Intertemporal Futures Pricing with Different Opinions about Price Changes

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Abstract

As Harrison and Kreps (1978) noted, unless traders are all identical and obliged to hold a stock forever, speculation would not extinguish in market, and different opinions on future evolution of asset price yield whereby. This paper derives intertemporal futures pricing formulas accounting for such presumption in a partial-equilibrium sense with stochastic interest rate and changing opinions. The closed-form solutions show that some indeterminate empirics such as converging patterns of Contango and normal backwardation result from complicated relationships among interacted dynamics of interest, spot price, and changing opinions simultaneously.

Keywords：Intertemporal futures pricing, different opinions, stochastic interest rate.

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I Introduction

Originated from simple no-arbitrage argument and presumptions of perfect and frictionless markets, the cost of carry model states that futures price $f(t)$ at time t with expiration date T should be:

$$f(t) = S(t)\, e^{(c-y)(T-t)} \tag{1}$$

where $S(t)$ denotes the price of some underlying asset, c is the carry cost and y is the convenience yield.[1] When futures market and cash market maintain an effective linkage by arbitragers, futures contract and spot price are viewed as perfect substitutes, and rational arbitragers sustain negligible mispricing series fluctuate randomly around zero theoretically. Nevertheless, much literature does not conclude the consistent empirical phenomenon from the cost of carry model. Some studies attribute this discrepancy to miscellaneous market frictions or asymmetric limitations. Others identify effects of time-varying market variables such as stochastic interest rate and dividend yield.

For example, Cornell and French (1983a, 1983b) argue that because futures traders lose tax timing options, the prices of stock index futures contracts may be less than that predicted by the cost of carry model.[2] Modest and Sunderesan (1983) and Klemkosky and Lee (1991) derive no-arbitrage bands allowing for asymmetric transaction costs of long and short positions in spot and futures markets. Gibson and Schwartz (1990) empirically test validity of a two-factor

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[1] For investment assets such as stock index, the carry cost is that riskless rate minus dividend yield, and the convenience yield is negligible. (Hull (1997))

[2] However, Cornell's (1985) later empirical test reveals that the timing option is not an important factor in pricing stock index futures.

model for pricing financial and real assets contingent on oil price. They assert that stochastic convenience yield is an important futures pricing factor. In contrast, Bhatt and Cakici (1990) find significant positive relationship between stock index dividend and mispricing from the cost of carry model. However, Yadav and Pope (1994) find that a 20% variation in dividend leads only to 0.1% mispricing, suggesting that such risk is not important for S&P 500 index futures.

On the other hand, it is well known that stochastic interest rate differentiates futures and forward prices. This issue is thoroughly discussed by Cox, Ingersoll, and Ross (1981), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981). Cakici and Chatterjee (1991) use S&P 500 index futures data and make empirical comparisons amongst constant and stochastic interest rate models including Cox, Ingersoll, and Ross' (1981) square root model and Vasick's (1977) mean-reverting one. They find that the stochastic interest rate models yield better pricing results particularly when spot rate is far away from its long-term mean and that this superiority is not sensitive to specifications of the interest rate model. In addition, Richard and Sundaresan (1981), Cox, Ingersoll, and Ross (1981), and Hemler and Longstaff (1991) derive general-equilibrium futures pricing formulas with different combinations of various stochastic factors including volatility, interest rate, and spot price or real output. Ramaswamy and Sundaresan (1985), Schwartz (1997), and Hilliard and Reis (1998) simultaneously consider different combinations of stochastic properties of spot price, interest rate and convenience yield to develop closed-form solutions in a partial-equilibrium sense. In particular, Hilliard and Reis (1998) consider jump effect in their model additionally.

Except the frictions and the stochastic factors mentioned above affect futures pricing, some market phenomena should be taken into consideration as well, different opinions, say. For instance, Harrison and Kreps (1978) assert that security prices are aggregation results of diverse investor assessments on information. Speculation, market re-opening, and actively portfolio management are needed by investors as results. Figlewski (1978) proves that different expectations on asset price among investor deteriorate the efficient market hypothesis proposed by Fama. Harris and Raviv (1993) assume that traders interpret common information in different ways, and each of them believes absolutely in the validity of his or her interpretation. Their model helps to explain some empirical regularities including positive correlation between absolute price changes and volume, negative serial correlation of consecutive price changes, and positive autocorrelation of volume series.

Empirical evidence supports the hypothesis of different opinions as well. For example, Frankel and Froot (1990) find that standard macroeconomic models can not explain dollar path, especially from June, 1984 to February, 1985. In particular, the unexpected deviations are so large to be explained by rational revision such as taste or technology change. Ederington and Lee (1995) find that volatility remains higher after news releases than normal times in interest rate and foreign exchange futures markets. And such volatility effect is irrelevant with initial price change. It means that disagree among participants exists even in filtering common macroeconomic news. Frechette and Weaver (2001) reject the representative agent hypothesis in U.S. soybean futures market at the 99% level of confidence. They comment that although the homogeneity assumption has been maintained in the past to ensure model tractability, it is incompatible with what we know to be true about markets.

More specifically, dynamics of market variables for arbitrage activities may differ among traders because they hold different opinions. Thus, this study considers divergent opinions on underlying asset price as a given premise within a partial-equilibrium framework and derives intertemporal futures pricing formulas with stochastic interest rate.[3] Furthermore, because traders may change their original perspectives to some extent or to a new landmark as time goes by, possible changing paths between different opinions are also discussed.

There are at least two similar papers on the same topic. Fremault (1991) analyzes arbitraging between stock index and futures markets in a rational expectations economy with three types of traders different on trading motives.

[3] General-equilibrium futures pricing models such as Richard and Sundaresan (1981), Cox, Ingersoll, and Ross (1981), and Hemler and Longstaff (1991) offer another complete analytical framework allowing for interaction among different economic departments simultaneously.

They are two types of one-market traders who activate only in spot market or futures market separately and index arbitragers who arbitrage between spot and futures markets. What Fremault's two-period model emphasizes is impacts of index arbitrage on price, volatility, and welfare. Linn and Stanhouse (2003) examine economic advantages of learning in a futures market. They develop a discrete dynamic model consisting of producers, consumers, and speculators differentiated by different predictive equations when formulating forecasts of next period's spot price. Their model shows that learned traders enjoy an economic advantage in futures market. In contrast to our model, both studies are rational expectation equilibrium models without intertemporal setting and derivation of futures pricing formula explicitly.

The paper is structured as follows. We first specify our preliminary conditions and develop partial differential equations with different combinations of important factors in Section Ⅱ. Then we derive closed-form solutions and discuss comparative static results amongst important variables in Section Ⅲ. At last, Section IV concludes our study.

II Model Specifications

1. Different opinions

Figlewski (1978) addressed that many interesting rational expectation models are based on separating traders of several types according to some criteria such as different informativeness, degrees of risk aversion, predictive ability, wealth, etc. And many heterogeneity settings appear in later studies help to articulate academic reasoning with practical phenomena.[4] In particular,

[4] For example, Constantinides (1982) discusses importance about heterogeneous wealth distributions in complete market. Models proposed by Kyle (1985), Holden and Subrahmanyam (1992), Foster and

continuous-time models assuming only one stochastic process or single belief of underlying asset for every trader may not adequately capture such characteristics. (Chen and Epstein (2002)) Here we take heterogeneity as different opinions on future evolution of underlying asset price. That is, traders are alike with the same perspectives about price dynamics in the same group, but with divergent viewpoints among different groups.

It is common to use different price dynamics to proxy traders' different opinions in literature.[5] Second, no matter what specific issues or designs involved, equilibrium market price of many models is proved to be of a linear-combination function of heterogeneous types.[6] Third, other theoretical models departing from classical volume-price of demand-supply setting also assert that equilibrium price has the similar linear characteristic.[7] Thus, we assume that all traders can be divided into two groups and combined by some value-weighted proportion function $\xi(t)$. And each trader views himself as trading in a perfectly competitive market to exclude possibility of information exploitation from different group of traders. The assumption of just two opinions is made for expositional convenience. The model developed below can be extended to a multi-type case.

Let $(\mu_1, \delta_1, \sigma_1)$ and $(\mu_2, \delta_2, \sigma_2)$ denote different opinions made by two types of traders, where $\mu_\bullet$ are drift terms, $\delta_\bullet$ are instantaneous dividend yields, and $\sigma_\bullet$ are diffusion terms of the two processes. Note that we allow a co-varying random part of the two different expectations by a correlation coefficient ρ_{12} in addition to the adjusting dynamics $\xi(t)$. As for dz_1 and dz_2, they are two mutually independent Wiener processes. That is,

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Viswanathan (1994, 1996) emphasize the heterogeneous informational contents with sequential auctions to resemble a rational expectation equilibrium. Harris and Raviv (1993) allege that market participants have different opinions on price perspective. Chen and Epstein (2002) develop equilibrium results with multiple-priors utility.

[5] Please refer to Harrison and Kreps (1978), Frankel and Froot (1990), Harris and Raviv (1993), Ederington and Lee (1995), Ahn, Boudoukh, Richardson, and Whitelaw (2002), Frechette and Weaver (2001), etc.

[6] Please refer to Kyle (1985), Holden and Subrahmanyam (1992), and Foster and Viswanathan (1996).

[7] Please refer to Figlewski (1978) and Kogan, Ross, Wang, and Westerfield (2004).

$$\frac{dS_1}{S} = \left(\mu_1 - \delta_1\right) dt + \sigma_1 \, dz_1$$
$$\frac{dS_2}{S} = \left(\mu_2 - \delta_2\right) \, dt + \sigma_2 \begin{bmatrix} \rho_{12} & \sqrt{1-\rho_{12}^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix} \tag{2}$$

After value-weighted summation, the resulting dynamics of underlying asset price with different opinions can be written as:

$$\frac{dS}{S} = \left[\xi(t)\left(\mu_1 - \delta_1\right) + \left(1 - \xi(t)\right)\left(\mu_2 - \delta_2\right)\right] dt$$
$$+ \left[\left(\xi(t)\,\sigma_1 + \left(1-\xi(t)\right)\sigma_2 \, \rho_{12}\right) dz_1 + \left(1-\xi(t)\right)\sigma_2 \sqrt{1-\rho_{12}^2} \, dz_2\right] \tag{3}$$

2. Changing opinions

Other than holding different opinions, market participants adjust or update their opinions recursively. This is highlighted in rational-expectation-economy literature. In addition, Brennan, Jegadeesh, and Swaminathan (1993) find that the number of investment analysts following a firm can be taken as a proxy of adjusting speed. Lo and Mackinlay (1990) relate firm size to adjustment speed in their empirical work. Merton (1987) asserts that large firm will attract more individual interesting. Hemler and Longstaff (1991) comment that new information tends to affect traders with identical position in the same direction as time to maturity is decreasing, thus reducing basis risk. MacKinlay and Ramaswamy (1988) and Yadav and Pope (1994) find that basis is path-dependent and decreasing with shorter time to maturity. These studies all address the importance of adjustment effect.

Here we use a deterministic time-varying coefficient $\xi\left(t\right)$ to proxy the opinions changing path. It adjusts traders' original perspectives on drift and diffusion terms during some time span Δt, i.e., $\xi\left(t\right)\mu_{\square} \rightarrow \xi\left(t+\Delta t\right)\mu_{\square}$ and $\xi\left(t\right)\sigma_{\square} \rightarrow \xi\left(t+\Delta t\right)\sigma_{\square}$. And the solution of stochastic differential equation of underlying asset price now can be stated as:

$$
\begin{aligned}
S_t = S_0 \exp\Bigg\{ & \int_0^t \Big[\ \Big\{ \xi(t)\big[(\mu_1-\mu_2)-(\delta_1-\delta_2)\big]+(\mu_2-\delta_2) \Big\} \\
& -\frac{1}{2}\Big[\big[\xi(t)\,\sigma_1+(1-\xi(t))\,\sigma_2\,\rho_{12}\big]^2+\Big[(1-\xi(t))\,\sigma_2\sqrt{1-\rho_{12}^2}\Big]^2\Big]\ \Big]dt \\
& +\int_0^t \big[\xi(t)\,\sigma_1+(1-\xi(t))\,\sigma_2\,\rho_{12}\big]\,dz_1 \\
& +\int_0^t \Big[(1-\xi(t))\,\sigma_2\sqrt{1-\rho_{12}^2}\Big]\,dz_2 \ \Bigg\}
\end{aligned} \tag{4}
$$

3. Stochastic interest rate

Stochastic interest rate plays critical role on differentiating prices of forward and futures contracts.[8] Many famous futures pricing models use different stochastic models to describe interest rate dynamics. For instance, Cox, Ingersoll, and Ross (1981), Ramaswamy and Sundaresan (1985), and Hemler and Longstaff (1991) use square root process to describe interest rate dynamics. Schwartz (1997) uses Vasicek's (1977) interest rate model. And Hilliard and Reis (1998) uses HJM model. Because Cakici and Chatterjee (1991) find that the stochastic property of interest rate matters for pricing futures contracts while the exact specification is insensitive to the empirical results, we use Vasicek's (1977) Ornstein-Uhlenbeck stochastic process for modeling tractability. Here we assume that the instantaneous risk free rate follows Vasicek's (1977) Ornstein-Uhlenbeck stochastic process. That is,

$$dr = \alpha(m-r)\,dt + \sigma_r\,dz_r \tag{5}$$

where α represents the coefficient of adjusting speed of interest rate reverting to a long-run mean m, and σ_r is the diffusion term of interest rate dynamics. We take them as constant parameters for simplicity. Under the assumption that there exist enough traded assets to eliminate various risks, we can obtain a

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[8] See Richard and Sundaresan (1981), Jarrow and Oldfield (1981), Levy (1989), and Flesaker (1991) etc.

fundamental partial differential equation:[9]

$$
\begin{aligned}
& f_S\ S\left[r-\xi(t)\left(\delta_1-\delta_2\right)-\delta_2\right]+f_r\left[\alpha(m-r)-\lambda_r\ \sigma_r\right]+\frac{1}{2} f_{rr}\ \sigma_r^2 \\
& +\frac{1}{2} f_{SS}\ S^2\left\{\left[\xi(t)\,\sigma_1+(1-\xi(t))\,\sigma_2\ \rho_{12}\right]^2\right. \\
& \qquad\qquad \left.+\left[(1-\xi(t))\,\sigma_2\sqrt{1-\rho_{12}^2}\right]^2\right\} \qquad (6) \\
& +f_{Sr}\ S\,\sigma_r\left\{\xi(t)\ \sigma_1\ \rho_{1r}+\left[1-\xi(t)\right]\sigma_2\ \rho_{1r}\ \rho_{12}\right. \\
& \qquad\qquad \left.+\left[1-\xi(t)\right]\sigma_2\ \rho_{2r}\sqrt{1-\rho_{12}^2}\right\} \\
& =f_\tau
\end{aligned}
$$

where $\tau=T-t$, ρ_{1r} and ρ_{2r} are constant correlation coefficients of different opinions and interest rate, and λ_r denotes an unit market risk price of stochastic interest rate. After the construction of partial differential equation, we need boundary condition:

$$f(T)=S(T) \qquad (7)$$

Nevertheless, the closed-form solution of this partial differential equation can not be obtained unless that the explicit functional form of $\xi(t)$ is prescribed. We next discuss several possible specifications.

III Closed-form Solutions and Comparative Statics

1. When interest rate is constant and opinions are not time-varying

If $\xi(t)$ is constant, then the cost of carry model is the case. We can derive

[9] Please refer to Appendix A for the proof.

the corresponding partial different equation from Equation (6):

$$f_S\ S\left\{\ r-\left[\xi\left(\delta_1-\delta_2\right)+\delta_2\right]\right\}+\frac{1}{2}f_{SS}\ S^2\left[\xi\left(\sigma_1-\sigma_2\ \rho\right)+\sigma_2\ \rho\right]^2 \tag{8}$$

$$+\frac{1}{2}f_{SS}\ S^2\left[\left(1-\xi\right)\sigma_2\ \sqrt{1-\rho^2}\right]^2=f_\tau$$

With constant interest and fixed holding period $T-t$, the futures pricing formula becomes:[10]

$$f\left(t\right)=S\left(t\right)e^{\left[r-\xi\left(\delta_1-\delta_2\right)-\delta_2\right]\left(T-t\right)} \tag{9}$$

Equation (9) provides a concise illustration about heterogeneous effects. In particular, different perspectives of traders affect futures pricing through ξ and the difference in estimated dividend yields but not the drift and diffusion terms. Furthermore, if we define $\Delta_\delta=\left|\delta_1-\delta_2\right|$ as a proxy of "dividend heterogeneity", then the results of partial differentiation is as follows:

$$\begin{cases} \dfrac{1}{f}\dfrac{\partial f}{\partial\Delta_\delta}=-\xi\left(T-t\right)<0 & for\quad \delta_1\geq\delta_2 \\ \dfrac{1}{f}\dfrac{\partial f}{\partial\Delta_\delta}=-\left(1-\xi\right)\left(T-t\right)<0 & for\quad \delta_1<\delta_2 \end{cases} \tag{10}$$

$$\frac{1}{f}\frac{\partial f}{\partial\xi}=-\left(\delta_1-\delta_2\right)\left(T-t\right) \tag{11}$$

It is obvious that a larger Δ_δ reduces the futures price because of the negative differentiation results. In addition, if $\delta_1=\delta_2=\delta$, then the formula degenerates to the cost of carry model. ξ is irrelevant because that expectations between the two types of traders are homogeneous.[11]

2. When interest rate is stochastic and opinions are not time-varying

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[10] The Feynman-Kac theorem can be used to solve the partial differential equations. Please refer to Duffie (1996) for expositions of the theorem and its associated regularity conditions.

[11] We thank the anonymous referee for this correction.

Stochastic interest rate differentiates futures and forward pricing through the marking-to-market mechanism. In contrast to Equation (6), the partial differential equation allowing for stochastic interest rate now becomes:

$$
\begin{aligned}
& f_S\, S \left\{ r - \left[\xi \left(\delta_1 - \delta_2 \right) + \delta_2 \right] \right\} \\
& + f_r \left[\alpha (m - r) - \lambda_r\, \sigma_r \right] + \frac{1}{2} f_{SS}\, S^2 \left[\xi\, \sigma_1 + (1-\xi)\, \sigma_2\, \rho_{12} \right]^2 \\
& + \frac{1}{2} f_{SS}\, S^2 \left[(1-\xi)\, \sigma_2 \sqrt{1-\rho_{12}^2} \right]^2 + \frac{1}{2} f_{rr}\, \sigma_r^2 + f_{Sr}\, \xi\, S\, \sigma_1\, \sigma_r\, \rho_{1r} \\
& + f_{Sr} (1-\xi)\, S\, \sigma_2\, \sigma_r\, \rho_{1r}\, \rho_{12} + f_{Sr} (1-\xi)\, S\, \sigma_2\, \sigma_r\, \rho_{2r} \sqrt{1-\rho_{12}^2} \\
& = f_\tau
\end{aligned}
\tag{12}
$$

And the corresponding closed-form pricing formula is

$$
\begin{aligned}
f(t) = S(t) \exp \Big\{ & \left[m^* - \left[\xi \left(\delta_1 - \delta_2 \right) + \delta_2 \right] \right] (T - t) \\
& + L_1(t) + L_2(t) + L_3(t) \Big\}
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
L_1(t) = \Bigg\{ & \frac{\sigma_r^2}{2\,\alpha^2} \\
& + \frac{\sigma_r \left[\xi\, \sigma_1\, \rho_{1r} + (1-\xi)\, \sigma_2\, \rho_{1r}\, \rho_{12} + (1-\xi)\, \sigma_2\, \rho_{2r} \sqrt{1-\rho_{12}^2} \right]}{\alpha} \Bigg\} (T - t)
\end{aligned}
$$

$$
\begin{aligned}
L_2(t) = \Bigg\{ & r_t - m^* - \frac{\sigma_r^2}{2\alpha^2} \\
& - \frac{\sigma_r \left[\xi\, \sigma_1\, \rho_{1r} + (1-\xi)\, \sigma_2\, \rho_{1r}\, \rho_{12} + (1-\xi)\, \sigma_2\, \rho_{2r} \sqrt{1-\rho_{12}^2} \right]}{\alpha} \Bigg\} H
\end{aligned}
$$

$$
L_3(t) = -\frac{\sigma_r^2}{4\,\alpha} H^2
$$

$$H(t)=\frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$m^{*}=m-\frac{\lambda_{r}\ \sigma_{r}}{\alpha}$$

This solution can be verified easily by substituting back to Equation (12). Note that the constant proportion ξ, the correlation coefficients $\rho_{\xi r}$ between different spot price perspectives and interest rate, the co-varying expectations between two groups of traders represented by ρ_{12}, market risk price λ_r of stochastic interest rate, the parameters of interest rate process, and the heterogeneous cash volatility components σ_1 and σ_2 all take effects in this formula, except drift terms μ_1 and μ_2 in spot price dynamics. This is because that profits-or-losses from the marking-to-market mechanism must be replicated to form a no-arbitrage portfolio and invoke asset pricing, fluctuating interest rate brings about stochastic pending positions therefrom. Consequently, not only do the parameters of interest rate dynamics come into the closed-form solution, but also the parameters of spot price dynamics appear in Equation (13) owning to the correlation coefficient $\rho_{\xi r}$. If interest rate is constant, then the pricing of a futures contract is equivalent to a forward contract, and the volatilities of spot returns will not affect the cost of carry model anymore. This is what Equation (9) has shown.

On the other hand, we can observe effects of various important parameters on futures price by partially differentiation, that is, [12]

$$\frac{1}{f}\frac{\partial f}{\partial\sigma_{S}}=\frac{\sigma_{r}\ \rho_{rS}}{\alpha}(\tau+H) \tag{14}$$

$$\frac{1}{f}\frac{\partial f}{\partial\rho_{rS}}=\frac{1}{\alpha}(H+\tau)\,\sigma_{r}\ \sigma_{S} \tag{15}$$

---

[12] Note that the mean-reverting coefficient α of interest rate process is greater than 0 and less than 1 definitely and $0<H<1-e^{-1}$ consequently.

$$\frac{1}{f}\frac{\partial f}{\partial m} = \tau - H \tag{16}$$

$$\frac{1}{f}\frac{\partial f}{\partial \sigma_r} = \frac{1}{2\alpha}\left[\left(2-2H+H^2\right)\sigma_r + 2\left(1+H\right)\rho_{Sr}\ \sigma_S\right] \tag{17}$$

$$\frac{1}{f}\frac{\partial f}{\partial \alpha} = \frac{1}{4\alpha^2}\Big\{ \left[H\left(4+3H\right) - 2\left(2+H+H\left(1+H\right)\alpha\right)\tau\right]\sigma_r^2 \\ -4H\left(2-\alpha\tau\right)\rho_{Sr}\ \sigma_r\ \sigma_S + 4\alpha\left(\tau - H\left(1-\alpha\tau\right)\right)\left(r-m^*\right) \Big\} \tag{18}$$

$$\frac{1}{f}\frac{\partial f}{\partial \xi} = \Delta_\delta\ \tau + \frac{1}{\alpha}\left(H+\tau\right) \\ \left[Cov\left(dr, \frac{dS_1}{S_1}\right)\times\left(1-\rho_{12}\frac{\sigma_2}{\sigma_1}\right) - Cov\left(dr, \frac{dS_2}{S_2}\right)\times\left(\sqrt{1-\rho_{12}^2}\right)\right] \tag{19}$$

$$\frac{1}{f}\frac{\partial f}{\partial \tau} = \frac{1}{2\alpha}H\left[-1+\alpha\left(1+H\right)\right]\sigma_r^2 \\ +\frac{1}{\alpha}\left[\rho_{Sr}\ \sigma_r\ \sigma_S\left(2-\alpha H\right)\right] - \left[\Delta_\delta + r + \alpha H\left(r-m^*\right)\right] \tag{20}$$

where

$$\rho_{Sr}\ \sigma_S = \xi\ \sigma_1\ \rho_{1r} + \left(1-\xi\right)\sigma_2\ \rho_{1r}\ \rho_{12} + \left(1-\xi\right)\sigma_2\ \rho_{2r}\sqrt{1-\rho_{12}^2}$$

Equations (14) ~ (20) present the comparative static results between futures price and important variables. For examples, cash market volatility (σ_S) is determined completely by the correlation between spot price and interest rate (ρ_{Sr}). Thus, the negative ρ_{Sr} implies that increased spot price volatility will reduce futures price and basis (here the basis is defined as $f-S$). In addition, the relationship between futures price and ρ_{Sr} is positive in unanimity. However, the relationship between futures price and the characteristics of stochastic interest rate (m, σ_r, α) are complicated with other parameters. Such phenomena reflect complexities in reality that market participants can not forecast futures price evolution without simultaneous considerations of other important factors. Likewise, effect of adjustment trending is dependent on the heterogeneities of dividends (Δ_δ) and volatilities ($\frac{\sigma_2}{\sigma_1}$), the differences of interest rate sensitivities between the two types of traders ($Cov\left(dr, dS_1/S_1\right)$ versus $Cov\left(dr, dS_2/S_2\right)$) and the correlation coefficient of the two heterogeneous perspectives. (ρ_{12}) In addition, the converging pattern of futures price evolution is determined by the dividend heterogeneity and the level of present interest rate relative to its long-run mean ($r - m^*$). It is not just a monotonic path of converging pattern between futures and cash markets.

In addition, if the homogeneous expectation prevails between the two types of traders, that is, $\mu_1 = \mu_2 = \mu$, $\delta_1 = \delta_2 = \delta$, $\sigma_1 = \sigma_2 = \sigma_S$, $\rho_{1r} = \rho_{2r} = \rho_{Sr}$, and $\rho_{12} = 1$, Equation (13) becomes,

$$F(t) = S(t)\exp\left[\left(m^* - \delta\right)(T-t) + L_1(t) + L_2(t) + L_3(t)\right] \tag{21}$$

where

$$L_1(t) = \left(\frac{\sigma_r\ \sigma_S\ \rho_{Sr}}{\alpha} + \frac{\sigma_r^2}{2\ \alpha^2}\right)\ \tau$$

$$L_2(t) = \left(r_t - m^* - \frac{\sigma_r^2}{2\alpha^2} - \frac{\sigma_r\ \sigma_S\ \rho_{Sr}}{\alpha}\right) H$$

$$L_3(t) = -\frac{\sigma_r^2}{4\ \alpha} H^2$$

Note that ξ disappears in this formula because that expectations between the two types of traders are homogeneous. This formula can be used for later comparison to check the consistency and rightness of mathematical derivations.

3. When interest rate is stochastic and opinions are time-varying

MacKinlay and Ramaswamy (1988) pinpoint that the arbitrage risk or mispricing from the cost of carry model exacerbates with longer time to maturity for three reasons. First, an unanticipated variability of dividend payment erodes anticipated arbitrage profit with longer time to expiration. Second, longer term means a larger amount of unexpected interest earning or cost from financing mark-to-market flows on a futures position. Third, when traders attempt to use less than the full basket of stocks in the index for arbitrage, replicating errors and adjustment costs become more serious and more expensive owing to a longer maturity. That is, as time to maturity gets shorter, the degrees of uncertainties become smaller. Hemler and Longstaff (1991) also comment that traders tend to hold same opinions about new information with decreasing time to maturity. Hence, a natural converging tendency exists between different expectations as time goes by. If the adjustment process is controlled by a monotonic function, say, $\xi(t)=\frac{T-t}{T}$, representing a plain converging tendency toward the second perspective. The partial differential equation allowing for stochastic interest rate and adjustment effect is as follows:

$$
\begin{aligned}
& f_S\, S\left\{ r-\left[\delta_1+\frac{t}{T}\left(\delta_2-\delta_1\right)\right]\right\}+f_r\left[\alpha(m-r)-\lambda_r\,\sigma_r\right] \\
& +\frac{1}{2}f_{SS}\,S^2\left\{\left[\sigma_1+\frac{t}{T}\left(\sigma_2\,\rho_{12}-\sigma_1\right)\right]^2+\left(\frac{t}{T}\sigma_2\sqrt{1-\rho_{12}^2}\right)^2\right\} \\
& +f_{Sr}\,\rho_{Sr}\,S\,\sigma_r\left\{\left[\sigma_1+\frac{t}{T}\left(\sigma_2\,\rho_{12}-\sigma_1\right)\right]^2+\left(\frac{t}{T}\sigma_2\sqrt{1-\rho_{12}^2}\right)^2\right\}^{\frac{1}{2}} \\
& +\frac{1}{2}f_{rr}\,\sigma_r^2=f_\tau
\end{aligned}
\tag{22}
$$

And the pricing formula becomes:[13]

$$
f(t)=S(t)\exp\left\{\left[m^*-\left[\frac{1}{2T}\left(\delta_1-\delta_2\right)(T-t)+\delta_2\right]\right](T-t)+K_1(t)+K_2(t)+K_3(t)\right\}
\tag{23}
$$

where

[13] Please refer to Appendix B for the proof of this futures pricing formula.

$$K_1(t)=\frac{1}{2T\alpha^2}\Big\{ 2(1+T\alpha)\rho_{12}\rho_{1r}\sigma_2\sigma_r$$
$$-\alpha^2\left[T\rho_{12}^2\sigma_1^2+T\left(T^2-1\right)\left(1-\rho_{12}^2\right)\sigma_2^2\right]+T\alpha^2\rho_{12}^4\sigma_2$$
$$+2(1+T\alpha)\sqrt{1-\rho_{12}^2}\,\rho_{2r}\sigma_2\sigma_r+\sigma_r\left(T\sigma_r-2\rho_{1r}\sigma_1\right)\Big\}\tau$$
$$+\frac{1}{2T\alpha}\Big\{\alpha\left[\left(2T^2-1\right)\left(1-\rho_{12}^2\right)+\rho_{12}^4\right]\sigma_2^2$$
$$-\alpha\rho_{12}\sigma_1\left[\left(1-\rho_{12}\right)\sigma_1-\sigma_2\right]$$
$$+\sqrt{1-\rho_{12}^2}\,\rho_{2r}\sigma_2\sigma_r-\rho_r\left(\sigma_1-\rho_{12}\sigma_2\right)\sigma_{1r}\Big\}\tau^2$$
$$+\frac{1}{6T^2}\Big\{\left(2-\rho_{12}\right)\rho_{12}\sigma_1^2-2\rho_{12}\sigma_1\sigma_2$$
$$+\left[\left(1-3T^2\right)\left(1-\rho_{12}^2\right)+\rho_{12}^4\right]\sigma_2^2\Big\}\tau^3$$
$$K_2(t)=\frac{1}{2T\alpha^2}\Big\{2\left(r_t-m^*\right)T\alpha^2$$
$$-\sigma_r\left[2(1+T\alpha)\left(\rho_{2r}\sqrt{1-\rho_{12}^2}+2\rho_{1r}\rho_{12}\right)\sigma_2-2\rho_{1r}\sigma_1+T\sigma_r\right]\Big\}H$$
$$K_3(t)=-\frac{\sigma_r^2}{4\alpha}H^2$$

Similarly, we can observe effects of various important parameters on futures price by partially differentiation, that is,

$$\frac{1}{f}\frac{\partial f}{\partial m}=\tau-H \tag{24}$$

$$\frac{1}{f}\frac{\partial f}{\partial\sigma_r}=\frac{\left[\alpha-(1-H)\right]}{\alpha^2}\rho_{1r}\sigma_1+\frac{\left[(\alpha-1)+H\alpha\right]}{\alpha^3}\rho_{2r}\sigma_2$$
$$+\frac{2(1-H)}{\alpha}\rho_{2r}-\frac{\left[H^2\alpha^2-2\alpha(1-H)\right]}{2\alpha^2}\sigma_r \tag{25}$$

$$\frac{1}{f}\frac{\partial f}{\partial\alpha}=\frac{\alpha-4(1-H)}{\alpha^3}\sigma_r\rho_{1r}\sigma_1-\frac{2(1-H)}{\alpha^2}\sigma_r\rho_{2r}$$
$$+\frac{3-2(1+H)\alpha}{\alpha^4}\sigma_r\rho_{2r}\sigma_2+\frac{H(4+H\alpha)-4}{4\alpha^3}\sigma_r^2 \tag{26}$$

$$\frac{1}{f}\frac{\partial f}{\partial \tau} = 2\left[m - \frac{1}{2}\left(\delta_1 + \delta_2\right)\right] + \frac{\sigma_r^2}{2\alpha^2}$$
$$+\frac{\sigma_r}{\alpha^3}\left[\left(\alpha^2 - 2\alpha H\right)\sigma_1 \rho_{1r} + 2\left(\sigma_2 + \alpha^2\right)\rho_{2r}\right] \qquad (27)$$

Because the signs of the comparative statics results are indeterminate, we take numerical examples of Equation (23) with covariance matrices subjected to be positive definite in Table 1 and Table 2. The specific influence on futures pricing resulting from one key opinion parameter can be observed in the column "*f*". Note that we set $S = 1$. Thus, if $f < 1$, then the basis (which is defined as futures price minuses spot price) is negative. Otherwise, the basis is positive for $f > 1$. In Table 1, traders of different groups hold exactly the same opinion on the three sets of key parameters $\rho_{\square r}$, $\sigma_{\square}$ and $\delta_{\square}$. In other words, there is no disagree among traders and zeros occupy in the column "opinions difference". It is obvious that the futures price increases and the basis transforms to be positive as the individual expectations on $\sigma_{\square}$ and $\delta_{\square}$ become smaller, and as the co-varying degrees ($\rho_{\square}$) between cash and futures markets lower, i.e., the smaller absolute values of correlation coefficients.

In contrast, Table 2 shows that the larger divergent degrees of opinions on various parameters, that is, the larger $\left|\rho_{1r} - \rho_{2r}\right|$, $\left|\sigma_1 - \sigma_2\right|$, and $\left|\delta_1 - \delta_2\right|$ terms, all make the futures price lower and make the basis tend to be more smaller or negative. As a matter of fact, if we take the divergent opinions as one source of volatility (Frankel and Froot (1990) and Ederington and Lee (1995)), such outcome coincides with the empirical evidence that increased volatility lowers basis. (Chen, Cuny, and Haugen (1995)).

Table 1. Numerical Examples of Equation (23)
—— identical opinions on ρ_{1r}/ρ_{2r}, σ_1/σ_2, and δ_1/δ_2

(The futures price f increases and the basis tends to be more positive as the individual expectations on $\sigma_{\square}$ and $\delta_{\square}$ become smaller and as the co-varying degrees ($\rho_{\square r}$) between cash and futures markets become lower.)

f	ρ_{1r}	ρ_{2r}	opinions difference
0.9818	-55.0%	-55.0%	0
0.9911	-40.0%	-40.0%	0
1.0006	-25.0%	-25.0%	0
1.0101	-10.0%	-10.0%	0
1.0198	5.0%	5.0%	0

f	ρ_{1r}	ρ_{2r}	opinions difference
0.9971	5.0%	5.0%	0
0.9989	4.5%	4.5%	0
1.0006	4.0%	4.0%	0
1.0023	3.5%	3.5%	0
1.0041	3.0%	3.0%	0

f	ρ_{1r}	ρ_{2r}	opinions difference
0.9906	5.0%	5.0%	0
0.9956	4.0%	4.0%	0
1.0006	3.0%	3.0%	0
1.0056	2.0%	2.0%	0
1.0106	1.0%	1.0%	0

(Here we set $S=1$ to clarify the symbol of basis. If $f<1$ then the basis which is defined as futures price minuses spot price is negative. The median values of ρ_{1r}/ρ_{2r}, σ_1/σ_2, and δ_1/δ_2 are used to be fixed parameters when allowing for variation of only one key parameter accordingly. Other parameter settings include $\alpha=-0.3$, $r=3\%$, $m=3\%$, $\sigma_r=3$, $T=1$, $t=0.5$, $\rho_{12}=0$ and $\lambda_r=3\%$.)

Table 2. Numerical Examples of Equation (23)
— divergent opinions on ρ_{1r}/ρ_{2r}, σ_1/σ_2, and δ_1/δ_2

(The futures price f decreases and the basis tends to be more negative as the band between different opinions on $\rho_{\square r}$, $\sigma_{\square}$ and $\delta_{\square}$ become wider.)

f	ρ_{1r}	ρ_{2r}	opinions difference
1.0115	0.0%	0.0%	0.0%
1.0079	5.0%	-5.0%	10.0%
1.0044	10.0%	-10.0%	20.0%
1.0009	15.0%	-15.0%	30.0%
0.9974	20.0%	-20.0%	40.0%
f	ρ_{1r}	ρ_{2r}	opinions difference
1.0093	1.0%	1.0%	0.0%
1.0072	2.0%	2.5%	0.5%
1.0044	3.0%	4.5%	1.5%
1.0023	4.0%	6.0%	2.0%
1.0002	5.0%	7.5%	2.5%
f	ρ_{1r}	ρ_{2r}	opinions difference
1.0196	1.0%	1.0%	0.0%
1.0120	2.0%	3.0%	1.0%
1.0044	3.0%	5.0%	2.0%
0.9969	4.0%	7.0%	3.0%
0.9895	5.0%	9.0%	4.0%

(Here we set $S=1$ to clarify the symbol of basis. If $f<1$ then the basis which is defined as futures price minuses spot price is negative. The median values of ρ_{1r}/ρ_{2r}, σ_1/σ_2, and δ_1/δ_2 are used to be fixed parameters when allowing for variation of only one key parameter accordingly. Other parameter settings include $\alpha=-0.3$, $r=3\%$, $m=3\%$, $\sigma_r=3$, $T=1$, $t=0.5$, $\rho_{12}=0$ and $\lambda_r=3\%$.)

In addition, if the homogeneous expectation prevails between the two types of traders, that is, $\mu_1=\mu_2=\mu$, $\delta_1=\delta_2=\delta$, $\sigma_1=\sigma_2=\sigma$, $\rho_{1r}=\rho_{2r}=\rho_{Sr}$, and $\rho_{12}=1$, Equation (23) is identical Equation (13). In specific,

$$m^* - \left[\frac{1}{2T} \left(\delta_1 - \delta_2 \right) \left(T - t \right) + \delta_2 \right] = m^* - \delta$$

$$L_i \left(t \right) = K_i \left(t \right), \ \ i = 1, 2, 3$$

That is, identical opinions on parameters make divergent settings vain no matter what case is. It completes verifications of consistency and rightness of mathematical derivations between the two Cases.

VI Concluding Remarks

Information revelation plays a predominant role on security price determination. However, traders may not hold same opinion when they encounter same information to formulate their perspectives on price of a security. Furthermore, they adjust their original perspectives to some extent depending on their own judgments as time passing. Consequently, it is wiser for a trader to consider other market participants' expectations and adjust recursively to advance his perspective.

This study derives closed-form futures pricing solutions within an intertemporal framework allowing for perspective heterogeneities, adjustment behaviors, and stochastic interest rate. We find significant differences as compared with the cost of carry model. In particular, the cost of carry model is a degenerated case of our model, and the existence of divergent opinions lowers futures price basis. Moreover, the relationship between futures price and cash market volatility is associated with co-variation degree of spot price and interest rate, and the effect of changing opinions is complicated with divergent degrees of opinions on dividends, volatilities, interest rate sensitivities, and correlation of random parts between the two types of traders.

Appendix A — Proof of Equation (6)

Under assumptions of complete market and enough traded assets to eliminate various risks, we can form a portfolio

$$\Pi = \sum_j k_j \, v_j$$

where v_j is a traded asset and k_j is its corresponding weight in this portfolio. Because the dynamics of two state variables are Itô processes, we can obtain the dynamics of v_j by Itô lemma, Equation (3), and Equation (5),

$$\begin{aligned}
dv_j = {} & \frac{\partial v_j}{\partial t} dt + \frac{\partial v_j}{\partial S}\Big[\xi(t)\left(\mu_1-\delta_1\right)+\left(1-\xi(t)\right)\left(\mu_2-\delta_2\right)\Big] S \, dt \\
& + \frac{\partial v_j}{\partial r}\Big[\,\alpha\left(m-r\right)\Big] dt + \frac{1}{2}\frac{\partial^2 v_j}{\partial r^2}\sigma_r^2 \, dt \\
& + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left\{\Big[\xi(t)\,\sigma_1 + \left(1-\xi(t)\right)\sigma_2\,\rho_{12}\Big]^2 + \Big[\left(1-\xi(t)\right)\sigma_2\sqrt{1-\rho_{12}^2}\,\Big]^2\right\} S^2 \, dt \\
& + \frac{\partial^2 v_j}{\partial S\,\partial r}\Big\{\,\xi(t)\, S\,\sigma_1\,\sigma_r\,\rho_{1r} + \Big[1-\xi(t)\Big] S\,\sigma_2\,\sigma_r\,\rho_{1r}\,\rho_{12} \\
& \qquad + \Big[1-\xi(t)\Big] S\,\sigma_2\,\sigma_r\,\rho_{2r}\sqrt{1-\rho_{12}^2}\,\Big\} dt \\
& + \frac{\partial v_j}{\partial S} S\left\{\Big[\xi(t)\,\sigma_1 + \left(1-\xi(t)\right)\sigma_2\,\rho_{12}\Big] dz_1 + \left(1-\xi(t)\right)\sigma_2\sqrt{1-\rho_{12}^2}\, dz_2\right\} \\
& + \frac{\partial v_j}{\partial r}\sigma_r \, dz_r
\end{aligned}$$

Then we can obtain the dynamic of the portfolio,

$$
\begin{aligned}
d\Pi = \sum_j k_j \Bigg\{ & \frac{\partial v_j}{\partial t} + \frac{\partial v_j}{\partial r}\left[\alpha\left(m - r\right)\right] + \frac{1}{2}\frac{\partial^2 v_j}{\partial r^2}\sigma_r^2 \\
& + \frac{\partial v_j}{\partial S}\left[\xi(t)\left(\mu_1 - \delta_1\right) + \left(1-\xi(t)\right)\left(\mu_2 - \delta_2\right)\right] S \\
& + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\xi(t)\sigma_1 + \left(1-\xi(t)\right)\sigma_2\ \rho_{12}\right]^2 S^2 \\
& + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\left(1-\xi(t)\right)\sigma_2\sqrt{1-\rho_{12}^2}\right]^2 S^2 \\
& + \frac{\partial^2 v_j}{\partial S\,\partial r}\Big[\ \xi(t) S\,\sigma_1\,\sigma_r\,\rho_{1r} + \left[1-\xi(t)\right] S\,\sigma_2\,\sigma_r\,\rho_{1r}\,\rho_{12} \\
& \qquad + \left[1-\xi(t)\right] S\,\sigma_2\,\sigma_r\,\rho_{2r}\sqrt{1-\rho_{12}^2}\ \Big] \Bigg\} dt \\
= \sum_j & \left(r - \delta_j\right) k_j\, v_j\, dt
\end{aligned}
$$

where δ_j denotes yield of the *j*-th position. That is,

$$
\begin{cases}
\begin{aligned}
\sum_j k_j \Bigg\{ & \frac{\partial v_j}{\partial t} + \frac{\partial v_j}{\partial S}\left[\xi(t)\left(\mu_1 - \delta_1\right) + \left(1-\xi(t)\right)\left(\mu_2 - \delta_2\right)\right] S \\
& + \frac{\partial v_j}{\partial r}\left[\alpha\left(m - r\right)\right] + \frac{1}{2}\frac{\partial^2 v_j}{\partial r^2}\sigma_r^2 \\
& + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\xi(t)\sigma_1 + \left(1-\xi(t)\right)\sigma_2\ \rho_{12}\right]^2 S^2 \\
& + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\left(1-\xi(t)\right)\sigma_2\sqrt{1-\rho_{12}^2}\right]^2 S^2 \\
& + \frac{\partial^2 v_j}{\partial S\,\partial r}\Big[\ \xi(t) S\,\sigma_1\,\sigma_r\,\rho_{1r} + \left[1-\xi(t)\right] S\,\sigma_2\,\sigma_r\,\rho_{1r}\,\rho_{12} \\
& \qquad + \left[1-\xi(t)\right] S\,\sigma_2\,\sigma_r\,\rho_{2r}\sqrt{1-\rho_{12}^2}\ \Big] - \left(r - \delta_j\right) v_j \Bigg\} = 0 \\
\sum_j k_j \Bigg\{ & \frac{\partial v_j}{\partial r}\sigma_r \\
& + \frac{\partial v_j}{\partial S} S \left[\left(\xi(t)\sigma_1 + \left(1-\xi(t)\right)\sigma_2\ \rho_{12}\right) + \left(1-\xi(t)\right)\sigma_2\sqrt{1-\rho_{12}^2}\right] \Bigg\} = 0
\end{aligned}
\end{cases}
$$

Thus, nontrivial solutions of the simultaneous linear equations will satisfy the following relationship:

$$
\begin{aligned}
&\lambda_S \frac{\partial v_j}{\partial S} S\left[\xi(t)\,\sigma_1 + \left(1-\xi(t)\right)\sigma_2\ \rho_{12}\right] \\
&+\lambda_S \frac{\partial v_j}{\partial S} S\left[\left(1-\xi(t)\right)\sigma_2 \sqrt{1-\rho_{12}^2}\ \right] + \lambda_r \frac{\partial v_j}{\partial r}\sigma_r \\
&= \frac{\partial v_j}{\partial t} + \frac{\partial v_j}{\partial S}\left[\xi(t)\left(\mu_1-\delta_1\right) + \left(1-\xi(t)\right)\left(\mu_2-\delta_2\right)\right] S \\
&\qquad + \frac{\partial v_j}{\partial r}\left[\alpha\left(m-r\right)\right] + \frac{1}{2}\frac{\partial^2 v_j}{\partial r^2}\sigma_r^2 + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\xi(t)\,\sigma_1 + \left(1-\xi(t)\right)\sigma_2\ \rho_{12}\right]^2 S^2 \\
&\qquad + \frac{1}{2}\frac{\partial^2 v_j}{\partial S^2}\left[\left(1-\xi(t)\right)\sigma_2 \sqrt{1-\rho_{12}^2}\right]^2 S^2 \\
&\qquad + \frac{\partial^2 v_j}{\partial S\,\partial r}\Big[\ \xi(t)\, S\, \sigma_1\, \sigma_r\, \rho_{1r} + \left[1-\xi(t)\right] S\, \sigma_2\, \sigma_r\, \rho_{1r}\, \rho_{12} \\
&\qquad\qquad + \left[1-\xi(t)\right] S\, \sigma_2\, \sigma_r\, \rho_{2r} \sqrt{1-\rho_{12}^2}\ \Big] - \left(r-\delta_j\right) v_j
\end{aligned}
$$

Finally, let $v_j = F$ without any disbursement needed as constructing the portfolio and $\lambda_{\square}$ be unit market prices of risks, then the partial differential equation is readily obtained.

Appendix B — Proof of Equation (23)

Let $G=\ln S$, thus the conditional distribution of G at time T follows a normal one with mean and variance:

$$E_t\left[G(T)\right]=\ln S(t)+\left[r_t\ H+m^*(\tau-H)\right]$$

$$-\left[\frac{1}{2T}\left(\delta_1+\delta_2\right)(T-t)^2+\frac{t}{T}\delta_2(T-t)\right]$$

$$-\frac{1}{2}\left\{\frac{(T-t)^3\ \sigma_1^2}{3T^2}+\frac{(T-t)^2(2t+T)\rho_{12}\ \sigma_1\ \sigma_2}{3T^2}\right.$$

$$\left.+(T-t)\,t^2\sigma_2^2\left(1-\rho_{12}^2\right)+\frac{(T-t)\left((T-t)^2+3tT\right)\rho_{12}^2\ \sigma_2^2}{3T^2}\right\}$$

$$Var\left[G(T)\right]=\frac{-\sigma_r^2}{2\,\alpha^3}\left[2\,H\ \alpha+H^2\ \alpha^2-2\,\alpha\,(T-t)\right]$$

$$+\left\{\frac{(T-t)^3\ \sigma_1^2}{3\,T^2}+\frac{(T-t)^2\ (2t+T)\ \rho\ \sigma_1\ \sigma_2}{3T^2}\right.$$

$$\left.+(T-t)\,t^2\sigma_2^2\left(1-\rho^2\right)+\frac{(T-t)\left[(T-t)^2+3tT\right]\rho^2\ \sigma_2^2}{3T^2}\right\}$$

$$+\sigma_r\ \rho_{1r}\left\{\frac{\sigma_1\left[2H-2\tau+\alpha(T-t)^2+3tT^2\right]}{\alpha^2\ T}\right.$$

$$-\frac{\rho_{12}\ \sigma_2}{\alpha^2\ T}\left[\ 2H+2\alpha H\ T-2(T-t)^2\right.$$

$$\left.\left.+3tT+\alpha(T+t)(T-t)^2+3tT\ \right]\right\}$$

$$+\sigma_r\ \rho_{2r}\left\{\frac{2\sqrt{1-\rho_{12}^2}\left[-H+(T-t)^2+3tT\right]}{\alpha}\right.$$

$$\left.+\frac{\sigma_2\left[-2(1+\alpha T)H+\alpha(T+t)-H+(T-t)^2+3tT\right]}{\alpha^3 T}\right\}$$

Then the futures price $f(t)$ is readily obtained,

$$f(t) = \hat{E}(S(T)) = e^{\hat{E}(G(T)) + \frac{1}{2}\hat{V}ar(G(T))}$$

where the symbol $\frown$ denoting operations under risk-neutral probability measure. Equation (23) is obtained after tedious algebraic operations.

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