

國立政治大學應用數學系

碩士學位論文

指導教授：李明融 博士

Existence Result of Mild Solution
of Boltzmann Equation

波茲曼方程式柔和解的存在性

碩士班學生：魏照誠 撰

中華民國九十九年六月

Contents

謝辭	i
Abstract	ii
中文摘要	iii
1 Introduction	1
2 Basic Kinetic Theory of Gases and the Boltzmann Equation	4
2.1 Binary elastic collisions	4
2.2 The distribution functions and macroscopic fluid variables	7
2.3 The Boltzmann equation	11
3 Element Properties of the Collision Operator	15
3.1 Collision invariants	15
3.2 The H-Theorem and the Maxwellian distribution	17
3.3 Grad's angular cutoff potential	20
4 Hydrodynamical Limits:Expansions and Decomposition	22
4.1 A formal discussion on conservation laws	22
4.2 Hilbert expansion	23
4.3 Chapman-Enskog expansion	25
4.4 Micro-macro decomposition	29

5 Solutions in a Weighted Lebesgue Space	35
5.1 Preliminaries	35
5.2 Existence of the mild solution	41
5.3 Uniform stability of solutions	42
References	44



謝辭

碩士班兩年的日子很快地已經接近了尾聲。在這兩年中，首先我要感謝我的指導教授李明融老師細心的關懷與指導，使得此篇論文能順利完成。李老師開明的作風無論在學術或是待人處事上都令我受益良多。能跟隨著李老師學習真的是感到非常榮幸。再者，非常感謝陳俊全老師、洪盟凱老師、謝宗翰老師、及符聖珍老師於百忙中抽空來參加我的論文口試，並於口試期間熱心的提供諸多寶貴意見，使得此篇論文更趨於完善，亦讓我收穫良多。另外，在此也要感謝中興大學應用數學系及政大應用數學系的所有老師們在於我修業學士與碩士學位期間教授我數學知識，並帶領我走入數學的世界。老師們的諄諄教誨我會永銘於心。

接下來我要感謝的是我親愛的家人與女朋友琬婷，是你們給予我最大的支持與鼓勵，並在我落寞無助時給予我最溫暖的安慰與陪伴。也是因為有你們的陪伴，讓我在這段時間能好好安心專注於我的學業中。真的很謝謝你們，也好愛你們。

再來當然要感謝我多年的好友們：耀天、醇達、迪倫、俊棠、嘉德、嘉偉、柏翰、富元，我的人生中有你們相互的扶持與鼓勵讓我感到從來不曾孤單過，謝謝你們，希望我們都可以一起邁向更璀璨的未來。

最後，還要感謝這兩年在政大曾經幫助與關心過我的所有同學、學長姐、還有學弟妹們，謝謝你們的鼓勵與祝福。其實還有很多人想感謝，但真的無法一一寫出來，很抱歉。總之，謝謝所有關心愛我的人。

～謹以此篇論文獻給所有關心愛我的人～

魏照誠 謹誌于
國立政治大學應用數學系
中華民國九十九年六月

Abstract

In this thesis, we consider the initial-value problem for the Boltzmann equation of the form

$$\begin{cases} f_t + \xi \cdot \nabla_{\mathbf{x}} f = Q(f, f), & (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, \mathbf{x}, \xi) = f_0(\mathbf{x}, \xi), & (\mathbf{x}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{cases}$$

We prove that the existence of mild solution in the weighted Lebesgue space

$$L^1_{\omega}(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \equiv \left\{ f \in L^1(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \mid f\omega \in L^1(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \right\}$$

by using Banach's fixed point theorem, and that the uniform stability of solution with respect to the weighted norm. Here $\omega \equiv \mathbf{M}(\xi)^{\frac{-1}{2}}$, and $\mathbf{M}(\xi)$ is any Maxwellian.

Key words: Boltzmann equation, Maxwellian, existence of mild solutions, uniform stability.

中文摘要

在這篇論文裡，我們考慮以下型式的波茲曼方程之起始值問題

$$\begin{cases} f_t + \xi \cdot \nabla_{\mathbf{x}} f = Q(f, f), & (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, \mathbf{x}, \xi) = f_0(\mathbf{x}, \xi), & (\mathbf{x}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{cases}$$

我們利用 Banach 固定點定理證明了柔和解在加權魯貝格空間

$$L^1_{\omega}(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \equiv \left\{ f \in L^1(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \mid f\omega \in L^1(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\xi}^3) \right\}$$

上的存在性，並證明其解在如此的範數下之均勻穩定性。這裡的 $\omega \equiv M(\xi)^{-\frac{1}{2}}$ ，而 $M(\xi)$ 是任意的馬氏分佈函數。

關鍵詞：波茲曼方程，馬氏分佈函數，柔和解的存在性，均勻穩定。

1 Introduction

The kinetic theory, introduced by Boltzmann [17] at the end of the nineteenth century, provides a description of gases at an intermediate level between the hydrodynamic description which does not allow to take into account phenomena far from thermodynamic equilibrium, and the atomistic description which is often too complex. Nevertheless, a gas flow may be modelled at either the macroscopic or the microscopic level. The macroscopic model regards the gas as a continuous medium and the description is in terms of the spatial and temporal variations of the familiar flow properties such as the velocity, density, pressure, and temperature. The Euler and Navier-Stokes equations provide the conventional mathematical model of a gas as a continuum.

On the other hand, the microscopic or molecular model recognizes the particulate structure of the gas as a myriad of discrete molecules and ideally provides information on the position, velocity, and state of every molecule at all times. The mathematical model at this level is the celebrated Boltzmann equation, which is a mathematical model of the phenomenological kinetic theory of gases which describes the evolution, in time and space, of the one-particle distribution function for a simple monatomic gas of a large number of identical particles.

Furthermore, the degree of rarefaction of a gas is generally expressed through the Knudsen number, denoted by κ (or Kn), which is the ratio of the mean free path λ to some characteristic length L of the domain containing the gas, i.e.,

$$\kappa = \frac{\lambda}{L}. \quad (1.1)$$

It plays an essential role in the analysis of asymptotic relations between the Boltzmann equation and macroscopic fluid equations.

In this paper, we study the existence theory and stability of solutions to the Boltzmann equation. The first existence theory of the solutions to the Boltzmann equation goes back to 1932 when T.S.Carleman [18] proved the existence of global (in time) solutions to the Cauchy problem for the spatially homogeneous case.

It should be stressed that this is 2 years before the incompressible Navier-Stokes equation was solved by J.Leray on the existence of global weak solutions. On the other hand, the research on the spatially inhomogeneous Boltzmann equation started much later. It is only in 1965 when Grad [11] constructed the first local solutions near the Maxwellian, and it is in 1974 when the first author of this note constructed global solutions that are also near the Maxwellian, extending Grad's mathematical framework, [19].

However, the process made afterward was remarkable. Up to the present, three different methods have been developed for establishing the global existence theory. The difference is due to difference of function spaces used for solving the Boltzmann equation, and consequently, the methods of proof employed are also different. The following is a short summary of the solutions established so far :

1. Solutions in L^∞ framework: Grad's scheme was extended to construct global solutions in the L^∞ space for various initial and initial boundary value problems. See, [19], etc.
2. Solutions in L^1 framework: DiPerna-Lions [5] constructed global L^1 solutions without smallness assumption on initial data. These solutions are called the renormalized solutions because they are the weak solutions to the Boltzmann equation in the renormalized form.
3. Solutions in L^2 framework: Recently, the L^2 energy method, which is familiar in the theory of nonlinear PDE's, because available for the Boltzmann equation by introducing a new decomposition of the equation and solutions, called the macro-micro decomposition. This was developed by Liu-Yang-Yu [16], Liu-Yu [12], Yang-Zhao [22] and independently by Guo [6, 7, 8, 9, 10]. Also, the green function for the Cauchy problem was constructed by Liu-Yu [13].
4. Solutions in L^∞_β framework: The first global existence theorem was established in the space $L^\infty_\beta(\mathbb{R}^3_\xi; H^k(\mathbb{R}^3_x))$, $\beta > \frac{5}{2}$, $k \geq 2$, by using the spectral analysis

by Nishida-Imai [14], where

$$L_\beta^\infty(\mathbb{R}_\xi^3) \equiv \left\{ f : (1 + |\xi|)^\beta f \in L^\infty(\mathbb{R}_\xi^3) \right\}.$$

The same result was obtained by Shizuta [15] for the torus case with the space $L_\beta^\infty(\mathbb{R}_\xi^3; C^k(\mathbb{T}_x^3))$, $\beta > \frac{5}{2}$, $k \in \mathbb{Z}^+$. Recently, Yang-Ukai [20] presented a function space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap L_\beta^\infty(\mathbb{R}_\xi^3; L^\infty(\mathbb{R}_x^3))$, $\beta > \frac{3}{2}$, in which the Cauchy problem is globally well-posed in a mild sense without any regularity conditions.

This thesis contains five sections. In section 1, we give an introduction. In section 2, we review the binary elastic collisions and the well-known Boltzmann equation. In section 3, we introduce collision invariants, which aid in obtaining fluid dynamic conservation equations and the H-Theorem. This theorem in turn leads to the well-known Maxwellian distribution. We end section 3 by introducing the Grad's angular cutoff potential, which plays a crucial role of the existence theory. Furthermore, thanks to [16, 21, 22], we study a presentation of the few rigorous results on the fluid dynamical limit available in section 4. These are Hilbert expansion, Chapman-Enskog expansion, and macro-micro decomposition. They explain how the compressible Euler equations and compressible Navier-Stokes equations arise in suitable limits from the Boltzmann equation. Finally, in section 5, we study the existence of mild solution to the Boltzmann equation in the weighted Lebesgue space $L_\omega^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ by using the Banach's fixed point theorem, and the uniform L_ω^1 -type stability of solution.

2 Basic Kinetic Theory of Gases and the Boltzmann Equation

This section begins with a review of scattering concepts important to derivation of the Boltzmann equation. More details can be found in [1, 2, 3].

2.1 Binary elastic collisions

Intermolecular collisions in dilute gases are overwhelmingly likely to be binary collisions involving just two molecules. An elastic collision is defined as one in which there is no interchange of translational and internal energy. We may denote the pre-collision velocities of the two collision partners in a typical binary collision by ξ and ξ_* . Given the physical properties of the molecules and the orientation of the trajectories, our task is to determine the post-collision velocities ξ' and ξ'_* .

Linear momentum and energy must be conserved in the collision. This requires

$$m_1\xi + m_2\xi_* = m_1\xi' + m_2\xi'_* = (m_1 + m_2)\xi_m \quad (2.1)$$

and

$$m_1|\xi|^2 + m_2|\xi_*|^2 = m_1|\xi'|^2 + m_2|\xi'_*|^2, \quad (2.2)$$

where m_1 and m_2 are the masses of the two molecules and ξ_m is the velocity of the center of mass of the pair of molecules. Equation (2.1) shows that this center of mass velocity is not affected by the collision. The pre-collision and post-collision values of the relative velocity between the molecules may be defined by

$$\begin{cases} \xi_r = \xi - \xi_*, \\ \xi'_r = \xi' - \xi'_*. \end{cases} \quad (2.3)$$

Equations (2.1) and (2.3) give

$$\begin{cases} \xi = \xi_m + \frac{m_2}{m_1+m_2}\xi_r, \\ \xi_* = \xi_m - \frac{m_2}{m_1+m_2}\xi_r. \end{cases} \quad (2.4)$$

The pre-collision velocities relative to the center of mass are $\xi - \xi_m$ and $\xi_* - \xi_m$. Equation (2.4) shows that these velocities are antiparallel in this frame of reference and, if the molecules are point centers of force, the force between them remains in the plane containing the two velocities. The collision is therefore planer in the center of mass frame. Similarly, equations (2.1) and (2.3) imply

$$\begin{cases} \xi' = \xi_m + \frac{m_2}{m_1+m_2}\xi'_r, \\ \xi'_* = \xi_m - \frac{m_2}{m_1+m_2}\xi'_r. \end{cases} \quad (2.5)$$

This shows that the post-collision velocities are also anti-parallel in the center of mass frame.

Furthermore, equations (2.4) and (2.5) show that

$$\begin{cases} m_1 |\xi|^2 + m_2 |\xi_*|^2 = (m_1 + m_2) |\xi_m|^2 + m_r |\xi_r|^2, \\ m_1 |\xi'|^2 + m_2 |\xi'_*|^2 = (m_1 + m_2) |\xi_m|^2 + m_r |\xi'_r|^2. \end{cases} \quad (2.6)$$

where $m_r := \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass. A comparison of equation (2.6) with the energy equation (2.2) shows that the magnitude of the relative velocity is unchanged by collision, i.e.,

$$\xi'_r = \xi_r. \quad (2.7)$$

Since both ξ_m and ξ_r may be calculated from the pre-collision velocities, the determination of the post-collision velocities reduces to the calculation of the change in direction \mathbf{n} of the relative velocity vector.

Now we rewrite (2.1) and (2.2) as

$$\begin{cases} \xi + \xi_* = \xi' + \xi'_*, \\ |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2. \end{cases} \quad (2.8)$$

Introduce a unit vector \mathbf{n} directed along $\xi - \xi'$; this direction bisects the directions of ξ and $-\xi'$. Because of our definition of \mathbf{n} we have

$$\xi' = \xi + \mathbf{n}C, \quad (2.9)$$

where C is a scalar to be determined. The first of equations (2.8) gives

$$\xi'_* = \xi_* - \mathbf{n}C. \quad (2.10)$$

Substituting (2.9) and (2.10) into (2.8) we obtain

$$|\xi|^2 + 2\mathbf{n} \cdot \xi + C^2 + |\xi_*|^2 - 2\mathbf{n} \cdot \xi'_* C + C^2 = |\xi|^2 + |\xi_*|^2,$$

which implies that

$$\mathbf{n} \cdot (\xi_* - \xi_*)C + C^2 = 0.$$

Hence, dismissing the case $C = 0$ which corresponds to a trivial solution of the conservation equations, we have

$$C = -\mathbf{n} \cdot (\xi - \xi_*). \quad (2.11)$$

Thus, we have

$$\begin{cases} \xi' = \xi - \mathbf{n}[\mathbf{n} \cdot (\xi - \xi_*)], \\ \xi'_* = \xi_* + \mathbf{n}[\mathbf{n} \cdot (\xi - \xi_*)]. \end{cases} \quad (2.12)$$

On the other hand, if we consider the case on a hard sphere

$$\mathbb{S}_+^2 \equiv \left\{ \Omega \in \mathbb{S}^2 \mid (\xi - \xi_*) \cdot \Omega \geq 0 \right\},$$

then (2.12) becomes

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\Omega. \end{cases} \quad (2.13)$$

In this case, we use $B(|\xi - \xi_*|, \vartheta)$ to denote the collision cross section, where

$$\vartheta = \arccos \left(\frac{(\xi - \xi_*) \cdot \Omega}{|\xi - \xi_*|} \right) \in \left[0, \frac{\pi}{2} \right] \quad (2.14)$$

is the scattering angle between $\xi - \xi_*$ and Ω . The definition of B depends on the physics of collision. In fact, for the inverse power interaction potential, B takes the form of

$$B(|\xi - \xi_*|, \vartheta) = b_\gamma(\vartheta)|\xi - \xi_*|^\gamma, \quad -2 < \gamma \leq 1, \quad (2.15)$$

where $b_\gamma(\vartheta)$ is the collision kernel. However, different values of γ corresponds to different kinds of interactions, namely, $\gamma \geq 0$ corresponds to the hard intersection potential, and $\gamma < 0$ corresponds to the soft intersection potential. Moreover, the hard sphere model satisfies

$$B(|\xi - \xi_*|, \vartheta) = \sigma|\xi - \xi_*| \cos \vartheta$$

with σ being the radius of the hard sphere.

2.2 The distribution functions and macroscopic fluid variables

A gas flow would be completely described, in the classical sense, by listings of the position, velocity, and internal state of every molecule at a particular instant. The number of molecules in a real gas is so large that such a description is unthinkable, and we must resort to a statistical description in terms of probability distributions.

To see this, consider a sample of gas which is homogeneous in physical space and contains N identical molecules. A typical molecule has a velocity ξ with component ξ^1, ξ^2, ξ^3 in the direction of the Cartesian axes $x^1, x^2, \text{ and } x^3$. Note that $x^1, x^2, \text{ and } x^3$ define a space called physical space, and a volume element in physical space may be denoted by $d\mathbf{x}$. Similarly, $\xi^1, \xi^2, \text{ and } \xi^3$ define velocity space, and a volume element may be denoted by $d\xi$. The product $d\mathbf{x}d\xi$ then denotes a volume element in phase space, which is the 6-dimensional (in general, $6N$ -dimensional) space formed by the combination of physical space and velocity space. The state of the gas is therefore modelled by a distribution function in phase space. In the case

of a monatomic gas, the function

$$f(t, \mathbf{x}, \xi), (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$$

is termed the velocity distribution function. Its definition involves probability concepts; any result in which it appears will be a result as to the probable, or average, behavior of the gas; in addition,

$$f(t, \mathbf{x}, \xi) d\mathbf{x} d\xi \quad (2.16)$$

represents the number of particles, which at time t , have position \mathbf{x} and velocity ξ . Notice that f can never be negative and must either have finite bounds in velocity space or tend to zero as $|\xi| \rightarrow \infty$.

Furthermore, in order to relate the macroscopic properties to this distribution function, we must determine the relationship between the function and the average value of any molecule quantity \mathbf{q} . This quantity is either a constant or a function of the molecule velocity. The mean value principle gives

$$\bar{\mathbf{q}} = \int_{\mathbb{R}^3} \mathbf{q} f d\xi. \quad (2.17)$$

Now, we introduce some macroscopic fluid variables, and f will be assumed to be an expected mass density in phase space. Define the mass density

$$\rho(t, \mathbf{x}) \equiv \int_{\mathbb{R}^3} f(t, \mathbf{x}, \xi) d\xi. \quad (2.18)$$

The fluid velocity (or bulk velocity of gas) $u(t, \mathbf{x})$ is the average of the molecular velocities ξ at a certain point \mathbf{x} and time instant t ; since f is proportional to the probability for a molecular to have a given velocity, u is given by

$$u(t, \mathbf{x}) \equiv \frac{\int_{\mathbb{R}^3} \xi f(t, \mathbf{x}, \xi) d\xi}{\int_{\mathbb{R}^3} f(t, \mathbf{x}, \xi) d\xi} = \frac{\int_{\mathbb{R}^3} \xi f(t, \mathbf{x}, \xi) d\xi}{\rho(t, \mathbf{x})}. \quad (2.19)$$

Equation(2.19) can also be written as follows:

$$\rho(t, \mathbf{x}) u(t, \mathbf{x}) = \int_{\mathbb{R}^3} \xi f(t, \mathbf{x}, \xi) d\xi =: m(t, \mathbf{x}), \quad (2.20)$$

or, using components:

$$m^i(t, \mathbf{x}) \equiv \rho(t, \mathbf{x})u^i(t, \mathbf{x}) \equiv \int_{\mathbb{R}^3} \xi^i f(t, \mathbf{x}, \xi) d\xi, \quad i = 1, 2, 3. \quad (2.21)$$

The fluid velocity u is what we can directly perceive of the molecular motion by means of macroscopic observation; it is zero for a gas in equilibrium in a box at rest. Every molecule has its own velocity ξ , which can be decomposed into the sum of u and other velocity

$$\mathbf{c} \equiv \xi - u \quad (2.22)$$

called the random or peculiar velocity. By (2.17), we have

$$\bar{\mathbf{c}} = \int_{\mathbb{R}^3} \mathbf{c} f d\xi = \int_{\mathbb{R}^3} \xi f d\xi - \int_{\mathbb{R}^3} u f d\xi = \int_{\mathbb{R}^3} \xi f d\xi - u\rho = 0,$$

i.e., the average of \mathbf{c} is zero.

The quantity m^i that appears in equation (2.21) is the i th component of the mass flow or of the momentum density of the gas. Other quantities of similar nature are the momentum flow:

$$m_{ij} \equiv \int_{\mathbb{R}^3} \xi^i \xi^j f d\xi \quad (i, j = 1, 2, 3); \quad (2.23)$$

the energy density per unit volume:

$$w \equiv \frac{1}{2} \int_{\mathbb{R}^3} |\xi|^2 f d\xi; \quad (2.24)$$

and the energy flow:

$$r^i \equiv \frac{1}{2} \int_{\mathbb{R}^3} \xi^i |\xi|^2 f d\xi \quad (i = 1, 2, 3). \quad (2.25)$$

Equation (2.25) shows that the momentum flow is described by the components of a symmetric tensor of second order, because we need to describe the flow in the i th direction of the momentum in the j th direction. Note also that

$$\begin{aligned} \int_{\mathbb{R}^3} c^i c^j f d\xi &= \int_{\mathbb{R}^3} (\xi^i - u^i)(\xi^j - u^j) f d\xi \\ &= m_{ij} - \rho u^i u^j, \end{aligned}$$

which implies that

$$m_{ij} = \rho u^i u^j + \int_{\mathbb{R}^3} \mathbf{c}^i \mathbf{c}^j f d\xi = \rho u^i u^j + p_{ij}, \quad (2.26)$$

where

$$p_{ij} \equiv \int_{\mathbb{R}^3} c^i c^j f d\xi \quad (i, j = 1, 2, 3) \quad (2.27)$$

plays the role of stress tensor.

Furthermore, we define the internal energy E per unit mass (associated with random motions) by

$$\rho E = \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{c}|^2 f d\xi. \quad (2.28)$$

Note that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{c}|^2 f d\xi &= \frac{1}{2} \int_{\mathbb{R}^3} |\xi - u|^2 f d\xi \\ &= w - \frac{1}{2} \rho |u|^2. \end{aligned}$$

Thus the energy density per unit volume can also be written as follows:

$$w = \frac{1}{2} \rho |u|^2 + \rho E = \rho \left(\frac{1}{2} |u|^2 + E \right). \quad (2.29)$$

Similarly, we define

$$q^i = \frac{1}{2} \int_{\mathbb{R}^3} c^i |\mathbf{c}|^2 f d\xi, \quad (i = 1, 2, 3), \quad (2.30)$$

called the components of the so-called heat-flow vector, and we can obtain

$$r^i = \rho u^i \left(\frac{1}{2} |u|^2 + E \right) + \sum_{i=1}^3 u^j p_{ij} + q^i \quad (i = 1, 2, 3). \quad (2.31)$$

The decomposition in equation (2.31) shows that the microscopic energy flow is a sum of a macroscopic flow of energy (both kinetic and internal), of the work (per unit area and unit time) done by stresses, and of the heat flow.

We end this subsection with the definition of pressure p in terms of f ; p is nothing other than 1/3 of the spur or trace (i.e. the sum of the three diagonal terms) of p_{ij} and thus is given by

$$p = \frac{1}{3} \int_{\mathbb{R}^3} |\mathbf{c}|^2 f d\xi. \quad (2.32)$$

If we compare this with the definition of the specific internal energy E , given in equation (2.28), we obtain the relation:

$$p = \frac{2}{3}\rho E. \quad (2.33)$$

In order to complete the connection, as a simple mathematical consequence of the Boltzmann equation, we can derive five differential relations satisfied by the macroscopic quantities introduced above; these relations describe the balance of mass, momentum, and energy and have the same form as in continuum mechanics, and we will introduce these in the following sections.

2.3 The Boltzmann equation

Now, we can drive the well-known Boltzmann equation. At a particular instant, the number of molecules in the phase space element $d\mathbf{x}d\xi$ is given by equation (2.16). The rate of change of the number of molecules in the element is

$$\frac{\partial}{\partial t} f d\mathbf{x}d\xi. \quad (2.34)$$

The processes that contribute to the change in the number of molecules within $d\mathbf{x}d\xi$, which are

- (i) The convection of molecules across the face of $d\mathbf{x}$ by the molecular velocity ξ . The representation of the phase space element as separate volume elements in physical and velocity space emphasizes the fact that \mathbf{x} and ξ are treated as independent variables.
- (ii) The 'convection' of molecules across the surface of $d\xi$ as a result of the external force per unit mass \mathbf{F} .
- (iii) The scattering of molecules into and out of $d\mathbf{x}d\xi$ as a result of intermolecular collisions. The gas is assumed to be dilute. One consequence of this assumption is that a collision may be assumed to be an instantaneous event at a

fixed location in physical space. A second major consequence of the dilute gas assumption is that all collisions may be assumed to be binary collisions.

First consider process (i) which is a conservative process across the surface $d\mathbf{x}$. The number of molecules in the phase space element is $f d\mathbf{x} d\xi$, so the number density of class ξ molecules within $d\mathbf{x}$ is $f d\xi$. Note that the flux of the quantity \mathbf{q} across the element per unit area per unit time in the direction of \mathbf{e} is $\Delta \mathbf{q} \xi \cdot \mathbf{e}$. So, the total flux is obtained by summing over all velocity classes and can be written

$$\overline{\mathbf{q} \xi \cdot \mathbf{e}}. \quad (2.35)$$

Hence, equations (2.35) and (2.17) then enable the net inflow of molecules of this class ξ across the surface of $d\mathbf{x}$ to be written as

$$- \int_{S_{\mathbf{x}}} f \xi \cdot \mathbf{e}_{\mathbf{x}} dS_{\mathbf{x}} d\xi, \quad (2.36)$$

where $S_{\mathbf{x}}$ is the total area of the surface of $d\mathbf{x}$, $dS_{\mathbf{x}}$ is the element of this surface, and $\mathbf{e}_{\mathbf{x}}$ is the unit normal vector of this unit element. By the Gauss-divergence Theorem, we have

$$- \int_{S_{\mathbf{x}}} f \xi \cdot \mathbf{e}_{\mathbf{x}} dS_{\mathbf{x}} d\xi = - \int_{d\mathbf{x}} \nabla \cdot (f \xi) d(d\mathbf{x}) d\xi = - \nabla \cdot (f \xi) d\mathbf{x} d\xi \quad (2.37)$$

because f and ξ are constants within $d\mathbf{x}$.

Also, since we are considering only molecules of class ξ , the velocity ξ may be taken outside the divergence in physical space. Therefore, the inflow of molecules of class ξ across the surface of $d\mathbf{x}$ due to the velocity ξ is

$$-\xi \cdot \nabla_{\mathbf{x}} f d\mathbf{x} d\xi, \quad (2.38)$$

and the inflow of molecules across the surface of $d\xi$ due to the external force \mathbf{F} per unit mass is

$$-\mathbf{F} \cdot \nabla_{\xi} f d\mathbf{x} d\xi. \quad (2.39)$$

Furthermore, the volume swept out in physical space by the collision cross section $B(|\xi - \xi_*|, \vartheta)$ in the hard sphere is $B(|\xi - \xi_*|, \vartheta) d\Omega$, and the number of class

ξ_* molecules per unit volume in physical space is $f_* d\xi_*$, where f_* denotes the value of f at ξ_* . The number of collisions in the hard sphere per unit time is

$$f_* B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* \quad (2.40)$$

Since the number of class ξ molecules in the phase space element is $f d\xi d\mathbf{x}$, the number of class $\xi, \xi_* \rightarrow \xi', \xi'_*$ collisions per unit time in the element is

$$f f_* B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x} \quad (2.41)$$

Just as f denotes the value of the velocity distribution function f at ξ , f_* denotes the value of f at ξ_* . Similarly, f' and f'_* denote the values of f at ξ' and ξ'_* , respectively.

The existence of inverse collisions means that an analysis, exactly similar to that leading to equation (2.40) may be made for the collisions of class $\xi, \xi_* \leftarrow \xi', \xi'_*$ that scatter molecules into class ξ . This yields

$$f' f'_* B(|\xi' - \xi'_*|, \vartheta) d\Omega' d\xi'_* d\xi' d\mathbf{x} \quad (2.42)$$

for the collision rate in the phase space element $d\xi' d\mathbf{x}$. By equation (2.7), we have

$$| B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi | = | B(|\xi' - \xi'_*|, \vartheta) d\Omega' d\xi'_* d\xi' | \quad (2.43)$$

So, equation (2.42) may be written

$$f' f'_* B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x} \quad (2.44)$$

The rate of increase of molecules of class ξ in the phase space element $d\xi d\mathbf{x}$ as a result of the combined direct and inverse collisions of class $\xi, \xi_* \longleftrightarrow \xi', \xi'_*$ is obtained by the subtracting the loss rate (equation (2.40)) from the rate of gain (equation (2.43)). This gives

$$(f' f'_* - f f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x} \quad (2.45)$$

The total rate of increase of molecules of class ξ in the element as a result of collisions is given by

$$\int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} (f' f'_* - f f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x} \quad (2.46)$$

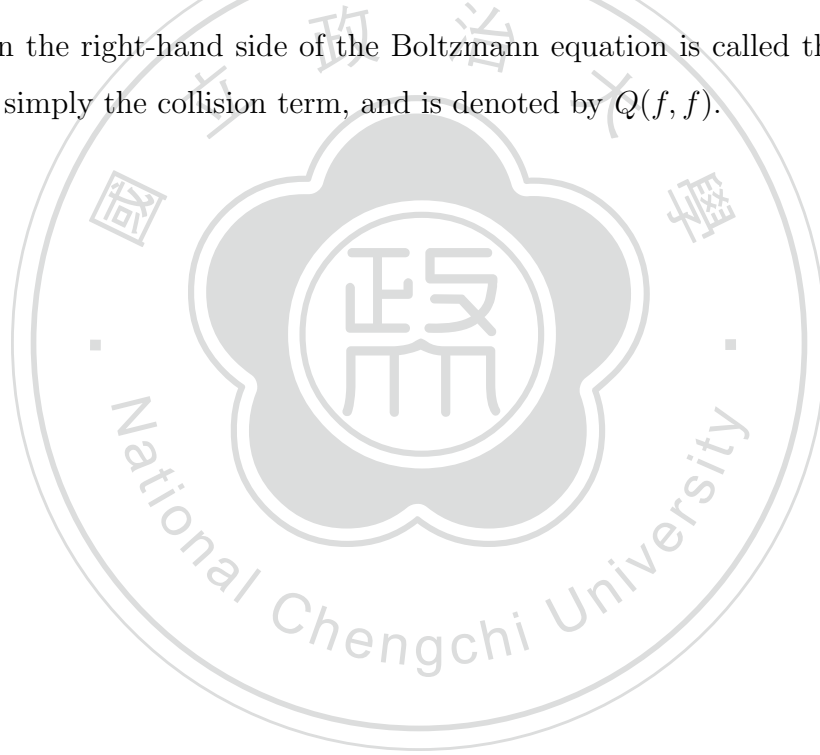
Hence, equations (2.34),(2.38),(2.39), and (2.46) give

$$\begin{aligned} \frac{\partial}{\partial t} f d\xi d\mathbf{x} &= -\xi \cdot \nabla_{\mathbf{x}} f d\xi d\mathbf{x} - \mathbf{F} \cdot \nabla_{\xi} f d\xi d\mathbf{x} \\ &+ \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} (f' f'_* - f f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x}. \end{aligned} \quad (2.47)$$

If the latter terms are transferred to the left-hand side and the complete equation is divided by $d\xi d\mathbf{x}$, we have the Boltzmann equation for a sample dilute gas:

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\xi} f = \int_{\mathbb{R}^3} \int_{\mathbb{S}_+^2} (f' f'_* - f f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_*. \quad (2.48)$$

The term on the right-hand side of the Boltzmann equation is called the collision operator or simply the collision term, and is denoted by $Q(f, f)$.



3 Element Properties of the Collision Operator

In this section, we review some element properties of the collision operator and introduce the celebrated H-Theorem. Finally, we introduce the Grad's angular cutoff potential, which plays a role of the existence theory.

3.1 Collision invariants

In this subsection, we study some elementary properties of the collision integral on a hard sphere

$$Q(f, f) = \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} (f' f'_* - f f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* \quad (3.1)$$

where

$$\mathbb{S}_+^2 \equiv \left\{ \Omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0 \right\}.$$

Actually it turns out that it is more convenient to study the slightly more general bilinear expression associated with $Q(f, f)$, i.e.,

$$Q(f, g)(\xi) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} (f' g'_* + g' f'_* - f g_* - g f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_*. \quad (3.2)$$

It is clear that when $g \equiv f$, equation (3.2) reduces to equation (3.1) and

$$Q(f, g) = Q(g, f). \quad (3.3)$$

Our first aim is to study the eightfold integral:

$$\int_{\mathbb{R}^3} Q(f, g) \psi(\xi) d\xi = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f' g'_* + g' f'_* - f g_* - g f_*) \psi(\xi) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi, \quad (3.4)$$

where f , g , and ψ are functions such that the indicated integrals exist and the order of the integration does not matter. Furthermore, by the Fubini's Theorem and the

fact that the Jacobian $|\frac{\partial(\xi, \xi_*)}{\partial(\xi', \xi'_*)}| = 1$, we have

$$\int_{\mathbb{R}^3} Q(f, g)\psi(\xi)d\xi = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f'g'_* + g'f'_* - fg_* - gf_*)\psi(\xi)B(|\xi - \xi_*|, \vartheta)d\Omega d\xi d\xi_* \quad (3.5)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (fg_* + gf_* - f'g'_* - g'f'_*)\psi(\xi')B(|\xi' - \xi'_*|, \vartheta)d\Omega d\xi' d\xi'_* \quad (3.6)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (fg_* + gf_* - f'g'_* - g'f'_*)\psi(\xi')B(|\xi - \xi_*|, \vartheta)d\Omega d\xi d\xi_* \quad (3.7)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (fg_* + gf_* - f'g'_* - g'f'_*)\psi(\xi_*)B(|\xi - \xi_*|, \vartheta)d\Omega d\xi d\xi_* \quad (3.8)$$

Taking the sum of equations (3.4), (3.5), (3.7), and (3.8) and dividing by four we obtain

$$\int_{\mathbb{R}^3} Q(f, g)\psi(\xi)d\xi = \frac{1}{8} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f'g'_* + g'f'_* - fg_* - gf_*)(\psi + \psi_* - \psi' - \psi'_*)B(|\xi - \xi_*|, \vartheta)d\Omega d\xi d\xi_*. \quad (3.9)$$

This relation expresses a basic property of the collision term, which is frequently used. In particular, when $g \equiv f$, equation (3.9) becomes

$$\int_{\mathbb{R}^3} Q(f, f)\psi(\xi)d\xi = \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f'f'_* - ff_*)(\psi + \psi_* - \psi' - \psi'_*)B(|\xi - \xi_*|, \vartheta)d\Omega d\xi d\xi_*. \quad (3.10)$$

Observe that the integral in (3.9) is zero independent of the particular functions f and g , if

$$\psi + \psi_* = \psi' + \psi'_* \quad (3.11)$$

is valid almost everywhere in velocity space. Since the integral appearing in the left-hand side of equation (3.10) is the rate of change of the average value of the function ψ due to collisions, the functions satisfying (3.11) is called collision invariants.

Definition 3.1. A function $\psi(\xi)$ is called the collision invariant if it satisfies equation (3.11), or, equivalently, if it satisfies

$$\int_{\mathbb{R}^3} Q(f, g)\psi(\xi)d\xi = 0. \quad (3.12)$$

for all $f, g \in C_0^\infty(\mathbb{R}_\xi^3, \mathbb{R}_+)$.

Remark 3.1.

(i) In fact, Q has five collision invariants in \mathbb{R}^3 , which are

$$\psi_0(\xi) = 1, \quad \psi_i(\xi) = \xi^i \quad (i = 1, 2, 3), \quad \psi_4(\xi) = \frac{1}{2}|\xi|^2. \quad (3.13)$$

This leads to the conservation laws of the Boltzmann equation (2.48) introduced latter.

(ii) The first discussion of equation (3.11) is due to Boltzmann [17], who assumed ϕ to be differentiable twice and arrived at the result that the most general solution of equation (3.11) is given by

$$\psi(\xi) = A + B \cdot \xi + C |\xi|^2, \quad (3.14)$$

where A , B , and C are constants.

3.2 The H-Theorem and the Maxwellian distribution

Now consider the case that the external force $\mathbf{F} = 0$, and let f be a nonnegative solution to (2.48) with rapid decay properties in (\mathbf{x}, ξ) . Then the Boltzmann equation (2.48) becomes

$$\frac{\partial f}{\partial t} = Q(f, f). \quad (3.15)$$

Furthermore, if we let f be a density function of a gas, and since it is nonnegative, we may define the H-function by

$$H(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \log(f) d\mathbf{x}d\xi. \quad (3.16)$$

Theorem 3.1 (The Boltzmann's H-Theorem). *If f is a nonnegative solution to the equation (3.15), i.e., the Boltzmann equation (2.48) is under the assumptions that f is a nonnegative solution to (2.48) with rapid decay properties in (\mathbf{x}, ξ) and $\mathbf{F} = 0$, then*

$$\frac{dH}{dt} \leq 0. \quad (3.17)$$

Proof. Note that

$$\frac{dH}{dt} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial}{\partial t} \left(f \log(f) \right) d\mathbf{x}d\xi \quad (3.18)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial f}{\partial t} (1 + \log(f)) d\mathbf{x}d\xi \quad (3.19)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q(f, f)(\xi) \left(1 + \log(f) \right) d\mathbf{x}d\xi \quad (3.20)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q(f, f)(\xi) \log(f) d\mathbf{x}d\xi \quad (3.21)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q(f, f)(\xi_*) \log(f_*) d\mathbf{x}d\xi_* \quad (3.22)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} -Q(f, f)(\xi') \log(f') d\mathbf{x}d\xi \quad (3.23)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} -Q(f, f)(\xi'_*) \log(f'_*) d\mathbf{x}d\xi_* \quad (3.24)$$

because $\psi \equiv 1$ is a collision invariant. From (3.21) to (3.24), we obtain

$$\frac{dH}{dt} = \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f'_* f' - f f_*) \log \left(\frac{f f_*}{f' f'_*} \right) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi d\mathbf{x}. \quad (3.25)$$

Since the logarithm function is increasing on $(0, \infty)$, then

$$(f'_* f' - f f_*) \log \left(\frac{f f_*}{f' f'_*} \right) \leq 0. \quad (3.26)$$

Thus, from (3.25) and (3.26), we conclude that $\frac{dH}{dt} \leq 0$.

□

In above proof, we also obtain

Corollary 3.1. *If f is a nonnegative function such that $\log(f)Q(f, f)$ is integrable and the manipulations of the previous section when $\psi \equiv \log(f)$, then the Boltzmann inequality*

$$\int_{\mathbb{R}^3} \log(f)Q(f, f)d\xi \leq 0. \quad (3.27)$$

holds. In particular, the equality holds if and only if $\log(f)$ is a collision invariant by equation (3.26), or equivalently,

$$f = \exp(a + b \cdot \xi + c|\xi|^2), \quad (3.28)$$

where a, b , and c are constants.

Remark 3.2.

(i) The integral

$$D(t, \mathbf{x}) := - \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (f'_* f' - f f_*) \log\left(\frac{f f_*}{f' f'_*}\right) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* d\xi \quad (3.29)$$

in the equation (3.25) is called the *entropy dissipation integral*. It is non-negative as seen (3.26).

(ii) In the equation (3.28) c must be negative, since $f \in L^1(\mathbb{R}^3)$. If we rewrite (3.28) as

$$f \equiv \frac{\rho}{(2\pi\mathbf{R}\theta)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2\mathbf{R}\theta}\right), \quad (3.30)$$

where $\theta > 0$ is the temperature and \mathbf{R} is the gas constant. The function appearing in equation (3.30) is the celebrated *Maxwell distribution or Maxwellian*, and is always denoted by $\mathbf{M}(\xi)$ or $\mathbf{M}_{[\rho, u, \theta]}(\xi)$.

(iii) The Maxwellian is known to describe the velocity distribution of a gas in an equilibrium state. Here (ρ, u, θ) are taken to be parameters, and if they are constants, \mathbf{M} is called a global Maxwellian while if they are functions of (t, \mathbf{x}) , it is called a local Maxwellian.

(iv) Without loss of generality, we may take a global Maxwellian as

$$\mathbf{M} \equiv \mathbf{M}_{[1,0,1]}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|\xi|^2}{2}\right) \quad (3.31)$$

in \mathbb{R}^3 . Evidently, the global Maxwellian is a stationary solution of the Boltzmann equation (2.48) if the external force \mathbf{F} is absent.

(v) By the Corollary 3.1, we conclude that

$$Q(f, f) = 0 \iff \int_{\mathbb{R}^3} \log(f) Q(f, f) d\xi = 0 \iff f \equiv \mathbf{M}(\xi).$$

Remark 3.3. The H-Theorem provides many physical implications as follows:

(i) It says that the entropy is increasing with time.

(ii) As far as the total entropy dissipation integral

$$\int_{\mathbb{R}^3} D d\mathbf{x}$$

is bounded in t , the H-function may play a role of the Lyapounov function, to prove that the solution of the Boltzmann equation converges to a limit. Corollary 3.1 then asserts that this limit should be a Maxwellian; that is, the Maxwellian is the only possible asymptotically stable stationary solution of the Boltzmann equation.

(iii) Physically, this can be rephrased as the equilibrium state of the gas is uniquely described by the Maxwellian, not by any other distribution functions. Thus, one can say that the Maxwellian is built-in in the Boltzmann equation.

3.3 Grad's angular cutoff potential

Note that the properties of the collision operator Q stated in the previous section are valid only when the relevant integrals converge. However, there are two different singular properties of the collision kernel, which are the strong singularity at $\vartheta = \frac{\pi}{2}$ in (2.15) due to the grazing collision and the unboundedness as $|\xi| \rightarrow \infty$.

The first singularity does not guarantee the convergence of integral over \mathbb{S}^2 in (3.1) under a mild assumption on f such that it is bounded. To avoid this difficulty,

Grad [11] introduced an idea to cutoff the singularity at $\vartheta = \frac{\pi}{2}$ assuming that $b_\gamma(\vartheta)$ vanishes near $\vartheta = \frac{\pi}{2}$. The assumption was highly successful for existence theory of the Boltzmann equation in the sense that almost all progresses made after Grad owe to his idea. This is known as the Grad's angular cutoff assumption.

In the following sections, we assume that $b_\gamma(\vartheta)$ is a nonnegative measurable function satisfying

$$\int_{\mathbb{S}_+^2} b_\gamma(\vartheta) d\Omega \geq b_0, \quad b_\gamma(\vartheta) \leq b_1 |\cos \vartheta| \quad (3.32)$$

for some constants $b_0, b_1 > 0$.

Under this assumption, the collision operator Q becomes well-defined. To see this, let $\mathbf{M} = \mathbf{M}(\xi)$ be any Maxwellian and introduce the function

$$\nu_{\mathbf{M}}(\xi) = \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} B(|\xi - \xi_*|, \vartheta) \mathbf{M}(\xi_*) d\Omega d\xi_*, \quad (3.33)$$

which satisfies under the assumption (3.32),

$$\nu_0 (1 + |\xi|)^\gamma \leq \nu_{\mathbf{M}}(\xi) \leq \nu_1 (1 + |\xi|)^\gamma, \quad (3.34)$$

for some positive constants ν_0 and ν_1 . Notice that (3.34) only holds in the case on hard potentials, i.e., $\gamma \in [0, 1]$.

4 Hydrodynamical Limits: Expansions and Decomposition

In this section, we will use the decomposition of the solution into the macroscopic and microscopic components to reformulate the Boltzmann equation

$$f_t + \xi \cdot \nabla_{\mathbf{x}} f = \frac{1}{\kappa} Q(f, f), \quad (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (4.1)$$

where κ is the Knudsen number introduced in section 1, as a system of conservation laws for the macroscopic components coupled with an equation for the microscopic components. This kind of thinking is similar to the Hilbert and Chapman-Enskog expansions where the leading term is a local Maxwellian with its macroscopic components governed by the conservation laws, either the Euler equations or Navier-Stokes equations.

4.1 A formal discussion on conservation laws

Note that if we multiply both sides of (4.1) by one of the element collision invariants ψ_α ($\alpha = 0, 1, 2, 3, 4$) which are given by (3.13), and integrate with respect to ξ , we have

$$\int_{\mathbb{R}^3} \psi_\alpha (f_t + \xi \cdot \nabla_{\mathbf{x}} f) d\xi = 0$$

or

$$\partial_t \int_{\mathbb{R}^3} \psi_\alpha f d\xi + \sum_{j=1}^3 \partial_{x_j} \int_{\mathbb{R}^3} \xi^j \psi_\alpha f d\xi = 0 \quad (\alpha = 0, 1, 2, 3, 4). \quad (4.2)$$

If $\alpha = 0$, then equation (4.2) becomes

$$\rho_t + \sum_{j=1}^3 \partial_{x_j} \int_{\mathbb{R}^3} \xi^j f d\xi = 0,$$

which implies

$$\rho_t + \operatorname{div}_{\mathbf{x}} m = 0.$$

If $\alpha = i$, $i = 1, 2, 3$, then equation (4.2) becomes

$$m_t^i + \sum_{j=1}^3 \partial_{x^j} \int_{\mathbb{R}^3} \xi^j \xi^i f d\xi = 0.$$

By (2.23) and (2.26), we get

$$m_t^i + \sum_{j=1}^3 \partial_{x^j} (m^i u^j + p_{ij}) = 0.$$

Similarly, if $\alpha = 4$ we have

$$\partial_t \left(\frac{1}{2} \rho |u|^2 + \rho E \right) + \sum_{j=1}^3 \partial_{x^j} \left[\rho u^j \left(\frac{1}{2} |u|^2 + E \right) + \sum_{i=1}^3 u^i p_{ji} + q_j \right] = 0.$$

Hence, from above discussion, we have the system of conservation laws:

$$\left\{ \begin{array}{l} \rho_t + \operatorname{div}_{\mathbf{x}} m = 0, \\ m_t^i + \left(\sum_{j=1}^3 (m^i u^j + p_{ij}) \right)_{x^j} = 0, \quad i = 1, 2, 3, \\ \left[\rho \left(\frac{|u|^2}{2} + E \right) \right]_t + \left\{ \sum_{j=1}^3 \left[\rho u^j \left(\frac{|u|^2}{2} + E \right) + \sum_{i=1}^3 u^i p_{ji} + q_j \right] \right\}_{x^j} = 0. \end{array} \right. \quad (4.3)$$

These equations have the so-called conservation form because they express the circumstance that a certain quantity neither created nor destroyed in a certain region $D \subseteq \mathbb{R}^3$ because something is flowing through the boundary ∂D .

4.2 Hilbert expansion

In this subsection, we assume that κ is a small constant and use it as the parameter for the expansion. In 1912, Hilbert [4] introduced the following famous expansion of the solution to the Boltzmann equation:

$$f = \sum_{n=0}^{\infty} \kappa^n f_n. \quad (4.4)$$

By putting this expansion into the Boltzmann equation (4.1) and comparing the terms by the order of κ , we have the following equations for f_n :

$$\begin{cases} Q_0 = 0, \\ (f_{n-1})_t + \xi \cdot \nabla_{\mathbf{x}} f_{n-1} = Q_n, \quad n \geq 1, \end{cases} \quad (4.5)$$

where

$$Q_0 = Q(f_0, f_0), \quad Q_n = 2Q(f_0, f_n) + \sum_{k=1}^{n-1} Q(f_k, f_{n-k}), \quad n \geq 1. \quad (4.6)$$

Hence, by the Remark 3.2 (iv), the first equation in (4.6) implies that f_0 is a local Maxwellian, i.e.,

$$f_0 = \mathbf{M}_0 \equiv \mathbf{M}_{[\rho^0, u^0, \theta^0]} = \frac{\rho^0}{(2\pi\mathbf{R}\theta^0)^{3/2}} \exp\left(-\frac{|\xi - u^0|^2}{2\mathbf{R}\theta^0}\right), \quad (4.7)$$

where ρ^0 , u^0 , and θ^0 are functions of (t, \mathbf{x}) , and f_0 satisfies

$$(f_0)_t + \xi \cdot \nabla_{\mathbf{x}} f_0 = Q_1. \quad (4.8)$$

Here, Q_1 is a microscopic component which is orthogonal to the five collision invariants $\psi_\alpha(\xi)$, $\alpha = 0, 1, \dots, 4$. The solvability condition for (4.8) gives the system of conservation laws

$$\int_{\mathbb{R}^3} \psi_\alpha((f_0)_t + \xi \cdot \nabla_{\mathbf{x}} f_0) d\xi = 0, \quad (4.9)$$

which are exactly the compressible Euler equations

$$\begin{cases} \rho_t^0 + \nabla_{\mathbf{x}} \cdot (\rho^0 u^0) = 0, \\ (\rho^0 u^0)_t + \nabla_{\mathbf{x}} \cdot (\rho^0 u^0 \otimes u^0) + \nabla_{\mathbf{x}} p^0 = 0, \\ [\rho^0(E^0 + \frac{1}{2}|u^0|^2)]_t + \nabla_{\mathbf{x}} \cdot \{[\rho^0(E^0 + \frac{1}{2}|u^0|^2) + p^0]u\} = 0, \end{cases} \quad (4.10)$$

where the pressure function is given by $p^0 = \mathbf{R}\rho^0\theta^0$ and the internal energy $E^0 = \frac{3}{2}\mathbf{R}\theta^0$.

For $n \geq 1$, let

$$S_n = \sum_{k=1}^{n-1} Q(f_k, f_{n-k}), \quad (4.11)$$

and

$$L_{\mathbf{M}_0}h = 2Q(h, f_0), \quad (4.12)$$

which is the linearized collision operator with respect to the local Maxwellian \mathbf{M}_0 .

Then under the solvability condition for the second equation in (4.5)

$$\int_{\mathbb{R}^3} \psi_\alpha((f_{n-1})_t + \xi \cdot \nabla_{\mathbf{x}} f_{n-1}) d\xi = 0, \quad (4.13)$$

f_n can be represented in terms of f_k for $k = 0, 1, \dots, n-1$ by

$$f_n = \sum_{\alpha=0}^4 c_\alpha \psi_\alpha \mathbf{M}_0 + L^{-1}((f_{n-1})_t + \xi \cdot \nabla_{\mathbf{x}} f_{n-1} - S_n). \quad (4.14)$$

Thus, the conservation laws

$$\int_{\mathbb{R}^3} \psi_\alpha((f_n)_t + \xi \cdot \nabla_{\mathbf{x}} f_n) d\xi = 0, \quad \alpha = 0, 1, \dots, 4, \quad (4.15)$$

are the systems of linearized Euler equations around the fluid variables (ρ^0, u^0, θ^0) for the macroscopic components in f_n .

Since to determine the value of f_n in the Hilbert expansion involves the differentiation of f_{n-1} , by induction, the convergence of this expansion can only be expected when the solution is infinitely differentiable and bounded with respect to the Knudsen number κ . Therefore, in general, the Hilbert expansion does not converge, especially in the present of initial layer, shock layer and boundary layer where the value of the differentiation grows when κ decreases.

4.3 Chapman-Enskog expansion

The Chapman-Enskog expansion was introduced by Chapman and Enskog in 1916 and 1917 independently. The advantage of this expansion is that the first order correction yields the Navier-Stokes equations for the macroscopic components so that the viscosity and heat conductivity are correctly represented.

Formally, we can write

$$\frac{\partial f}{\partial t} = \sum_{n=0}^{\infty} \kappa^n \frac{\partial^{(n)} f_n}{\partial t}. \quad (4.16)$$

Since the conserved quantities are unexpanded, the consistency requires that for $n \geq 1$,

$$\int_{\mathbb{R}^3} \psi_\alpha f_n d\xi = 0, \quad \alpha = 0, 1, 2, 3, 4, \quad (4.17)$$

which implies that all the function f_n for $n \geq 1$ are microscopic. Substituting (4.4) and (4.16) into the Boltzmann equation (4.1), we have

$$\begin{cases} Q_0 = Q(f_0, f_0) = 0, \\ \sum_{k=0}^{n-1} \frac{\partial^{(k)} f_{n-k-1}}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f_{n-1} = 2Q(f_0, f_n) + S_n, \quad n \geq 1, \end{cases} \quad (4.18)$$

where the notation has the same meaning as in the last subsection. However, notice that here each f_n is a functional of the conserved quantities which are not expanded. Again, the first equation in (4.18) implies that f_0 must be a local Maxwellian, i.e.,

$$f_0 = \mathbf{M} \equiv \mathbf{M}_{[\rho, u, \theta]} = \frac{\rho}{(2\pi \mathbf{R}\theta)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2\mathbf{R}\theta}\right). \quad (4.19)$$

Furthermore, for $n = 1$, the second equation in (4.18) can be written as

$$\frac{\partial^{(0)} f_0}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f_0 = L_{\mathbf{M}} f_1. \quad (4.20)$$

The solvability condition for (4.20) immediately gives the following Euler equations

$$\begin{cases} \frac{\partial^{(0)} \rho}{\partial t} = -\frac{\partial}{\partial x^i} (\rho u^i), \\ \frac{\partial^{(0)} u^i}{\partial t} = -u^j \frac{\partial u^i}{\partial x^j} - \frac{1}{\rho} \frac{\partial p}{\partial x^i}, \quad i, j = 1, 2, 3, \\ \frac{\partial^{(0)} \theta}{\partial t} = -u^i \frac{\partial \theta}{\partial x^i} - \frac{2}{3} \theta \frac{\partial u^i}{\partial x^i}, \end{cases} \quad (4.21)$$

where $p = \mathbf{R}\rho\theta$, here and in what follows, the summation is over any repeated indices. By plugging the expression of the local Maxwellian of f_0 into the equation (4.20), we have

$$\frac{1}{\rho} \mathbf{M} \mathbf{B}^0 \rho + \frac{1}{\theta} \left(\frac{|\mathbf{c}|^2}{2\mathbf{R}\theta} - \frac{3}{2} \right) \mathbf{M} \mathbf{B}^0 \theta + \frac{1}{\mathbf{R}\theta} c^j \mathbf{B}^0 u^j = L_{\mathbf{M}} f_1, \quad (4.22)$$

where $\mathbf{c} = \xi - u$ is the random velocity introduced in (2.22) and B^0 is the following linear operator

$$\mathbf{B}^0 \equiv \frac{\partial^{(0)}}{\partial t} + \xi \cdot \nabla_{\mathbf{x}}. \quad (4.23)$$

Now we can substitute the time derivative $\frac{\partial^{(0)}}{\partial t}$ of (4.21) into the equation (4.22) to obtain

$$\left(\frac{|\mathbf{c}|^2}{2\mathbf{R}\theta} - \frac{5}{2} \right) \mathbf{M} \frac{c^i}{\theta} \frac{\partial \theta}{\partial x^i} + \frac{1}{\mathbf{R}\theta} \left(c^i c^j - \frac{1}{3} |\mathbf{c}|^2 \delta_{ij} \right) \mathbf{M} \frac{\partial u^i}{\partial x^j} = L_{\mathbf{M}} f_1. \quad (4.24)$$

By using the Burnett functions defined by

$$\begin{cases} \mathbf{A}_j(\xi) = \frac{|\xi|^2 - 5}{2} \xi^j, & j = 1, 2, 3, \\ \mathbf{B}_{ij}(\xi) = \xi^i \xi^j - \frac{1}{3} \delta_{ij} |\xi|^2, & i, j = 1, 2, 3, \end{cases} \quad (4.25)$$

we have

$$f_1 = L_{\mathbf{M}}^{-1} \left(\sqrt{\mathbf{R}} \mathbf{A}_i \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R}\theta}} \right) \mathbf{M} \frac{\partial \theta}{\partial x^i} + \mathbf{B}_{ij} \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R}\theta}} \right) \mathbf{M} \frac{\partial u^i}{\partial x^j} \right). \quad (4.26)$$

Notice that we have used the fact that the operator $L_{\mathbf{M}}$ is invertible in the microscopic space which is the space orthogonal to the null space of $L_{\mathbf{M}}$.

Before going further, we review the properties of the Burnett functions.

Proposition 4.1. *Let*

$$A' = L_{\mathbf{M}}^{-1} \mathbf{A}, \quad B' = L_{\mathbf{M}} \mathbf{B}. \quad (4.27)$$

Then there exist positive functions $a(r)$ and $b(r)$ defined on $[0, \infty)$ such that

$$A'(\xi) = -a(|\xi|) \mathbf{A}(\xi), \quad B'(\xi) = -b(|\xi|) \mathbf{B}(\xi). \quad (4.28)$$

And the following properties hold, where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R}^3)$.

- (1) $\langle -A_i, A'_i \rangle$ is positive and independent of i .
- (2) $\langle A_i, A'_j \rangle = 0$ for any $i \neq j$.
- (3) $\langle A_i, B'_{jk} \rangle = 0$ for any i, j, k .

(4) $\langle B_{ij}, B'_{kl} \rangle = \langle B_{kl}, B'_{ij} \rangle = \langle B_{ji}, B'_{kl} \rangle$ holds and is independent of i, j for any fixed k, l .

(5) $-\langle B_{ij}, B'_{ij} \rangle$ is positive and independent of i, j when $i \neq j$.

(6) $-\langle B_{ii}, B'_{jj} \rangle$ is positive and independent of i, j when $i \neq j$.

(7) $-\langle B_{ii}, B'_{ii} \rangle$ is positive and independent of i .

(8) $\langle B_{ij}, B'_{kl} \rangle = 0$ unless either $(i, j) = (k, l)$ or (l, k) , or $i = j$ and $k = l$.

(9) $\langle B_{ii}, B'_{ii} \rangle - \langle B_{ii}, B'_{jj} \rangle = 2\langle B_{ij}, B'_{ij} \rangle$ holds for any $i \neq j$.

These proofs are quite technical and can be found in [23].

With the Burnett functions, the viscosity $\mu(\theta)$ and heat conductivity coefficient $\varkappa(\theta)$ can be represented by

$$\begin{cases} \mu(\theta) = -\kappa \mathbf{R} \theta \int_{\mathbb{R}^3} \mathbf{B}_{ij} \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R} \theta}} \right) L_{\mathbf{M}}^{-1} \left(\mathbf{B}_{ij} \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R} \theta}} \right) \mathbf{M} \right) d\xi > 0, \quad i \neq j, \\ \varkappa(\theta) = -\kappa \mathbf{R}^2 \theta \int_{\mathbb{R}^3} \mathbf{A}_l \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R} \theta}} \right) L_{\mathbf{M}}^{-1} \left(\mathbf{A}_l \left(\frac{\mathbf{c}}{\sqrt{\mathbf{R} \theta}} \right) \mathbf{M} \right) d\xi > 0, \end{cases} \quad (4.29)$$

Note that these coefficients are independent of the density function ρ .

Now, if we put f_1 into the conservation laws to include the first order approximation, then the conservation laws take the form

$$\int_{\mathbb{R}^3} \psi_\alpha ((f_0)_t + \xi \cdot \nabla_{\mathbf{x}} (f_0 + \kappa f_1)) d\xi = 0. \quad (4.30)$$

Since f_1 is microscopic, its contribution to the conservation of mass is zero. And its contribution to the equations of conservation of momentum and energy is represented by the stress tensor and heat flux:

$$p_{ij}^{(1)} = \kappa \int_{\mathbb{R}^3} c^i c^j f_1 d\xi, \quad q_i^{(1)} = \frac{\kappa}{2} \int_{\mathbb{R}^3} c^i |\mathbf{c}|^2 f_1 d\xi. \quad (4.31)$$

With the proposition 4.1, it is straightforward to calculate the stress tensor and heat flux in terms of the fluid variables:

$$\begin{cases} p_{ij}^{(1)} = -\mu(\theta) \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) + \frac{2}{3} \mu(\theta) \frac{\partial u^k}{\partial x^k} \delta_{ij}, \\ q_i^{(1)} = -\aleph(\theta) \frac{\partial \theta}{\partial x^i}. \end{cases} \quad (4.32)$$

In summary, the first order approximation in the Chapman-Enskog expansion is the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \operatorname{div}_{\mathbf{x}} m = 0, \\ m_t^i + \operatorname{div}_{\mathbf{x}}(m^i u) + p_{,x^i} = [\mu(\theta)(u_{,x^j}^i + u_{,x^i}^j - \frac{2}{3} \delta_{ij} \operatorname{div}_{\mathbf{x}} u)]_{,x^j}, \quad i, j = 1, 2, 3, \\ \left[\rho \left(E + \frac{1}{2} |u|^2 \right) \right]_t + \operatorname{div}_{\mathbf{x}} \left\{ \left[\rho \left(E + \frac{1}{2} |u|^2 \right) + p \right] u \right\} \\ = \mu(\theta) u^i \left(u_{,x^j}^i + u_{,x^i}^j - \frac{2}{3} \delta_{ij} \operatorname{div}_{\mathbf{x}} u \right) + (\aleph(\theta) \theta_{,x^i})_{,x^j}. \end{cases} \quad (4.33)$$

Again, similar but tedious calculation can be used to find the next terms, f_2, f_3, \dots , in the Chapman-Enskog expansion, however, without good mathematical theory.

4.4 Micro-macro decomposition

In this subsection, we study a micro-macro decomposition of the Boltzmann equation (4.1), which is based on the decomposition of the solution into its macroscopic (fluid dynamic) and microscopic (kinetic) component, we can reformulate the Boltzmann equation into a system of conservation laws for the time evolution of the macroscopic variables and an equation for the time evolution of the microscopic variable. The main idea is not to have any approximation, but a complete description of the solutions to the Boltzmann equation so that the analytic techniques from the theory of conservation laws can be applied in the study of the Boltzmann equation.

To see this, let $f(t, \mathbf{x}, \xi)$ be the solution to the Boltzmann equation (4.1), we will decompose it into the macroscopic fluid part, the local Maxwellian $\mathbf{M} = \mathbf{M}(t, \mathbf{x}, \xi) = \mathbf{M}_{[\rho, u, \theta]}(\xi)$, and the microscopic, non-fluid part $\mathbf{G} = \mathbf{G}(t, \mathbf{x}, \xi)$,

$$f(t, \mathbf{x}, \xi) = \mathbf{M}(t, \mathbf{x}, \xi) + \mathbf{G}(t, \mathbf{x}, \xi). \quad (4.34)$$

The local Maxwellian $\mathbf{M}(t, \mathbf{x}, \xi)$ is constructed from the fluid variables, the five conserved quantities, the mass density $\rho(t, \mathbf{x})$, momentum $m(t, \mathbf{x})$ and energy $E(t, \mathbf{x}) + \frac{1}{2} |u(t, \mathbf{x})|^2$ of the Boltzmann equation:

$$\left\{ \begin{array}{l} \rho(t, \mathbf{x}) \equiv \int_{\mathbb{R}^3} f(t, \mathbf{x}, \xi) d\xi, \\ m^i(t, \mathbf{x}) \equiv \int_{\mathbb{R}^3} \psi_i f(t, \mathbf{x}, \xi) d\xi, \quad (i = 1, 2, 3), \\ \rho(E + \frac{1}{2} |u|^2)(t, \mathbf{x}) \equiv \int_{\mathbb{R}^3} \psi_4 f(t, \mathbf{x}, \xi) d\xi, \end{array} \right. \quad (4.35)$$

and

$$\mathbf{M} \equiv \mathbf{M}_{[\rho, u, \theta]}(\xi) \equiv \frac{\rho}{(2\pi \mathbf{R}\theta)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2\mathbf{R}\theta}\right). \quad (4.36)$$

With respect to the local Maxwellian, we define an inner product in $\xi \in \mathbb{R}^3$ as

$$\langle h, g \rangle_{\mathbf{M}} \equiv \int_{\mathbb{R}^3} \frac{1}{\mathbf{M}} h(\xi) g(\xi) d\xi \quad (4.37)$$

for functions h, g of ξ such that the integral is well defined. Using this inner product with respect to this Maxwellian, the subspace spanned by the collision invariants has the following set of orthogonal basis:

$$\left\{ \begin{array}{l} \chi_0 \equiv \chi_0(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i \equiv \chi_i(\xi; \rho, u, \theta) \equiv \frac{\xi^i - u^i}{\sqrt{\mathbf{R}\theta\rho}} \mathbf{M}, \quad \text{for } i = 1, 2, 3, \\ \chi_4 \equiv \chi_4(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{6\rho}} \left(\frac{|\xi - u|^2}{\mathbf{R}\theta} - 3 \right) \mathbf{M}, \\ \langle \chi_\alpha, \chi_\beta \rangle = \delta_{\alpha\beta} \quad \text{for } \alpha, \beta = 0, 1, 2, 3, 4. \end{array} \right. \quad (4.38)$$

With this basis, we can define the macroscopic projection \mathbf{P}_0 and microscopic projection \mathbf{P}_1 as follows:

$$\mathbf{P}_0 h \equiv \sum_{\alpha=0}^4 \langle h, \chi_\alpha \rangle_{\mathbf{M}} \chi_\alpha, \quad (4.39)$$

$$\mathbf{P}_1 h = h - \mathbf{P}_0 h. \quad (4.40)$$

Note that the operators \mathbf{P}_0 and \mathbf{P}_1 are projections, that is,

$$\mathbf{P}_0 \mathbf{P}_0 = \mathbf{P}_0, \quad \mathbf{P}_1 \mathbf{P}_1 = \mathbf{P}_1, \quad \mathbf{P}_0 \mathbf{P}_1 = \mathbf{P}_1 \mathbf{P}_0 = 0.$$

We view the above decomposition of Boltzmann equation as the linearization around the local Maxwellian states so that the linear collision operator $L_{\mathbf{M}}$ is

$$L \equiv L_{[\rho, u, \theta]} g = Q(\mathbf{M} + g, \mathbf{M} + g) - Q(g, g). \quad (4.41)$$

Definition 4.1. A function $h(\xi)$ is called non-fluid if it gives raise to zero conserved quantities, that is,

$$\int_{\mathbb{R}^3} h(\xi) \psi_\alpha d\xi = 0, \quad \alpha = 0, 1, 2, 3, 4. \quad (4.42)$$

Note that functions in the range of the microscopic projection \mathbf{P}_1 are non-fluid. It is clear that for the solution $f(t, \mathbf{x}, \xi)$ of the form (4.34) of the Boltzmann equation (4.1), we have

$$\mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G}.$$

From the decomposition of the solution $f = \mathbf{M} + \mathbf{G}$, the Boltzmann equation (4.1) becomes

$$(\mathbf{M} + \mathbf{G})_t + \xi \cdot \nabla_{\mathbf{x}} (\mathbf{M} + \mathbf{G}) = \frac{1}{\kappa} (2Q(\mathbf{G}, \mathbf{M}) + Q(\mathbf{G}, \mathbf{G})). \quad (4.43)$$

We now decompose the Boltzmann equation. Using the same method as above, if we multiply equation (4.43) by ψ_α ($\alpha = 0, 1, 2, 3, 4$) and integrate with respect

to ξ we obtain the system of conservation laws:

$$\left\{ \begin{array}{l} \rho_t + \operatorname{div}_{\mathbf{x}} m = 0, \\ m_t^i + \left(\sum_{j=1}^3 m^i u^j \right)_{x^j} + p_{x^i} + \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} \mathbf{G}) d\xi = 0, \quad i = 1, 2, 3, \\ \left[\rho \left(\frac{|u|^2}{2} + E \right) \right]_t + \sum_{j=1}^3 \left\{ u^j \left[\rho \left(\frac{|u|^2}{2} + E \right) + p \right] \right\}_{x^j} + \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} \mathbf{G}) d\xi = 0. \end{array} \right. \quad (4.44)$$

Here p is the pressure for the monatomic gases same as Equation(2.33).

The microscopic equation is obtained by applying the microscopic projection \mathbf{P}_1 to the Boltzmann equation (4.43). Note that \mathbf{M}_t , as a function of ξ , is in the space spanned by χ_α , $\alpha = 0, 1, 2, 3, 4$. Hence

$$\mathbf{P}_0 \mathbf{M}_t = \mathbf{M}_t.$$

Note also that $\mathbf{P}_0 h = 0$ if

$$\int_{\mathbb{R}^3} h \psi_\alpha d\xi = 0, \quad \alpha = 0, 1, 2, 3, 4.$$

Thus the projection of collision terms under \mathbf{P}_0 is zero. We also have

$$\int_{\mathbb{R}^3} \mathbf{G}_t \psi_\alpha d\xi = \partial_t \int_{\mathbb{R}^3} \mathbf{G} \psi_\alpha d\xi = 0.$$

Therefore, we have

$$\mathbf{P}_0 \mathbf{G}_t = 0,$$

and so

$$\mathbf{P}_1(\mathbf{M}_t + \mathbf{G}_t) = (\mathbf{M}_t + \mathbf{G}_t) - \mathbf{P}_0(\mathbf{M}_t + \mathbf{G}_t) = \mathbf{G}_t.$$

With these, the microscopic equation is

$$\mathbf{G}_t + \mathbf{P}_1(\xi \cdot \nabla_{\mathbf{x}} \mathbf{G} + \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}) = \frac{1}{\kappa} L \mathbf{G} + \frac{1}{\kappa} Q(\mathbf{G}, \mathbf{G}). \quad (4.45)$$

From (4.45), we have

$$\begin{aligned} \mathbf{G} &= \kappa L^{-1} \left(\mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M} \right) + L^{-1} \left(\kappa (\partial_t \mathbf{G} + \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{G}) - Q(\mathbf{G}, \mathbf{G}) \right) \\ &= \kappa L^{-1} \left(\mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M} \right) + \Theta, \end{aligned} \quad (4.46)$$

where

$$\Theta := L^{-1} \left(\kappa (\partial_t \mathbf{G} + \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{G}) - Q(\mathbf{G}, \mathbf{G}) \right),$$

and substitute this into (4.44) yields the following fluid-type system for the macroscopic components:

$$\left\{ \begin{array}{l} \rho_t + \operatorname{div}_{\mathbf{x}} m = 0, \\ m_t^i + \left(\sum_{j=1}^3 m^i u^j \right)_{x^j} + p_{x^i} + \kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} L^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}) d\xi \\ \quad + \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} \Theta) = 0, \quad i = 1, 2, 3, \\ \left[\rho \left(\frac{|u|^2}{2} + E \right) \right]_t + \sum_{j=1}^3 \left\{ u^j \left[\rho \left(\frac{|u|^2}{2} + E \right) + p \right] \right\}_{x^j} + \kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} L^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}) d\xi \\ \quad + \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} \Theta) = 0. \end{array} \right. \quad (4.47)$$

In the above system, the terms

$$\begin{aligned} -\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} L^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}) d\xi &= -\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} L_{[\rho, u, \theta]}^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}_{[\rho, u, \theta]}) d\xi \\ &= -\kappa \int_{\mathbb{R}^3} \psi_i(\xi \cdot \nabla_{\mathbf{x}} L_{[1, u, \theta]}^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}_{[1, u, \theta]}) d\xi \end{aligned}$$

and

$$\begin{aligned} -\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} L^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}) d\xi &= -\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} L_{[\rho, u, \theta]}^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}_{[\rho, u, \theta]}) d\xi \\ &= -\kappa \int_{\mathbb{R}^3} \psi_4(\xi \cdot \nabla_{\mathbf{x}} L_{[1, u, \theta]}^{-1} \mathbf{P}_1 \xi \cdot \nabla_{\mathbf{x}} \mathbf{M}_{[1, u, \theta]}) d\xi \end{aligned}$$

are the viscosity and heat conductivity terms which are the same as those in the compressible Navier-Stokes equations; and they are independent of the density gradient $\nabla_x \rho$. In fact, with the Burnett functions defined by (4.25), and the viscosity coefficient $\mu(\theta)$ and heat conductivity coefficient $\aleph(\theta)$ defined by (4.29), the fluid-

type system (4.47) can be written as

$$\left\{ \begin{array}{l}
 \rho_t + \operatorname{div}_{\mathbf{x}} m = 0, \\
 m_t^i + \left(\sum_{j=1}^3 m^i u^j \right)_{x^j} + p_{x^i} \\
 = \sum_{j=1}^3 \left[\mu(\theta) \left(u_{x^j}^i + u_{x^i}^j - \frac{2}{3} \delta_{ij} \operatorname{div}_{\mathbf{x}} u \right) \right]_{x^j} - \int_{\mathbb{R}^3} \psi_i \left(\xi \cdot \nabla_{\mathbf{x}} \Theta \right) d\xi, \quad i = 1, 2, 3, \\
 \left[\rho \left(\frac{|u|^2}{2} + E \right) \right]_t + \sum_{j=1}^3 \left\{ u^j \left[\rho \left(\frac{|u|^2}{2} + E \right) + p \right] \right\}_{x^j} \\
 = \sum_{j=1}^3 \left[\mu(\theta) \left(u_{x^j}^i + u_{x^i}^j - \frac{2}{3} \delta_{ij} \operatorname{div}_{\mathbf{x}} u \right) \right]_{x^j} + \sum_{j=1}^3 \left(\aleph(\theta) \theta_{x^j} \right)_{x^j} - \int_{\mathbb{R}^3} \psi_4 \left(\xi \cdot \nabla_{\mathbf{x}} \Theta \right) d\xi.
 \end{array} \right. \quad (4.48)$$

From this fluid-type system, we can easily deduce the structure of the compressible Euler and the compressible Navier-Stokes equations. For example, when the Knudsen number κ is set to be zero, the system (4.48) becomes the compressible Euler equations; and when Θ is set to be zero in (4.48), it becomes the compressible Navier-Stokes equations. These fluid equations as derived through the Hilbert and Chapman-Enskog expansions are approximations to the Boltzmann equation. However, the above system is part of Boltzmann equation. Nevertheless, our approach is consistent in spirit with these expansions in that the higher order terms beyond first order in the expansions must satisfy a solvability condition, which means that these terms are microscopic. Furthermore, the above analysis also indicates that if we deduce the compressible Navier-Stokes equations from the Boltzmann equation, the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\aleph(\theta) > 0$ are smooth functions of the temperature θ .

5 Solutions in a Weighted Lebesgue Space

This section we are going to discuss the initial-value problem for the Boltzmann equation for the hard-sphere monatomic gas in the whole space \mathbb{R}^3 :

$$\begin{cases} f_t + \xi \cdot \nabla_{\mathbf{x}} f = Q(f, f), & (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \\ f(0, \mathbf{x}, \xi) = f_0(\mathbf{x}, \xi), & (\mathbf{x}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \quad (5.1)$$

where

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} (f' g'_* + g' f'_* - f g_* - g f_*) B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_*,$$

$$\mathbb{S}_+^2 = \left\{ \Omega \in \mathbb{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0 \right\},$$

and

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \Omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \Omega$$

as before.

5.1 Preliminaries

In this subsection, we give the definition of the mild solution, and present some lemmas which will be needed in the proofs of our main results.

Motivation for definition of mild solutions. We first consider the case for the collision term $Q(f, f) \equiv 0$ in (5.1) so that the Boltzmann equation in (5.1) becomes the free transport equation with initial data:

$$\begin{cases} f_t + \xi \cdot \nabla_{\mathbf{x}} f = 0, & (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, \mathbf{x}, \xi) = f_0(\mathbf{x}, \xi), & (\mathbf{x}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{cases} \quad (5.2)$$

Fix any point $(t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ and define

$$z(s) := f(t + s, \mathbf{x} + s\xi, \xi) \text{ for } s \in \mathbb{R}. \quad (5.3)$$

Then

$$\frac{d}{ds}z(s) = \nabla_{\mathbf{x}}f(t+s, \mathbf{x} + s\xi, \xi) \cdot \xi + f_t(t+s, \mathbf{x} + s\xi, \xi) = 0.$$

Hence, z is a constant function of s , and therefore for each point (t, \mathbf{x}, ξ) , f is constant on the line through (t, \mathbf{x}, ξ) with the direction $(1, \xi, 0) \in \mathbb{R}^5$. Furthermore, since $f(0, \mathbf{x} - t\xi, \xi) = f_0(\mathbf{x} - t\xi, \xi)$, we deduce

$$f(t, \mathbf{x}, \xi) = f_0(\mathbf{x} - t\xi, \xi), \quad (t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (5.4)$$

Hence, if (5.2) has a sufficiently regular solution f , it must certainly be given by (5.4). And conversely, we can check directly that if f_0 is C^1 , then f defined by (5.4) is indeed a solution of (5.2).

Now, we return to the I.V.P. (5.1). As above, fix $(t, \mathbf{x}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ and, inspired by the calculation above, set $z(s) := f(t+s, \mathbf{x} + s\xi, \xi)$ for $s \in \mathbb{R}$. Then

$$\frac{d}{ds}z(s) = \nabla_{\mathbf{x}}f(t+s, \mathbf{x} + s\xi, \xi) \cdot \xi + f_t(t+s, \mathbf{x} + s\xi, \xi) = Q(f, f)(t+s, \mathbf{x} + s\xi, \xi). \quad (5.5)$$

Thus,

$$\begin{aligned} f(t, \mathbf{x}, \xi) - f_0(\mathbf{x} - t\xi, \xi) &= z(0) - z(-t) \\ &= \int_{-t}^0 \frac{d}{ds}z(s) ds \\ &= \int_{-t}^0 Q(f, f)(t+s, \mathbf{x} + s\xi, \xi) ds \\ &= \int_0^t Q(f, f)(s, \mathbf{x} + (s-t)\xi, \xi) ds, \end{aligned}$$

which implies that

$$f(t, \mathbf{x}, \xi) = f_0(\mathbf{x} - t\xi, \xi) + \int_0^t Q(f, f)(s, \mathbf{x} + (s-t)\xi, \xi) ds. \quad (5.6)$$

The definition of mild solutions can be stated as follows:

Definition 5.1. *A nonnegative function $f \in C([0, T]; L_+^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ is a mild solution to (5.1) with a nonnegative initial data f_0 if and only if for all $t \in [0, T]$ and a.e. $(\mathbf{x}, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$, f satisfies the integral equation (5.6).*

Next, we define a weighted Banach space as follows: Let $\mathbf{M} \equiv \mathbf{M}(\xi)$ be any Maxwellian, and let $\omega := \mathbf{M}^{-\frac{1}{2}}$. Define

$$L^1_\omega(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\xi}) = \left\{ f \in L^1(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\xi}) \mid \|f\|_{L^1_\omega} < +\infty \right\}, \quad (5.7)$$

where

$$\|f\|_{L^1_\omega} := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f| \omega \, d\xi d\mathbf{x}. \quad (5.8)$$

To prove our main results, we now state the assumptions on cross section and the collision frequency as follows:

(A₁) The cross section $B(|\xi - \xi_*|, \vartheta)$ satisfies (2.15).

(A₂) $b_\gamma(\vartheta)$ is a nonnegative measurable function satisfying (3.32) for $\gamma \in [0, 1]$ and inequality (3.34) holds.

Under these assumptions, we have the following estimates for Q .

Lemma 5.1. *Under the assumptions (A₁) and (A₂), for any $p \in [1, \infty]$, $k \in [0, 1]$, and $f, g \in L^p(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\xi})$, there exists a constant $C > 0$ such that*

$$\begin{aligned} & \left\| \nu_{\mathbf{M}}^{-k} \omega (Q(f, f) - Q(g, g)) \right\|_{L^p} \\ & \leq C \left\{ \left(\|\nu_{\mathbf{M}}^{1-k} \omega f\|_{L^p} + \|\nu_{\mathbf{M}}^{1-k} \omega g\|_{L^p} \right) \|\omega(f - g)\|_{L^p} + \left(\|\omega f\|_{L^p} + \|\omega g\|_{L^p} \right) \|\nu_{\mathbf{M}}^{1-k} \omega(f - g)\|_{L^p} \right\}, \end{aligned} \quad (5.9)$$

where $\|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\xi})}$.

Proof. Write

$$Q(f, g) = \frac{1}{2} [Q^+(f, g) + Q^+(g, f) - Q^-(f, g) - Q^-(g, f)], \quad (5.10)$$

where

$$Q^+(f, g) = \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} f' g'_* B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_*, \quad (5.11)$$

and

$$Q^-(f, g) = \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} f g_* B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_*. \quad (5.12)$$

Then we can write

$$Q(f, f) - Q(g, g) = [Q^+(f, f) - Q^+(g, g)] + [Q^-(g, g) - Q^-(f, f)]. \quad (5.13)$$

First, we prove this lemma for Q^+ . To do this, put $f = \mathbf{M}^{\frac{1}{2}} h_1$ and $g = \mathbf{M}^{\frac{1}{2}} h_2$. By the conservation laws of collision (2.8) and the definition of Maxwellian (3.30), we have

$$\mathbf{M}(\xi') \mathbf{M}(\xi'_*) = \mathbf{M}(\xi) \mathbf{M}(\xi_*). \quad (5.14)$$

Hence, the Hölder inequality gives

$$\begin{aligned} & |Q^+(f, f) - Q^+(g, g)| \\ & \leq \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} |f' f'_* - g' g'_*| B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* \\ & = \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} \mathbf{M}(\xi)^{\frac{1}{2}} \mathbf{M}(\xi_*)^{\frac{1}{2}} |h_1(\xi') h_1(\xi'_*) - h_2(\xi') h_2(\xi'_*)| B(|\xi - \xi_*|, \vartheta) d\Omega d\xi_* \\ & \leq \left(\int_{\mathbb{R}^3 \times \mathbb{S}_+^2} \mathbf{M}(\xi)^{\frac{q}{2}} \mathbf{M}(\xi_*)^{\frac{q}{2}} B(|\xi - \xi_*|, \vartheta)^q d\Omega d\xi_* \right)^{1/q} \left(\int_{\mathbb{R}^3 \times \mathbb{S}_+^2} |h_1(\xi') h_1(\xi'_*) - h_2(\xi') h_2(\xi'_*)|^p d\Omega d\xi_* \right)^{1/p} \\ & \leq C \nu_{\mathbf{M}}(\xi) \mathbf{M}(\xi)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3 \times \mathbb{S}_+^2} |h_1(\xi') h_1(\xi'_*) - h_2(\xi') h_2(\xi'_*)|^p d\Omega d\xi_* \right)^{1/p} \\ & \leq C (\nu_{\mathbf{M}}(\xi') + \nu_{\mathbf{M}}(\xi'_*)) \mathbf{M}(\xi)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3 \times \mathbb{S}_+^2} |h_1(\xi') h_1(\xi'_*) - h_2(\xi') h_2(\xi'_*)|^p d\Omega d\xi_* \right)^{1/p}, \end{aligned}$$

where $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{\mathbb{R}^3 \times \mathbb{S}_+^2} \mathbf{M}(\xi_*)^{\frac{q}{2}} B(|\xi - \xi_*|, \vartheta)^q d\Omega d\xi_* \leq C (1 + |\xi|)^{\gamma q} \leq C \nu_{\mathbf{M}}(\xi)^q$$

by assumption (A_2) , and

$$\begin{aligned}
\nu_{\mathbf{M}}(\xi) &\leq C(1 + |\xi|)^\gamma \\
&= C(1 + |\xi' - [(\xi' - \xi'_*) \cdot \Omega]\Omega|)^\gamma \\
&\leq C(2 + |\xi'| + |\xi'_*|)^\gamma \\
&\leq C(\nu_{\mathbf{M}}(\xi') + \nu_{\mathbf{M}}(\xi'_*))
\end{aligned}$$

by (2.13) and (3.34). Here C is a positive constant depending on b_* , ν_0 , ν_1 , ρ , u , θ . Since the Jacobian of the change of variable $(\xi, \xi_*, \mathbf{x}, \Omega) \longleftrightarrow (\xi', \xi'_*, \mathbf{x}, \Omega)$ is unity, then by the Minkowski's inequality, we can obtain

$$\begin{aligned}
&\left\| \nu_{\mathbf{M}}^{-k} \omega(Q^+(f, f) - Q^+(g, g)) \right\|_{L^p} \\
&\leq C \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} (\nu_{\mathbf{M}}(\xi') + \nu_{\mathbf{M}}(\xi'_*))^{(1-k)p} |h_1(\xi')h_1(\xi'_*) - h_2(\xi')h_2(\xi'_*)|^p d\Omega d\xi'_* d\xi' d\mathbf{x} \right)^{1/p} \\
&\leq C \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}_+^2} \left[(\nu_{\mathbf{M}}(\xi')^{1-k} + \nu_{\mathbf{M}}(\xi'_*)^{1-k}) (|h_1(\xi')||h_1(\xi'_*) - h_2(\xi'_*)| + |h_1(\xi') - h_2(\xi')||h_2(\xi'_*)|) \right]^p d\Omega d\xi'_* d\xi' d\mathbf{x} \right\}^{1/p} \\
&\leq C \left\{ \left(\|\nu_{\mathbf{M}}^{1-k} \omega f\|_{L^p} + \|\nu_{\mathbf{M}}^{1-k} \omega g\|_{L^p} \right) \|\omega(f-g)\|_{L^p} + \left(\|\omega f\|_{L^p} + \|\omega g\|_{L^p} \right) \|\nu_{\mathbf{M}}^{1-k} \omega(f-g)\|_{L^p} \right\}.
\end{aligned}$$

This proves (5.9) for Q^+ for the case $p \in [1, \infty)$. The case $p = \infty$ can be proved similarly, and the proof for Q^- is also similar. Now, the proof of this lemma is complete. \square

Lemma 5.2. *Under the assumptions (A_1) and (A_2) , for any $f, g \in L_\omega^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, there exists a constant $C > 0$ such that*

$$\|Q(f, f) - Q(g, g)\|_{L_\omega^1} \leq C \|\nu_{\mathbf{M}}\|_{L^\infty} (\|f\|_{L_\omega^1} + \|g\|_{L_\omega^1}) \|f - g\|_{L_\omega^1}. \quad (5.15)$$

Proof. Setting $k = 0$ and $p = 1$ in (5.9), and using the Hölder inequality, we obtain

$$\begin{aligned} & \|Q(f, f) - Q(g, g)\|_{L^1_\omega} \\ & \leq C \left\{ \left(\|\nu_{\mathbf{M}}\omega f\|_{L^1} + \|\nu_{\mathbf{M}}\omega g\|_{L^1} \right) \|f - g\|_{L^1_\omega} + \left(\|f\|_{L^1_\omega} + \|g\|_{L^1_\omega} \right) \|\nu_{\mathbf{M}}\omega(f - g)\|_{L^1} \right\} \\ & \leq 2C \|\nu_{\mathbf{M}}\|_{L^\infty} (\|f\|_{L^1_\omega} + \|g\|_{L^1_\omega}) \|f - g\|_{L^1_\omega}, \end{aligned}$$

as desired. □

Furthermore, we recall that a mapping $F : \mathbf{X} \rightarrow \mathbf{X}$, where \mathbf{X} is a Banach space with norm $\|\cdot\|_{\mathbf{X}}$, is said to be a contraction of \mathbf{X} if there is a positive constant $\alpha < 1$ such that

$$\|F(\mathbf{u}) - F(\mathbf{v})\|_{\mathbf{X}} \leq \alpha \|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}. \quad (5.16)$$

Lemma 5.3 (Banach's Fixed Point Theorem). *A contraction F of a Banach space \mathbf{X} has a unique fixed point.*

Finally, in order to derive our main result for stability of mild solutions, we also need a fundamental inequality which is the well-known Gronwall's inequality :

Lemma 5.4 (Gronwall's Inequality). *Let $\mathbf{u} : [a, b] \rightarrow [0, \infty)$ and $\mathbf{v} : [a, b] \rightarrow \mathbb{R}$ be two continuous functions, and let C be a constant. If*

$$\mathbf{v}(t) \leq C + \int_a^t \mathbf{v}(s)\mathbf{u}(s) ds \quad (5.17)$$

for $t \in [a, b]$, then we have

$$\mathbf{v}(t) \leq C \exp \left(\int_a^t \mathbf{u}(s) ds \right) \quad (5.18)$$

for $t \in [a, b]$.

5.2 Existence of the mild solution

Now, we give the crucial estimate for the local existence of the mild solution to (5.1) by the Banach's fixed point theorem. To do this, given $T > 0$, let us denote the closed subset \mathbf{X}_0 of $L^1_\omega(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_\xi)$ by

$$\mathbf{X}_0 = \left\{ f \in \mathbf{X} \mid \|f\|_{L^1_\omega} \leq 2M_0 \text{ for some } M_0 > 0 \right\}, \quad (5.19)$$

and

$$C([0, T]; X_0) = \left\{ f(t, \mathbf{x}, \xi) \mid \|f\| < \infty \right\}, \quad (5.20)$$

where

$$\|f\| := \max_{t \in [0, T]} \|f\|_{L^1_\omega(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_\xi)}. \quad (5.21)$$

Theorem 5.1. *Under the assumptions (A_1) and (A_2) , if $f_0 \geq 0$, $f_0 \in L^1_\omega(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_\xi) \cap C(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_\xi)$ and $\|f_0\|_{L^1_\omega} \leq M_0$, then there exists a unique mild solution $f \in C([0, T]; X_0)$ to the initial-value problem (5.1).*

Proof. Let $\mathbf{X} := C([0, T]; X_0)$. We will apply Banach's fixed point theorem in the space \mathbf{X} . From the integral equation (5.6), we may define a mapping F on \mathbf{X} by

$$F(f) = f_0(\mathbf{x} - t\xi, \xi) + \int_0^t Q(f, f)(s, \mathbf{x} + (s-t)\xi, \xi) ds, \quad t \in [0, T]. \quad (5.22)$$

Assume $\|f_0\|_{L^1_\omega} \leq M_0$, then by lemma 5.2, we see that $F(f) \in \mathbf{X}$. Furthermore, for $f, g \in \mathbf{X}$, we have

$$F(f) - F(g) = \int_0^t (Q(f, f) - Q(g, g))(s, \mathbf{x} + (s-t)\xi, \xi) ds, \quad t \in [0, T]. \quad (5.23)$$

Apply lemma 5.2 again, we obtain

$$\begin{aligned} \|F(f) - F(g)\|_{L^1_\omega} &\leq \int_0^t \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |(Q(f, f) - Q(g, g))(s, \mathbf{x} + (s-t)\xi, \xi)| \omega d\xi d\mathbf{x} \right) ds \\ &\leq 4M_0 CT \|\nu_{\mathbf{M}}\|_{L^\infty} \|f - g\| \\ &= \alpha \|f - g\|, \end{aligned}$$

where $\alpha := 4M_0CT\|\nu_M\|_{L^\infty}$. Maximizing the left hand side over all $t \in [0, T]$, we get

$$\|F(f) - F(g)\| \leq \alpha \|f - g\|. \quad (5.24)$$

Thus F is a contraction, provided $T > 0$ is so small that $\alpha < 1$, and so F has a unique fixed point f in \mathbf{X} by Banach's fixed point theorem. Therefore, (5.1) has a unique mild solution f in \mathbf{X} . □

5.3 Uniform stability of solutions

In this subsection, we deal with the uniform L_ω^1 -type stability estimate.

Definition 5.2. *Suppose that the Banach space \mathbf{X} with norm $\|\cdot\|_{\mathbf{X}}$ is a solution space where the Cauchy problem is well-posed, and that $f(t), g(t) \in \mathbf{X}$, $t \geq 0$, are two solutions to the Cauchy problem with initial data $f_0, g_0 \in \mathbf{X}$, respectively. We say that the solution to the Cauchy problem is uniformly stable in time if there exists a constant C , which is independent of t , such that*

$$\|(f - g)(t)\|_{\mathbf{X}} \leq C\|f_0 - g_0\|_{\mathbf{X}}, \quad \forall t \geq 0. \quad (5.25)$$

Theorem 5.2. *Under the assumptions (A_1) and (A_2) , if $f(t, \mathbf{x}, \xi)$ and $g(t, \mathbf{x}, \xi)$ are mild solutions to (5.1) corresponding to initial data $f_0(\mathbf{x}, \xi)$ and $g_0(\mathbf{x}, \xi)$ in the closed subset X_0 of $L_\omega^1(\mathbb{R}_\mathbf{x}^3 \times \mathbb{R}_\xi^3)$, respectively, then, for any $t \geq 0$, there exists a constant $C > 0$ such that*

$$\|(f - g)(t)\|_{L_\omega^1} \leq C\|f_0 - g_0\|_{L_\omega^1}. \quad (5.26)$$

Proof. If f and g are mild solutions to (5.1), then we can write

$$f(t, \mathbf{x}, \xi) = f_0(\mathbf{x} - t\xi, \xi) + \int_0^t Q(f, f)(s, \mathbf{x} + (s - t)\xi, \xi) ds$$

and

$$g(t, \mathbf{x}, \xi) = g_0(\mathbf{x} - t\xi, \xi) + \int_0^t Q(g, g)(s, \mathbf{x} + (s - t)\xi, \xi) ds$$

for $t \geq 0$. Taking difference of the above two equations, we obtain

$$(f - g)(t, \mathbf{x}, \xi) = (f_0 - g_0)(\mathbf{x} - t\xi, \xi) + \int_0^t (Q(f, f) - Q(g, g))(s, \mathbf{x} + (s - t)\xi, \xi) ds.$$

Then by lemma 5.2, we have

$$\|(f - g)(t)\|_{L_\omega^1} \leq \|f_0 - g_0\|_{L_\omega^1} + \int_0^t \mathbf{u} \|f - g\|_{L_\omega^1}(s) ds,$$

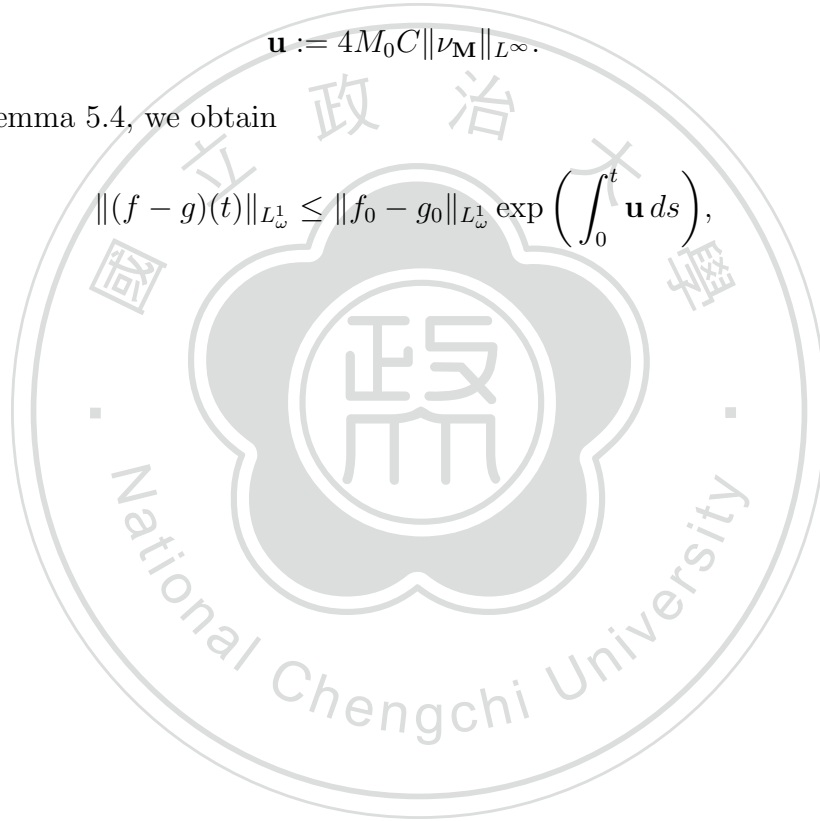
where

$$\mathbf{u} := 4M_0C\|\nu_M\|_{L^\infty}.$$

Hence, by lemma 5.4, we obtain

$$\|(f - g)(t)\|_{L_\omega^1} \leq \|f_0 - g_0\|_{L_\omega^1} \exp\left(\int_0^t \mathbf{u} ds\right),$$

as desired. □



References

- [1] G.A. Bird. *Molecular Gas Dynamics and the Direct simulation of Gas Flows*. Oxford, 1994.
- [2] Mario Pulvirenti Carlo Cercignani, Reinhard Illner. *The Mathematical Theory of Dilute Gases*. Springer, 1994.
- [3] Carlo Cercignani. *The Boltzmann Equation and Its Applications*. Springer, 1988.
- [4] D.Hilbert. Begründung der kinetischen gastheorie. *Mathematische Annalen* 72, pages 562–577, 1916/1917.
- [5] R.J. DiPerna and P.L. Lions. On the Cauchy problem for Boltzmann equation: Global existence and weak stability. *Ann. of Math.* 130, 1989. pp. 321 – 366.
- [6] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.* 231, 2002. pp. 391 – 434.
- [7] Y. Guo. The Vlasov -Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math.* 55 (9), 2002. pp. 1104 – 1135.
- [8] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.* 169, 2003. pp. 305 – 353.
- [9] Y. Guo. The Vlasov - Maxwell - Boltzmann system near Maxwellians. , *Invent. Math.* 153, 2003. pp. 593 – 630.
- [10] Y. Guo. The Boltzmann equation in the whole space. *Indiana Univ. Math. J.* 53, 2004. pp. 1081 – 1094.
- [11] H.Grad. Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations. *Proc.Symp.Appl.Math.* 17, pages p.154–183, 1965.

- [12] T.-P. Liu and S.-H. Yu. Boltzmann equation: Micro -Macro decompositions and positivity of shock profiles. *Comm. Math. Phys.* 246 (1), 2004. pp. 133 – 179.
- [13] T.-P. Liu and S.-H. Yu. Green’s function of Boltzmann equation, 3-D waves. *Bull. Inst. Math. Acad. Sin. (N.S.)* 1, 2006. pp. 1 – 78.
- [14] T. Nishida and K. Imai. Global solutions to the initial value problem for the nonlinear Boltzmann equation. *Publ. Res. Inst. Math. Sci.* 12, 1976/77. pp. 229 – 239.
- [15] Y. Shizuta. On the classical solutions of the Boltzmann equation. *Comm. Pure Appl. Math.* 36, 1983. pp. 705 – 754.
- [16] T. Yang T.-P. Liu and S.-H. Yu. Energy method for the Boltzmann equation. *Phys. D* 188 (3 – 4), 2004. pp. 178 – 192.
- [17] Ludwig Boltzmann ; translated by Stephen G. Brush. *Lectures On Gas Theory*. Dover Publications, 1995.
- [18] T.S.Carleman. Sur la théorie de l’équation intégro-differentielle de Boltzmann. *Acta.Math*,60, pages p.91–146, 1932.
- [19] S. Ukai. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.* 50, 1974. pp. 179 – 184.
- [20] S. Ukai and T. Yang. The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: Global and time-periodic solutions. *Anal. Appl.* 4, 2006. pp. 263 – 310.
- [21] S. Ukai and T. Yang. *Mathematical Theory of Boltzmann Equation*. Liu Bie Ju Centre for Mathematical Sciences, Lecture Notes Series, No. 8, 2006.
- [22] T. Yang and H.-J. Zhao. A new energy method for the Boltzmann equation. *J. Math. Phys.* 47, 2006. p. 053301.

- [23] Shih-Hsien Yu. Hydrodynamic limits with shock waves of the Boltzmann equation. *Commun. Pur. Appl. Math*, 58, pages 409–443, 2005.

