by definition

$$\Omega = \mathbb{P}^1(\mathbb{C}_q) \setminus \mathbb{P}^1(\mathbb{Q}_q).$$

It carries, in a natural way, the structure of a rigid $\mathbb{Q}_q$-space $\Omega^{\mathrm{rig}}$. The rigid $\mathbb{Q}_q$-space $\Omega^{\mathrm{rig}}$ determines a formal $\mathbb{Z}_q$-scheme up to admissible blowing ups in the special fiber [2]. There is a canonical choice of a formal $\mathbb{Z}_q$-scheme $\widehat{\Omega}$ whose associated rigid $\mathbb{Q}_q$-space is $\Omega^{\mathrm{rig}}$.

From the definition of $B'$, we have an anti-isomorphism

$$B^{\mathrm{opp}} \otimes_{\mathbb{Q}} \mathbb{A}^{q\infty} \simeq B' \otimes_{\mathbb{Q}} \mathbb{A}^{q\infty}. \tag{7.1}$$

We thus obtain a group isomorphism

$$B^{\times}(\mathbb{A}^{q\infty}) \simeq B'^{\times}(\mathbb{A}^{q\infty}) \tag{7.2}$$

after composition of (7.1) by the inversion $g \mapsto g^{-1}$. We write $\widehat{\Gamma}_0(M) = \Gamma_q^0 \cdot \Gamma_M^q (= \Gamma$ for short in this section), where $\Gamma_q^0$ denotes the group of units in the maximal ideal $\mathcal{O}_{B_q}$ and $\Gamma_M^q$ is an open compact subgroup of $B^{\times}(\mathbb{A}^q)$. We may consider $\Gamma_M^q$ as a subgroup of $B'^{\times}(\mathbb{A}^q)$ via (7.2). Define

$$Z_{\Gamma} = \Gamma_M^q \backslash \widehat{B'}^{\times} / B'^{\times}.$$

The group $\mathbf{GL}_2(\mathbb{Q}_q)$ acts naturally on the left on the Bruhat-Tits tree $\mathcal{T}$, the rigid analytic space $\Omega^{\mathrm{rig}}$, the formal scheme $\widehat{\Omega}$, and on $Z_{\Gamma}$. Let $\mathbb{Q}_q^{\mathrm{nr}}$ be the maximal unramified extension of $\mathbb{Q}_q$ and $\widehat{\mathbb{Z}}_q^{\mathrm{nr}}$ the completion of the ring of integers in $\mathbb{Q}_q^{\mathrm{nr}}$. An element $g \in \mathbf{GL}_2(\mathbb{Q}_q)$ acts on $\mathbb{Q}_q^{\mathrm{nr}}$ and $\widehat{\mathbb{Z}}_q^{\mathrm{nr}}$ via $\widehat{\mathrm{Frob}}_q^{-v_q(\det g)}$ where $\widehat{\mathrm{Frob}}_q$ denotes the arithmetic Frobenius automorphism.

Let $X^D(\Gamma)$ be the Shimura curve over $\mathbb{Z}_q$ corresponding to $\Gamma$. We denote by $\widehat{X^D(\Gamma)}$ the completion of $X^D(\Gamma)$ along its special fiber and by $X^D(\Gamma)^{\mathrm{rig}}$ the rigid analytic space over $\mathbb{Q}_q$ associates with $\widehat{X^D(\Gamma)}$.

Here comes the $q$-adic uniformization theorem (cf. [2] §5.2):

**Theorem 7.2** (Čerednik-Drinfel′d)**.** *There is a canonical isomorphism of formal schemes over* $\mathbb{Z}_q$

$$\mathbf{GL}_2(\mathbb{Q}_q)\backslash[(\widehat{\Omega}\widehat{\otimes}_{\mathbb{Z}_q}\widehat{\mathbb{Z}}_q^{\mathrm{nr}}) \times Z_{\Gamma}] \xrightarrow{\simeq} \widehat{X^D(\Gamma)}$$

*and a canonical isomorphism of rigid analytic spaces over* $\mathbb{Q}_q$

$$\mathbf{GL}_2(\mathbb{Q}_q)\backslash[(\Omega \otimes_{\mathbb{Q}_q} \mathbb{Q}_q^{\mathrm{nr}}) \times Z_{\Gamma}] \xrightarrow{\simeq} X^D(\Gamma)^{\mathrm{rig}}.$$

7.3. **Ribet's short exact sequence.** We let $X = X^D(M;r)$ and assume that $qq' \mid D$. Let $\Gamma \subset \widehat{B}^{\times}$ be the group of level $M$ defining $X$. We write $\Gamma = \Gamma_q\Gamma^q$, where $\Gamma_q = \mathcal{O}_{B_q}^{\times}$ and $\Gamma^q$ is an open compact subgroup of $B^{\times}(\mathbb{A}^q)$. We also insist that $\Gamma_{q'} = \mathcal{O}_{B_{q'}}^{\times}$. Let $\mathbf{GL}_2(\mathbb{Q}_q)_+$ (resp. $\mathbf{GL}_2(\mathbb{Q}_q)_+$) be the subset of elements in $\mathbf{GL}_2(\mathbb{Q}_q)$ whose reduced norm has even (resp. odd) valuation. In fact, $\mathbf{GL}_2(\mathbb{Q}_q)_+$ is a subgroup. Let $B'$ be the definite quaternion of discriminant $D/q$ obtaining from $B$ by exchanging the local invariants at $q$ and $\infty$. The Čerednik-Drinfel'd uniformization theorem gives a description of the dual graph $\mathcal{G}$ of the special fiber $Y$ of $X$ at $q$. That is, it can be described as $\mathbf{GL}_2(\mathbb{Q}_q)_+\backslash(\mathcal{T} \times Z_{\Gamma})$, where $Z_{\Gamma} = \Gamma^q\backslash\widehat{B}'^{\times}/B'^{\times}$.

We describe a bijection the set of edges of $\mathcal{T}$

$$\mathrm{Ed}(\mathcal{T}) \simeq \mathbf{GL}_2(\mathbb{Q}_q)_+/V_0(q)\mathbb{Q}_q^\times;$$

the set of vertices of $\mathcal{T}$

$$\mathrm{Ver}(\mathcal{T}) \simeq \mathbf{PGL}_2(\mathbb{Q}_q)/\,\mathbf{PGL}_2(\mathbb{Z}_q).$$

So the set of edges of $\mathcal{G}$ is:

$$\begin{aligned} \mathrm{Ed}(\mathcal{G}) &= \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash \mathrm{Ed}(\mathcal{T}) \times Z_\Gamma \\ &= \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash \left(\mathbf{GL}_2(\mathbb{Q}_q)_+/V_0(q)\mathbb{Q}_q^\times\right) \times Z_\Gamma \\ &= V_0(q)\Gamma^q\backslash \widehat{B}'^\times/B'^\times. \end{aligned}$$

Let us introduce

$$\mathscr{V}_+ := \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash\left(\mathbf{PGL}_2(\mathbb{Q}_q)_+/\,\mathbf{PGL}_2(\mathbb{Z}_q) \times Z_\Gamma\right)$$

and

$$\mathscr{V}_- := \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash\left(\mathbf{PGL}_2(\mathbb{Q}_q)_-/\,\mathbf{PGL}_2(\mathbb{Z}_q) \times Z_\Gamma\right).$$

Then the set of vertices of $\mathcal{G}$ is:

$$\begin{aligned} \mathrm{Ver}(\mathcal{G}) &= \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash \mathrm{Ver}(\mathcal{T}) \times Z_\Gamma \\ &= \mathbf{GL}_2(\mathbb{Q}_q)_+\backslash\left(\mathbf{PGL}_2(\mathbb{Q}_q)_+ \sqcup \mathbf{PGL}_2(\mathbb{Q}_q)_-\right)/\,\mathbf{PGL}_2(\mathbb{Z}_q) \times Z_\Gamma \\ &= \mathscr{V}_+ \sqcup \mathscr{V}_-. \end{aligned}$$

Define $\mathscr{V} := \mathbf{GL}_2(\mathcal{O}_q)\Gamma^q\backslash\widehat{B}'^\times/B'^\times = \mathbf{GL}_2(\mathcal{O}_q)\backslash Z_\Gamma$. Note that we have two degeneracy maps $\alpha, \beta$ from $\mathrm{Ed}(\mathcal{G})$ to $\mathscr{V}$ corresponding to the inclusion of $V_0(q)\Gamma^q$ into $\mathbf{GL}_2(\mathcal{O}_q)\Gamma^q$ and the conjugation by $W_q = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$. We have bijections between $\mathscr{V}$ and $\mathscr{V}_+$, and $\mathscr{V}$ and $\mathscr{V}_-$. Each edge $e$ connects $\alpha(e)$ in $\mathscr{F}_+$ to $\beta(e)$ in $\mathscr{V}_-$. In fact, we have

$$\begin{aligned} 1 : \mathscr{V} \to \mathscr{V}_+, &\qquad [x] \mapsto (1, x) \\ W_q : \mathscr{V} \to \mathscr{V}_-, &\qquad [x] \mapsto (W_q, W_q x). \end{aligned}$$

Using Honda-Tate theory, Carayol has described the set of singular points of $X^{D/qq'}(qq'M; r)$ mod$q'$ (i.e., the set $\Sigma_{qq'M;r}$ of supersingular points of $X^{D/qq'}(qq'M; r)$).

To simplify the notations and state Carayol's result in the sequel, we make the following assumptions: Let $B_1$ be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $D_1$, $p \nmid D_1$, $M_1 \geq 4$, and let $X^{D_1}(M_1; r) = G(\mathbb{Q})\backslash G(\mathbb{A}^\infty) \times \mathcal{H}_\infty/Z(\mathbb{A}^\infty)\Xi$ where $\Xi = \widehat{\Gamma}_0^{D_1}(M_1) \cap \widehat{\Gamma}_1^{D_1}(r) \subset G(\mathbb{A}^\infty)$. We denote by $K$ the restricted product of $(B_1 \otimes \mathbb{Q}_v)^\times$ for $v \neq p$.

**Proposition 7.3** (Carayol [4, 5])**.** *Let $\Sigma_{M_1;r}$ be the set of supersingular points of $X^{D_1}(M_1; r)$ mod $p$. Then the group $K \times \mathbb{Q}_p^\times$ acts transitively on $\Sigma_{M_1;r}$. For each $x \in \Sigma_{M_1;r}$, the stabilizer of $x$ is conjugate in $K \times \mathbb{Q}_p^\times$ to $\overline{Z(\mathbb{Q})}G'(\mathbb{Q})$ where $G' = B_2^\times$ obtaining from $B_1$ by changing the local invariants at $p$ and $\infty$ and $\overline{Z(\mathbb{Q})}$ is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}^\infty)$.*

*Remark* 7.4. For any $M_1 \geq 4$ as above, the set $\Sigma_{M_1;r}$ is in bijection with double coset:

$$\Sigma_{M_1;r} \simeq G'(\mathbb{Q})\backslash K \times \mathbb{Q}_p^\times/\Xi^p \times \mathbb{Z}_p^\times \simeq G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/\Xi^p \times \mathcal{O}_{B_{2p}}^\times.$$

Let us apply these information to our cases with $M_1 = qq'M, D_1 = D/qq', p = q', B = B_1$ and $B' = B_2$. Let $\Xi = \widehat{\Gamma}_0(qq'M) \cap \widehat{\Gamma}_1(r)$; consider the modulo $q'$ reduction of $X^{D/qq'}(qq'M;r)$, notice that

$$\begin{aligned}\left(\widehat{\Gamma}_0(qq'M) \cap \widehat{\Gamma}_1(r)\right)^{q'} \times \mathcal{O}^{\times}_{B'_{q'}} &= V_0(q)\left(\widehat{\Gamma}_0(qq'M) \cap \widehat{\Gamma}_1(r)\right)^{qq'} \times \mathcal{O}^{\times}_{B'_{q'}} \\ &\simeq V_0(q)\left(\widehat{\Gamma}_0(M) \cap \widehat{\Gamma}_1(r)\right)^{qq'} \times \mathcal{O}^{\times}_{B'_{q'}} \\ &= V_0(q)\Gamma^q,\end{aligned}$$

and this gives a bijective correspondence between $\mathrm{Ed}(\mathcal{G})$ and the set of singular points of $X^{D/qq'}(qq'M;r) \bmod q'$.

$$\begin{matrix}\mathrm{Ed}(\mathcal{G}) : \text{the edges of} \\ \text{the dual graph of } X^D(M;r) \bmod q\end{matrix} \Longleftrightarrow \begin{matrix}\Sigma_{qq'M;r} : \text{singular points of} \\ X^{D/qq'}(qq'M;r) \bmod q'\end{matrix} \tag{7.3}$$

Similarly, for vertices of the dual graph of the Shimura curve $X^D(M;r)$ we use Carayol's formula for $\mathscr{L} = \widehat{\Gamma}_0(q'M) \cap \widehat{\Gamma}_1(r)$. We find that

$$\left(\widehat{\Gamma}_0(q'M) \cap \widehat{\Gamma}_1(r)\right)^{q'} \times \mathcal{O}^{\times}_{B'_{q'}} \simeq \mathbf{GL}_2(\mathcal{O}_q)\Gamma^q.$$

The correspondence provides a bijection for $\mathscr{V}_?$ $(? = \emptyset, +, -)$:

$$\mathscr{V}_? \ (? = \emptyset, +, -) \Longleftrightarrow \begin{matrix}\Sigma_{q'M;r}\text{: singular points of} \\ X^{D/qq'}(q'M;r) \bmod q'\end{matrix}$$

Therefore, the map $1_*$ (resp. $W_{q,*}$) will correspond to $\alpha$ (resp. $\beta$).

Note that the number of irreducible components of the normalization is equal to the number of the vertices of the dual graph of the special fiber. Hence, for the lisse sheaf $\Lambda$ we let $\zeta$ be the composition of two Hecke-equivariant maps (1) and (2)

$$\begin{aligned}\zeta : \mathrm{H}^0(X_{\overline{s}}, \mu_*\mu^*\Lambda) &\overset{(1)}{\simeq} 1_*\left(\bigoplus_{y\in\Sigma_{q'M;r}} \mathscr{G}_y\right) \oplus W_{q,*}\left(\bigoplus_{y\in\Sigma_{q'M;r}} \mathscr{G}_y\right) \\ &\overset{(2)}{\twoheadrightarrow} 1_*\check{\mathbb{X}}_{q'}(q'M;r) \oplus W_{q,*}\check{\mathbb{X}}_{q'}(q'M;r),\end{aligned}$$

where (1) follows from Proposition 7.1 and (2) follows Proposition 4.3. We let $J = \ker(\zeta)$. Following from Proposition 7.1 $\mathrm{H}^0(X_{\overline{s}}, \mu_*\mu^*\Lambda)/J \simeq \mathrm{im}(\zeta)$ is a $\mathbb{T}_{qq'M;r}$-module. The Hecke-equivariant injection $\mathrm{H}^0(Y, \Lambda) \to \mathrm{H}^0(Y, \mu_*\mu^*\Lambda)$ induces an injection of $\mathrm{H}^0(Y,\Lambda)/(J \cap \mathrm{H}^0(Y,\Lambda))$ into $\mathrm{H}^0(X_{\overline{s}}, \mu_*\mu^*\Lambda)/J$. Hence, this map induces a $\mathbb{T}_{qq'M;r}$-module structure on $\mathrm{H}^0(Y,\Lambda)/(J \cap \mathrm{H}^0(Y,\Lambda))$.

**Lemma 7.5.** *We have* $\left(\mathrm{H}^0(Y,\Lambda)/J \cap \mathrm{H}^0(Y,\Lambda)\right)_{\mathfrak{m}} = 0$.

*Proof.* Since the restriction of $\Lambda$ to each irreducible component of $Y$ is constant, $\mathrm{H}^0(Y,\Lambda)$ is isomorphic to a direct sum of $\Lambda_y$'s, each corresponding to a connected component of $Y$. Hence, as the connected components of $X \otimes \overline{\mathbb{Q}}$ are defined over $\mathbb{Q}^{\mathrm{ab}}$, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $\mathrm{H}^0(X \otimes \overline{\mathbb{Q}}, \Lambda)$ factors through $\mathrm{Gal}(\overline{\mathbb{Q}}^{\mathrm{ab}}/\mathbb{Q})$. This gives rise to a reducible representation. So $\mathrm{H}^0(Y,\Lambda)/J \cap \mathrm{H}^0(Y,\Lambda)$ is Eisenstein. $\square$

By Proposition 4.3, we also have a surjective homomorphism of $\mathbb{Z}_p$-modules

$$\eta : \mathrm{H}^0(Y, \mathscr{G}) \to \check{\mathbb{X}}_q(qq'M; r).$$

Let $\mathscr{J} = \ker(\eta)$. On $\mathrm{H}^0(Y, \mathscr{G})/\mathscr{J}$, we have a $\widehat{\mathbb{T}}_{qq'M;r}$-module structure. Let $\theta$ be as in (4.4). By the previous lemma, we may identify $\left(\mathrm{H}^0(Y, \mu_*\mu^*\Lambda)/J\right)_{\mathfrak{m}}$ with $\left(\mathrm{H}^0(Y, \mu_*\mu^*\Lambda)/\theta^{-1}\mathscr{J}\right)_{\mathfrak{m}}$. By the definition of $\check{\mathbb{Y}}_q(q'M; r)$, we have the following exact sequence of $\widehat{\mathbb{T}}_{qq'M;r}$-modules

$$0 \to \left(\mathrm{H}^0(Y, \mu_*\mu^*\Lambda)/J\right)_{\mathfrak{m}} \to \left(\mathrm{H}^0(Y, \mathscr{G})/\mathscr{J}\right)_{\mathfrak{m}} \to \check{\mathbb{Y}}_q(q'M; r) \to 0.$$

Finally, we obtain the following exact sequence

$$0 \longrightarrow \check{\mathbb{X}}_{q'}(q'M; r)^2_{\mathfrak{m}} \xrightarrow{1_* \oplus W_{q,*}} \check{\mathbb{X}}_{q'}(qq'M; r)_{\mathfrak{m}} \longrightarrow \check{\mathbb{Y}}_q(q'M; r) \longrightarrow 0.$$

By (4.4), we see $\mathrm{H}^2(Y, \Lambda) \simeq \mathrm{H}^2(Y, \mu_*\mu^*\Lambda)$ and $\mathrm{H}^2(Y, \mu_*\mu^*\Lambda) = \mathrm{H}^2(\widetilde{Y}, \mu^*\Lambda)$. Since the components of $Y$ correspond to a disjoint union of two copies of $\Sigma_{q'M;r}$, $\mathrm{H}^2(\widetilde{Y}, \mu^*\Lambda)$ is isomorphic as a $\mathbb{Z}_p$-module to the direct sum of two copies of $\bigoplus_{x \in \Sigma_{q'M;r}} \mathrm{R}\Phi(\Lambda)_x$. Therefore, we have two injective homomorphisms of $\mathbb{Z}_p$-modules

$$f_1 : \mathbb{X}_{q'}(q'M; r)^2 \hookrightarrow \left( \bigoplus_{x \in \Sigma_{q'M;r}} \mathrm{R}\Phi(\Lambda)_x \right)^2 (1) \simeq \mathrm{H}^0(Y, \Lambda)(1)$$

and

$$f_2 : \mathbb{X}_{q'}(qq'M; r) \hookrightarrow \bigoplus_{x \in \Sigma_{qq'M;r}} \mathrm{R}\Phi(\Lambda)_x(1) \simeq \bigoplus_{x \in \Sigma_1} \mathrm{R}\Phi(\Lambda)_x(1).$$

Using similar argument above, we finally get the following exact sequence

$$0 \to \mathbb{Y}_q(q'M; r)_{\mathfrak{m}} \to \mathbb{X}_{q'}(qq'M; r)_{\mathfrak{m}} \to \mathbb{X}_{q'}(q'M; r)^2_{\mathfrak{m}} \to 0.$$

Since monodromy pairings are Hecke-equivariant, we deduce:

**Proposition 7.6.** *We have the following commutative diagram of $\widehat{\mathbb{T}}_{qq'M;r}$-modules where the rows are exact and the vertical maps come from monodromy pairings:*

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathbb{X}}_{q'}(q'M; r)^2_{\mathfrak{m}} \arrow[r, "1_* \oplus W_{q,*}"] & \check{\mathbb{X}}_{q'}(qq'M; r)_{\mathfrak{m}} \arrow[r] & \check{\mathbb{Y}}_q(M; r)_{\mathfrak{m}} \arrow[r] & 0 \\
0 & \mathbb{X}_{q'}(q'M; r)^2_{\mathfrak{m}} \arrow[l] \arrow[u] & \mathbb{X}_{q'}(qq'M; r)_{\mathfrak{m}} \arrow[l, "1^* \oplus W_q^*"] \arrow[u, "\lambda'_{q'}"'] & \mathbb{Y}_q(M; r)_{\mathfrak{m}} \arrow[l, "\iota"] \arrow[u, "\lambda_q"'] & 0 \arrow[l]
\end{tikzcd}$$

In our application, we will ignore the auxiliary prime $r$ in the proposition, since it is immaterial in our calculations below.

## 8. Proof of the main theorem

For a $\mathbb{T}$-module $L$, we will denote by $L_{\mathfrak{m},W}$ the module $L$ localized at $\mathfrak{m}$ and then tensored with $W$.

8.1. **The level-lowering.** We apply Ribet's short exact sequence to our descending induction by increasing $s$ via posing:

$$\begin{array}{ll} q' = q_{2m-2s} & q = q_{2m-2s-1} \\ \mathcal{P} = \ker\left(\pi : \widehat{\mathbb{T}}_Q^{D_s} \longrightarrow \widehat{\mathbb{T}}_Q^{D} \simeq W\right) & \\ \xi : \mathbb{Y}_q(M)_{\mathfrak{m}}[\mathcal{P}] \to \check{\mathbb{Y}}_q(M)_{\mathfrak{m}} & \xi^* \text{ dual of } \xi \\ \xi' : \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}] \to \check{\mathbb{X}}_{q'}(qq'M)_{\mathfrak{m}} & \xi'^* \text{ dual of } \xi' \end{array}$$

The maps $\xi^*\xi$ and $\xi'^*\xi'$ commute with the $W[G_{\mathbb{Q}}]$-action and can be regarded as given by multiplication by elements of $W$. We denote the corresponding ideals of $W$ by $(\xi^*\xi)$ and $(\xi'^*\xi')$.

Let $\mathcal{L} := \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}]$ and $\mathcal{L}' := \mathbb{Y}_q(M)_{\mathfrak{m}}[\mathcal{P}]$. Via the map $\iota$ in the Ribet's short exact sequence, we may identify $\mathcal{L}$ with $\mathcal{L}'$: Since $\mathcal{L} \cap \mathbb{Y}_q(M)_{\mathfrak{m},W}$ is of rank one and the Ribet's short exact sequence shows that $\mathcal{L}$ in $\mathbb{X}_{q'}(qM)^2_{\mathfrak{m},W}$ is finite. In fact, it is zero since $\mathbb{X}_{q'}(qM)^2_{\mathfrak{m},W}$ is torsion-free by the assumption $k < p$.

The following lemma and proposition are crucial in the proofs of the Theorem 5.2.

**Lemma 8.1.** (i) *$|\operatorname{coker}(\lambda_q)| = |W/(x_q)|$ for any $q \in Q$.*
(ii) *For any $\ell | Np$, $|\operatorname{coker}(\lambda_\ell)| = 1$.*
(iii) *$[\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}} : \mathcal{L}] = |x_{q'}|^{-1}$.*
(iv) *$[\mathbb{Y}_q(M)_{\mathfrak{m}} : \mathcal{L}] = 1$.*

*Proof.* (i) We see that the monodromy pairing acts from $\mathbb{Y}_q(M)_{\mathfrak{m}}$ to $\check{\mathbb{Y}}_q(M)_{\mathfrak{m}}$ as $\sigma - 1$ for $\sigma \in I$. Hence $|\operatorname{coker}(\lambda_q)| = |W/(x_q)|$ for any $q \in Q$.
(ii) Since we assume the minimality in our deformation of representations, for any $\ell \mid N$ the generator of the pro-$p$ part of the inertia at $\ell$ acts by $\begin{pmatrix} 1 & x_\ell \\ 0 & 1 \end{pmatrix}$ with $x_\ell \in W^\times$ so that $\operatorname{coker}(\lambda_\ell)$ is trivial. For $\ell = p$, since we assume $\overline{\rho}\mid_{I_p}$ is not split, the generator of the pro-$p$ part of the inertia at $p$ acts by $\begin{pmatrix} \chi_p^{k-1} & x_p \\ 0 & 1 \end{pmatrix}$ with $x_p \in W^\times$. Hence $\operatorname{coker}(\lambda_p)$ is trivial as well.
(iii) We have a surjection $\xi'_* : \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}} \to \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}]$. Using the monodromy parings, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}} \arrow[r] \arrow[d, "\xi'_*"] & \operatorname{Hom}_W(\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}, W) \arrow[r] \arrow[d, "f"] & \operatorname{coker}(\lambda'_{q'}) \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & \mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}] \arrow[r] & \operatorname{Hom}_W(\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}], W) \arrow[r] & \operatorname{coker}(\lambda_{q'}) \arrow[r] & 0 \end{tikzcd}$$

This implies the order of $\operatorname{coker}(g)$ equals to the order of $\operatorname{coker}(f)$. But the order of $\operatorname{coker}(f)$ is the order of the torsion subgroup of $\operatorname{coker}((\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}] \to (\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}})$. This derives that $\left|\operatorname{coker}(\operatorname{coker}(\lambda'_{q'}) \to \operatorname{coker}(\lambda_{q'}))\right| = [\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}} : \mathcal{L}]$. Recall that $\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}$ is self-dual; thus $\operatorname{coker}(\lambda'_{q'}) = 1$ and the result is deduced from (ii).
(iv) The fourth assertion is proved analogously.

□

**Proposition 8.2.** *We have*

$$(\xi'^{*}\xi') = (\xi^{*}\xi)(x_q x_{q'}),$$

*where when $s = m-1$ we declare $x_q$ to be a unit.*

*Proof.* Take a generator $\alpha$ of $\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}}[\mathcal{P}]$ and let $(\tau) = \langle \alpha, \alpha \rangle$. Let $\beta = \iota^{-1}(\alpha)$. By the adjoint property of $\xi'$ with respect to the monodromy pairing, we have

$$(\xi'^{*}\xi')(\langle \alpha, \alpha \rangle) = (\xi'^{*}\xi')(x_q) = (\langle \xi'(\alpha), \xi'(\alpha) \rangle) = \left( [\mathbb{X}_{q'}(qq'M)_{\mathfrak{m}} : \mathcal{L}]^{1/d} \right)^2 \cdot (\tau)$$

where $d = \operatorname{rank}_{\mathbb{Z}_p}(W)$. Similarly, we have $(\langle \beta, \beta \rangle) = (\tau)$ and

$$(\xi^{*}\xi)(x_q) = (\langle \xi'(\alpha), \xi'(\alpha) \rangle) = \left( [\mathbb{Y}_{q}(M)_{\mathfrak{m}} : \mathcal{L}]^{1/d} \right)^2 \cdot (\tau).$$

The result now follows easily from previous lemma. □

### 8.2. **Proof of Theorem 5.2.**

*Proof.* Using proposition inductively, we obtain

$$(\xi_m^{*}\xi_m) \subset \prod_{q \in Q} x_q.$$

Let $\mathcal{L} := \mathbb{Y}[\ker(\pi)]$, where $\mathbb{Y} = \mathbb{Y}_{q_1}(N)$. By the Lemma 8.1, we see that

$$[\mathbb{Y} : \mathcal{L}] = 1,$$

and this deduces a map $\xi_m$ from $\mathcal{L}$ to $\mathbb{Y}$ with torsion-free cokernel. Let $x$ be a generator of the free rank one $W$-module $\mathcal{L}$. By the adjoint property of monodromy pairing and Lemma 8.1, we see that $\xi_r^{*}(x)$ generates $\mathcal{L}$ and that

$$(\langle \xi_m(x), \xi_m(x) \rangle) = ((\xi_m^{*}\xi_m)\langle x, x \rangle) = (\xi_m^{*}\xi_m).$$

Define $I := \operatorname{Ann}_{\mathbb{T}_Q}(\ker(\pi))$ and $\eta_{\mathbb{T}_Q} := \pi(I)$. Since $\mathbb{Y}/\mathbb{Y}[I] \simeq \operatorname{Hom}_W(\mathcal{L}, W)$ we have

$$\mathbb{Y}/(\mathbb{Y}[\ker(\pi)] + \mathbb{Y}[I]) \simeq \operatorname{coker}(\mathcal{L} \to \operatorname{Hom}_W(\mathcal{L}, W))$$

as $W/\eta_{\mathbb{T}_Q}$-modules. Note that $\mathbb{Y}/(\mathbb{Y}[\ker(\pi)] + \mathbb{Y}[I])$ is also annihilated by $\eta_{\mathbb{T}_Q}$. Since we have

$$\left| W/\eta_{\mathbb{T}_Q} \right| \geq \left| \mathbb{Y}/(\mathbb{Y}[\ker(\pi)] + \mathbb{Y}[I]) \right| = \left| \operatorname{coker}(\mathcal{L} \to \operatorname{Hom}_W(\mathcal{L}, W)) \right|,$$

this implies $\eta_{\mathbb{T}_Q} \subset (\xi_m^{*}\xi_m) \subset (\prod_{q \in Q} x_q)$.

Applying the Numerical criterion and Proposition 6.3, we thus deduce the isomorphism

$$R_Q \simeq \widehat{\mathbb{T}}_Q.$$

□

8.3. **The method of Böckle.** In this section, we will give a proof of Theorem 3.10 which is originally due to Böckle. To use the results of Ramakrishna [22] §5, we need to assume $k < p$ which let us to compute the local Galois cohomology groups have exact dimensions at $p$.

Let $(\mathcal{O}, \pi, k)$ be a discrete valuation ring which is finite flat over $W(k)$. For a given continuous, absolutely irreducible residual representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathbf{GL}_2(k)$, let $\mathcal{D} = \{\mathcal{D}_\ell\}$ be a set of deformation conditions for deformations to $\underline{\mathbf{CNL}}_{\mathcal{O}}$-algebras where $\ell$ runs over all the places of $\mathbb{Q}$. For example:

- For almost all $\ell$, $\mathcal{D}_\ell$ is the condition that the deformation is unramified.
- For $\ell \neq p$ where ramification is allowed, $\mathcal{D}_\ell$ is the condition as in **(DC3)**.
- $\mathcal{D}_p$ is the condition imposed in **(DC4)**.
- If $\ell \in Q$, where $Q$ is a Ramakrishna-Khare system, then we require that $\mathcal{D}_\ell$ is as in **(DC5)**.

Note that we always fix the determinant of our deformations.

For any place $\ell$ of $\mathbb{Q}$, we denote by $R_{\mathcal{D}_\ell}$ the universal deformation ring which parametrizes deformations of $\overline{\rho}\,|_{G_\ell}$ subject to the condition $\mathcal{D}_\ell$. Also $R_{\mathcal{D}}$ denotes the universal deformation ring which satisfies all conditions of $\mathcal{D}$. Define $h^i(G_\ell, \mathrm{Ad}^0\,\overline{\rho}) := \dim_{\mathbb{F}} \mathrm{H}^i(G_\ell, \mathrm{Ad}^0\,\overline{\rho})$. Let $S$ be a finite set of places of $\mathbb{Q}$ such that deformations of type $\mathcal{D}$ are unramified outside $S$.

*Remarks* 8.3. (a) If $\rho$ is ramified at $\ell \in S \setminus \{p, \infty\}$ as in **(DC3)**, then $R_{\mathcal{D}_\ell}$ is of relative dimension $h^0(G_\ell, \mathrm{Ad}^0\,\overline{\rho})$.

(b) If $\ell \in Q$ as in **(DC5)**, then Ramakrishna in [23] has shown that $R_{\mathcal{D}_\ell} \simeq \mathcal{O}[\![T]\!]$ for some variable $T$. Clearly $h^0(G_\ell, \mathrm{Ad}^0\,\overline{\rho}) = 1$, and it is also shown that any deformation to $W(k)/(p^n)$ of type $\mathcal{D}_\ell$ can be lifted to a deformation to $W(k)/(p^{n+1})$ of the same type.

**Proposition 8.4.** *Given $\overline{\rho}$ and $\mathcal{D}$ as above. Suppose the following two conditions are satisfied:*

- *For $\ell \in S \setminus \{p, \infty\}$, the ring $R_{\mathcal{D}_\ell}$ is a complete intersection, flat over $\mathbb{Z}_p$, and of relative dimension $h^0(G_\ell, \mathrm{Ad}^0\,\overline{\rho})$.*
- *$R_{\mathcal{D}_p}$ is a complete intersection, flat over $\mathbb{Z}_p$, and of relative dimension $h^0(G_p, \mathrm{Ad}^0\,\overline{\rho})+1$.*

*Then*

$$R_{\mathcal{D}} \simeq \mathcal{O}[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_n)$$

*for suitable $n \in \mathbb{N}$ and $f_i \in \mathcal{O}[\![x_1, \ldots, x_n]\!]$.*

*Proof.* By Corollary 6.4 in [1], we can find a set of auxiliary primes $S_{\mathrm{aux}}$ such that:

- For $\ell \in S_{\mathrm{aux}}$, $R_{\mathcal{D}_\ell} \simeq \mathcal{O}[\![x_{\ell,1}, \ldots, x_{\ell,n_\ell}]\!]/J_\ell$ such that $n_\ell = h^1(G_\ell, \mathrm{Ad}^0\,\overline{\rho})$.
- For $\ell \in S \cup \{p\}$, $R_{\mathcal{D}_\ell} \simeq \mathcal{O}[\![x_{\ell,1}, \ldots, x_{\ell,n_\ell}]\!]/J_\ell$ such that $n_\ell = \dim_{\mathbb{F}} \mathfrak{m}_{R_{\mathcal{D}_\ell}}/(p, \mathfrak{m}^2_{R_{\mathcal{D}_\ell}})$, where $\mathfrak{m}_{R_{\mathcal{D}_\ell}}$ is the maximal ideal of $R_{\mathcal{D}_\ell}$.

Let $S' = S \cup S_{\mathrm{aux}}$, and Let $j_\ell$ be the minimal number of generators of $J_\ell$. By the assumptions on the prime $\ell \in S \cup \{p\}$ and by the remarks, we have

$$n_\ell = j_\ell + h^0(G_\ell, \mathrm{Ad}^0\,\overline{\rho}) + \begin{cases} 0 & \text{for } \ell \in S' \setminus \{p\}. \\ 1 & \text{for } \ell = p. \end{cases}$$

Let $\mathcal{D}'$ be the deformation problem with the same local contraints as $\mathcal{D}$ at primes $\ell \notin S_{\mathrm{aux}}$ and no local contraints at $\ell \in S_{\mathrm{aux}}$, and let $\mathfrak{m}_{R_{\mathcal{D}'}}$ be the maximal ideal of $R_{\mathcal{D}'}$. Then by Theorem 5.6 in [1] there exists a presentation

$$R_{\mathcal{D}} \simeq \mathcal{O}[\![x_1, \dots, x_n]\!]/J$$

where $n = \dim_{\mathbb{F}} \mathfrak{m}_{R_{\mathcal{D}'}}/(p, \mathfrak{m}_{R_{\mathcal{D}'}}^2)$ and $J$ is generated by at most $j := \sum_{\ell \in S'} j_\ell$ elements. By the Poitou-Tate Euler-Poincaré characteristic, we have

$$n = \sum_{\ell \in S'} (n_\ell - h^0(G_\ell, \mathrm{Ad}^0\,\overline{\rho})) = d + \sum_{\ell \in S'} j_\ell.$$

Hence, $n = j$. □

*Remark* 8.5. One does not know however that $R_{\mathcal{D}}$ is complete intersection.

**Corollary 8.6.** *For $\mathcal{D}$ of type $Q$ or $\alpha$-new, we have a presentation*

$$R_{\mathcal{D}} \simeq \mathcal{O}[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_m)$$

*for suitable $m \le n$ and $f_i \in \mathcal{O}[\![x_1, \dots, x_n]\!]$.*

**Lemma 8.7.** *Suppose that $R \in \underline{\mathbf{CNL}}_{\mathcal{O}}$ is a finitely generated $\mathcal{O}$-module. If $R$ has a presentation $R \simeq \mathcal{O}[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_m)$ where $m \le n$, then $R$ is complete intersection, finite flat over $\mathcal{O}$.*

*Proof.* Since $(\mathcal{O}, \pi, k)$ is a discrete valuation ring, $\dim \mathcal{O}[\![x_1, \dots, x_n]\!] = n + 1$. Let $\overline{R} := R/(\pi)$. Since $R$ is finitely generated over $\mathcal{O}$, $\overline{R}$ is a finite local ring over $\mathbb{F}$. Thus,

$$\mathrm{ht}(\pi, f_1, \dots, f_m) = \dim \mathcal{O}[\![x_1, \dots, x_n]\!] - \dim_{\mathbb{F}} \overline{R} = n + 1,$$

and hence the Krull intersection theorem yields $m = n$.

Since $\mathcal{O}[\![x_1, \dots, x_n]\!]$ is a regular local ring, and by Theorem 17.4 in [16], $(\pi, f_1, \dots, f_n)$ forms a regular sequence in $\mathcal{O}[\![x_1, \dots, x_n]\!]$. On the other hand, this also shows that $(f_1, \dots, f_n)$ is a regular sequence in $\mathcal{O}[\![x_1, \dots, x_n]\!]$, and therefore $R$ is a complete intersection. Note also that $(\pi)$ is a regular sequence in $R$, the multiplication by $\pi$ is injective which deduces the flatness of $R$ over $\mathcal{O}$. □

Hence we deduce the following:

**Corollary 8.8.** *If $R_\emptyset$ is finite over $\mathcal{O}$, then $R_\emptyset$ is complete intersection, finite flat over $\mathcal{O}$.*

We now can derive the following restatement of Theorem 3.10.

**Theorem 8.9.** *Suppose that for auxiliary set of primes $Q$ we have the isomorphism*

$$R_Q \xrightarrow{\sim} \widehat{\mathbb{T}}_Q$$

*of finite flat $W$-algebras. Then the canonical morphism $R_\emptyset \longrightarrow \widehat{\mathbb{T}}_\emptyset$ of minimal rings is an isomorphism as well.*

*Proof.* Consider the following diagram of finite flat $W$-algebras

$$\begin{tikzcd} R_Q \arrow[r, "\sim"] \arrow[d] & \widehat{\mathbb{T}}_Q \arrow[d] \\ R_\emptyset \arrow[r] & \widehat{\mathbb{T}}_\emptyset \end{tikzcd}$$

In order to show the bottom horizontal map is an isomorphism, it suffices to show this for a geometric generic fiber of $\operatorname{Spec} W$.

Let $K = \operatorname{Frac}(W)$, and let $\overline{K}$ be the algebraic closure of $K$. Tensoring the above diagram over $W$ with $\overline{K}$, we have

$$\begin{tikzcd} R_Q \otimes_W \overline{K} \arrow[r, "\sim"] \arrow[d] & \widehat{\mathbb{T}}_Q \otimes_W \overline{K} \arrow[r, "\sim"] \arrow[d] & \prod_{f\in M_Q} \overline{K} \arrow[d] \\ R_\emptyset \otimes_W \overline{K} \arrow[r, "(1)"] & \widehat{\mathbb{T}}_\emptyset \otimes_W \overline{K} \arrow[r, "\sim"] & \prod_{f\in M_\emptyset} \overline{K} \end{tikzcd}$$

where $M_Q$ denotes the set of normalized eigenforms for $\Gamma_0(pNQ)$ such that $\rho_f \bmod p \equiv \overline{\rho}$. (Similarly for $Q = 1$ we define $M_\emptyset$.) The isomorphisms on the right are consequence of the definitions of $\mathbb{T}_Q$ and $\mathbb{T}_\emptyset$. If the map (1) is not an isomorphism, then there is a modular form $f \in M_Q \backslash M_\emptyset$ and a non-trivial morphism $\alpha : R_\emptyset \to \overline{K}$ such that $\rho_f = \alpha \circ \rho_\emptyset$. By the definition of $R_\emptyset$, $\rho_f$ must be unramified at $Q$, and hence the conductor $N(\rho_f)$ of $\rho_f$ is prime to $Q$ by Carayol [6] which contradicts the fact $f \in M_Q \setminus M_\emptyset$. $\square$

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Report on
# *"Workshop on Galois Representations and Automorphic Forms"*

**Dates:** From 21 to 25 March, 2011

**Place:** School of Mathematics, Institute for Advanced Study (IAS), Princeton, USA

During the 2010-2011 academic year, Professor Richard Taylor of Harvard University lead a program on Galois Representations and Automorphic Forms.

The program will embrace all aspects of the conjectural relationship between automorphic forms and Galois representations: *functoriality and Langlands' conjectures*, analytic approaches (in particular *the trace formula*), algebraic approaches (those growing out of *Wiles's work on Fermat's Last Theorem*), *p-adic Hodge theory* (the so called *p-adic Langlands Program*) and applications to other problems in number theory.

There will be a weekly seminar and a week-long workshop during the week of 21 March, 2011, highlighting recent developments connected with the program.

The workshop will combine background talks aimed at graduate students and post-docs with research reports on the latest developments. The workshop will concentrate on the following topics:

1. Automorphy and potential automorphy of Galois representations and applications;

2. Integral models of cohomology of Shimura varieties and Rapoport-Zink spaces;

3. $p$-adic comparison theorems and applications to modular/automorphic forms;

4. $p$-adic $L$-functions and main conjectures, especially methods coming from automorphic forms;

5. The $p$-adic Langlands program.

We here describe and sketch the results of some interesting talks:

DeBacker and Reeder construct and parameterize $L$-packets on pure inner forms of unramified $p$-adic groups, that consist of depth zero supercuspidal representations. Tasho Kaletha generalizes their work to non-pure inner forms, by providing an alternative construction based on the theory of isocrystals with additional structure due to Kottwitz. Furthermore, Kaletha shows the stability and endoscopic transfer for these $L$-packets.

Adrian Iovita and Vincent Pilloni prove that a cuspidal automorphic form which occurs in the $\mathrm{H}^0$ of the coherent cohomology of some automorphic vector bundle on a Siegel variety and has tame level at $p$ can be $p$-adically deformed over the dimension $g$ weight space. Their methods certainly apply to any PEL Shimura variety and any prime $p$ which is split in the reflex field (this restriction on $p$ is imposed by the geometry: they need a non empty ordinary locus). The hypothesis that the automorphic form has tame level at $p$ is technical and will hopefully be removed in a near future.

Ana Caraiani strengthens the local-global compatibility of Langlands correspondences for $\mathbf{GL}_n$ in the case when $n$ is even and $l \neq p$. Let $L$ be a CM field and $\Pi$ be a cuspidal automorphic representation of $\mathbf{GL}_n(\mathbb{A}_L)$ which is conjugate self-dual. Under the assumption that $\Pi_\infty$ is cohomological and not "slightly regular", as defined by Shin, Chenevier and Harris constructed an $l$-adic Galois representation $R_l(\Pi)$ and proved the local-global compatibility up to semisimplification at primes $v$ not dividing $l$. Caraiani extends this compatibility by showing that the Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group at $v$ corresponds to the image of $\Pi_v$ via the local Langlands correspondence. She follows the strategy of Taylor-Yoshida, where it was assumed that $\Pi$ is square-integrable at a finite place. To make the argument work, she studies the action of the monodromy operator $N$ on the complex of nearby cycles on a scheme which is locally étale over a product of semistable schemes and derives a generalization of the weight-spectral sequence in this case.

Peter Scholze explains the theory of perfectoid spaces, which compares objects in characteristic $p$ with objects in characteristic 0. For example, a toric variety over $\mathbb{F}_p((t))$ can be regarded as a procover of the "same" toric variety over some infinite extension of $\mathbb{Q}_p$, in some precise sense. This generalizes the theorem of Fontaine, that the absolute Galois groups of $\mathbb{F}_p((t))$ and certain infinite extensions of $\mathbb{Q}_p$ are isomorphic, to higher dimensions.

The usual Katz-Mazur model for the modular curve $X(p^n)$ has horribly singular reduction. For large $n$ there isn't any model of $X(p^n)$ which has good reduction, but after extending the base one can at least find a semistable model, which means that the special fiber only has normal crossings as singularities. Jared Weinstein reveals a new picture of the special fiber of a semistable model of the entire tower of modular curves. Weinstein also indicates why this problem is important from the point of view of the local Langlands correspondence for $\mathbf{GL}(2)$.

Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor prove the compatibility of the local and global Langlands correspondences at places dividing $l$

for the $l$-adic Galois representations associated to regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of $\mathbf{GL}_n$ over an imaginary CM field or totally real field, under the assumption that the automorphic representations have Iwahori-fixed vectors at places dividing $l$ and have Shin-regular weight.

All lecturers included topics of interest to advanced students, but also took care to provide concrete examples that were accessible to non-experts.

# 國科會補助計畫衍生研發成果推廣資料表

日期:2011/12/23

<table>
<tr><td rowspan="3">國科會補助計畫</td><td>計畫名稱: 由GL_2與GSp_4所衍生的志村多樣體及其幾何</td></tr>
<tr><td>計畫主持人: 余屹正</td></tr>
<tr><td>計畫編號: 98-2115-M-004-007-MY2　　學門領域: 代數與數論</td></tr>
<tr><td colspan="2">無研發成果推廣資料</td></tr>
</table>

# 98年度專題研究計畫研究成果彙整表

| 計畫主持人：余屹正 | | | 計畫編號：98-2115-M-004-007-MY2 | | | | |
|---|---|---|---|---|---|---|---|
| 計畫名稱：由 GL_2 與 GSp_4 所衍生的志村多樣體及其幾何 | | | | | | | |
| 成果項目 | | | 量化 | | | 單位 | 備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等） |
| | | | 實際已達成數（被接受或已發表） | 預期總達成數(含實際已達成數) | 本計畫實際貢獻百分比 | | |
| 國內 | 論文著作 | 期刊論文 | 0 | 0 | 100% | 篇 | |
| | | 研究報告/技術報告 | 0 | 0 | 100% | | |
| | | 研討會論文 | 0 | 0 | 100% | | |
| | | 專書 | 0 | 0 | 100% | | |
| | 專利 | 申請中件數 | 0 | 0 | 100% | 件 | |
| | | 已獲得件數 | 0 | 0 | 100% | | |
| | 技術移轉 | 件數 | 0 | 0 | 100% | 件 | |
| | | 權利金 | 0 | 0 | 100% | 千元 | |
| | 參與計畫人力（本國籍） | 碩士生 | 0 | 0 | 100% | 人次 | |
| | | 博士生 | 0 | 0 | 100% | | |
| | | 博士後研究員 | 0 | 0 | 100% | | |
| | | 專任助理 | 0 | 0 | 100% | | |
| 國外 | 論文著作 | 期刊論文 | 0 | 1 | 100% | 篇 | |
| | | 研究報告/技術報告 | 0 | 0 | 100% | | |
| | | 研討會論文 | 1 | 1 | 100% | | |
| | | 專書 | 0 | 0 | 100% | 章/本 | |
| | 專利 | 申請中件數 | 0 | 0 | 100% | 件 | |
| | | 已獲得件數 | 0 | 0 | 100% | | |
| | 技術移轉 | 件數 | 0 | 0 | 100% | 件 | |
| | | 權利金 | 0 | 0 | 100% | 千元 | |
| | 參與計畫人力（外國籍） | 碩士生 | 0 | 0 | 100% | 人次 | |
| | | 博士生 | 0 | 0 | 100% | | |
| | | 博士後研究員 | 0 | 0 | 100% | | |
| | | 專任助理 | 0 | 0 | 100% | | |

| | |
|---|---|
| 其他成果<br>(無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。) | 無 |

| | 成果項目 | 量化 | 名稱或內容性質簡述 |
|---|---|---|---|
| 科教處計畫加填項目 | 測驗工具(含質性與量性) | 0 | |
| | 課程/模組 | 0 | |
| | 電腦及網路系統或工具 | 0 | |
| | 教材 | 0 | |
| | 舉辦之活動/競賽 | 0 | |
| | 研討會/工作坊 | 0 | |
| | 電子報、網站 | 0 | |
| | 計畫成果推廣之參與（閱聽）人數 | 0 | |

# 國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

| |
|---|
| 1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估<br>■達成目標<br>□未達成目標（請說明，以 100 字為限）<br>□實驗失敗<br>□因故實驗中斷<br>□其他原因<br>說明： |
| 2. 研究成果在學術期刊發表或申請專利等情形：<br>論文：□已發表 ■未發表之文稿 □撰寫中 □無<br>專利：□已獲得 □申請中 ■無<br>技轉：□已技轉 □洽談中 ■無<br>其他：（以 100 字為限） |
| 3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）<br>運用算術幾何的技術, 來了解 GSp(4)與 GL(2)的特殊纖維的幾何及其上同調群, 再藉由 p 進位單值化理論, 我們可以將這些上同調群轉譯成一些組合資料。經由這些訊息, 我們可以應用到模組提升定理的證明。 |