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Dynamic Asset Allocation Strategy for Intertemporal Pension Fund Management with Time-Varying Volatility

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ABSTRACT

In this paper, we examine the optimal dynamic asset allocation strategy in relation to a pension fund with a time-varying investment opportunity set. The sources of changes in the investment opportunity set in our model possess time-varying volatility and risk premia. We deal with the pension fund asset allocation problem, characterized by time-varying volatility, by using Sharpe and Tint's (1990) liability approach and Merton's (1993) intertemporal capital asset pricing methodology regarding the optimal investment strategies for university endowment funds. We develop an intertemporal model and derive a closed-form solution using perturbation methods. In contrast to Merton (1971, 1993), we propose a new liability hedging component in the dynamic asset allocation in addition to the intertemporal hedging component for pension funds in order to hedge against changes in the pension fund liability in our model.

1. INTRODUCTION

The investment and asset allocation policies in relation to pension funds have a profound effect on the world capital market. They affect the development of financial innovation, the behavior of security prices and rates of return, etc. Traditional models for pension fund asset allocation rarely consider the liability framework that provides a more comprehensive context of the fund's overall function.

In general, pension fund managers use the traditional asset-only optimization method to make investment strategies. Sharpe and Tint (1990) provide a new approach that avoids either asset-only or full-surplus optimization in relation to pension fund management. They argue that pension fund managers have always shown little interest in terms of giving consideration to liabilities when designing pension fund asset allocation strategies. One reason for this is that other methods require an all-or-nothing approach to considering liabilities. The Sharpe-Tint method allows for either a full or partial consideration of liabilities to meet practical needs. Surplus optimization is normally based on accepting the way in which liabilities are characterized in the financial statements, which is an accurate reflection of the true liability. In these two extreme cases, zero-liability consideration is the same as asset-only optimization, while full-liability consideration is the same as full-surplus optimization with regard to pension fund asset allocation strategies.

In dealing with the problem of investment strategies for university endowment funds,[1] Merton (1993) successfully applies the standard intertemporal consumption and investment model (Merton 1969, 1971, and 1973), which has been studied extensively in the finance literature. He assumes that the university endowment is invested in traded assets, including risky assets and a riskless asset. The dynamics of the return on the risky asset is expressed by $dP = P\alpha dt + P\sigma dZ$, in which P denotes the price of the risky asset, while the instantaneous expected return (α) and standard deviation of the return (σ) are assumed to be constant. The university's wealth is then defined as the sum of the endowment capital and capitalized present and future non-endowment

[1] Merton (1993) assumes that the purpose of a university is to be a collection of activities or outputs such as education, training, research, and the storage of knowledge. The intensities of these activities can be quantified and are subject to budget constraints. At the same time, there exists a preference ranking of intertemporal programs. In Merton's (1993) model, the dynamics of the unit activity cost and the non-endowment sources of cash flows are jointly Markov, while non-endowment income is spanned by the returns on financial assets.

income. The foremost objective of Merton's paper is to discuss the optimal portfolio allocation for university endowment funds under states of relative costs or prices of various activities. The optimal portfolio consists of a mean-variance efficient portfolio of the endowment plus a hedging portfolio to counter unanticipated fluctuations in the states.

In this paper, we build a dynamic asset allocation strategy model for pension fund management with time-varying expected return and volatility. To take into consideration liabilities in an asset mix, Sharpe and Tint (1990) develop the model using a traditional one-period static mean-variance optimization approach. In this paper, we use Merton's (1971, 1993) intertemporal model to find the optimal asset allocation strategies for pension funds and incorporate Sharpe and Tint's (1990) liabilities approach. Furthermore, we extend Merton's (1993) model in two directions. First, the return on risky assets is set as having mean-reverting expected returns with a time-varying conditional variance, while Merton (1993) sets the dynamics as having constant instantaneous expected returns and constant standard deviations over time. More recently, empirical studies on asset returns have exhibited time-varying volatility and mean-reverting asset returns,[2] implying the time-varying characteristic of the investment opportunities. Second, in contrast to Merton (1993), who arranges the relative costs of various activities as state variables, we do not use the pension liability as a state variable. Instead, this liability is one of the components of pension assets on the debit side.

In general, one cannot derive a closed-form solution with Merton's model (1971, 1993) when solving a nonlinear partial differential equation of the hedging portfolio with the stochastic opportunity. Recently, some researchers have tried to solve this problem, such as Liu (2001), Chacko and Viceira (2002) and Campbell and Viceira (2001).[3] In this paper, we follow the stochastic opportunity setting of Chacko and

[2] The time-varying volatility of asset returns, i.e. the conditional variance of asset returns, is not constant over time. This phenomenon has been evidenced by Campbell (1987), Harvey (1989, 1991), Glosten et al. (1993), and can be induced from this time variation in terms of investment opportunities. Moreover, asset returns are mean-reverting, as argued by Fama and French (1988), Poterba and Summers (1988), Cecchetti et al. (1990), Chen and Jeon (1998), and Balvers et al. (2000). This again implies that the investment opportunity is time-varying.

[3] Liu (2001) characterizes the closed-form solution of the optimal portfolio against a setting of stock returns exhibiting stochastic volatility or predictability. He shows that Merton's optimal dynamic portfolio selection problem, expressed by a nonlinear partial differential equation, may be reduced to a system of ordinary differential equations under a stochastic setting. Chacko and Viceira (2002) derive the optimal portfolio with stochastic volatility in incomplete markets by using the approximate solution method and find a solution around the unconditional mean of the log consumption-wealth ratio. Campbell and Viceira (2001) use a log-linear approximation to characterize the portfolio demand under a stochastic

Viceira (2002) to derive the optimal asset allocation strategy for pension funds. However, Chacko and Viceira (2002) mainly derive the solution on a log-linear expansion of the consumption-wealth ratio around its unconditional mean. Their solution is obtained around a particular point in the state space – the unconditional mean of the log consumption-wealth ratio. By contrast, we derive the approximate analytical solution using perturbation methods of approximation around a particular point in the preference space.

This paper is organized as follows. Section 2 describes the model used and environment assumed in this paper. Section 3 develops the optimal dynamic asset allocation strategy for pension funds. Section 4 discusses the optimal dynamic asset allocation strategy for intertemporal pension fund management. Finally, our conclusions are presented in Section 5.

2. A MODIFIED SHARPE-TINT MODEL (1990)

In this paper, we include "liabilities" in addition to "assets" (A), in discussing the dynamic asset allocation strategy for intertemporal pension fund management. We extend asset allocation to include a liability framework that provides a more comprehensive context for the fund's overall purpose, i.e. the pension fund's surplus (S).

In addition, we show that we can avoid the all-or-nothing approach, i.e. the two extremes of asset-only and full-surplus optimization. We allow the full or partial consideration of liabilities to meet practical needs and reflect the true liability. In the two extreme cases, zero-liability consideration is the same as asset-only optimization, while full-liability consideration is the same as surplus optimization for pension fund asset allocation strategies (Sharpe and Tint, 1990).

In Section 2.1, we review the approach of Sharpe and Tint (1990), which allows full or partial consideration of liabilities. In Sections 2.2–4, we build an intertemporal model in a time-varying environment setting based on their methodology.

2.1 Review of the Sharpe-Tint approach (1990)

Sharpe and Tint (1990) start by assuming that the pension fund has determined which liabilities it wishes to consider in determining its investment strategy. The pension

opportunity setting. They conclude that the ratio of bonds to stocks in the optimal portfolio increases with risk aversion.

fund's relevant measure of surplus (S) is equal to the subtraction of the relevant liability concept (L) multiplied by the importance to be attached to it (k) from the value of the pension fund's asset (A), i.e.

$$S = A - kL. \tag{1}$$

Then, the return on the fund's surplus $(\tilde{R}_S)$ can be defined as by Sharpe and Tint (1990):

$$\tilde{R}_S \equiv \frac{dS}{A} = \tilde{R}_A - k\frac{L}{A}\tilde{R}_L, \tag{2}$$

where $\tilde{R}_A$ is the return on pension assets, $\tilde{R}_L$ is the growth rate of the relevant liability concept and L/A is the inverse of the funding ratio, which is denoted by $1/F$.

This approach permits the pension fund manager to avoid the extreme of either asset-only or full-surplus optimization. If $k = 0$, the traditional asset-only optimization is being referred to; if $k = 1$, full-surplus optimization is being referred to; and if $0 < k < 1$, the value of k depends on the importance attached to the pension fund's liability.

2.2 Asset price and liability dynamics

The pension fund is assumed to comprise investments in traded assets only. There are two kinds of assets available for trading in the economy. The first kind is the risky asset of a market portfolio, where P_t denotes the price of the risky asset at time t. Then the return dynamics for the risky asset is given by,

$$\frac{dP_t}{P_t} = (r + \mu_{P_t})dt + \sigma_{P_t} dZ_P, \tag{3}$$

where μ_{P_t} is the time-varying instantaneous expected risk premium in relation to the risky asset; σ_{P_t} is the time-varying instantaneous standard deviation of the return on the risky asset; and dZ_P is a Wiener process. Another kind is a short-term risk-free bond with an interest rate r. We assume that the short rate is constant in order to focus on the stochastic volatility.

Next, we assume that the dynamics for the pension fund liability (L_t) growth rate is described in terms of the following stochastic differential equation,

$$\frac{dL_t}{L_t} = \mu_L dt + \sigma_{L_t} dZ_L, \tag{4}$$

where μ_L is the instantaneous expected growth rate of the pension liability; σ_{L_t} is the time-varying instantaneous standard deviation of the growth rate of the pension liability; and dZ_L is a Wiener process.[4]

We set the pension scheme models as stochastic in modelling the pension asset liability in this paper. The setting of the dynamics of the pension liability (L_t) growth rate, as described by the stochastic differential equation, seems too strong at first glance. However, we assume that the pension liability (which mainly consists of payroll) is positively correlated with the inflation rate. It follows that, if the inflation rate is stochastic (for example, Brennan and Xia, 2002), the pension liability is also stochastic.[5]

2.3 Volatility process

We denote stochastic variables with a subscript "t", and let the expected excess return and the conditional variance of the risky asset vary stochastically over time. For the above setting, the investment opportunity is time-varying. We assume that the instantaneous variance process is

$$\sigma^2_{P_t} = X_t^{-\frac{1}{\beta}}, \tag{5}$$

[4] Sharpe and Tint (1990) use the "economic value" for the projected benefit obligation to represent the value of the liability concept. Black (1989) also points out that different people can have very different views regarding pension liabilities. He gives two different definitions of pension liability: the narrow and the broad one. The former is defined as the present value of the vested benefits of the current work force, and the latter, as the present value of all benefits to be paid by the plan, i.e., the price of the claim to the benefits in the open market today. There are other definitions that lie between the narrow and the broad ones.

[5] There are discussions regarding the uncertainty associated with pension liabilities, in particular the concerns of pension fund managers regarding the effect of inflation on their pension obligations (Goodman and Marshall, 1988). For example, Winklevoss (1982) develops a pension liability and asset simulation model based on a Monte Carlo simulation of inflation, which is used to provide a stochastic projection of the pension liability. Black (1989) shows that both the narrowly-defined liability and the broadly-defined one will change over time like a security, and that the liability components will also be random. Tzeng et al. (2000) give another example based on the stochastic liability assumption. They discuss surplus management under a stochastic process by assuming that the returns on assets and liabilities follow the mean-reverting pattern suggested by Vasicek (1977).

and that the state variable X_t has the following mean-reverting process:

$$dX_t = \pi(m - X_t)dt + \sigma\sqrt{X_t}dZ_X, \tag{6}$$

where m is the long-term mean and π is the reversion parameter of the mean-reverting process. The time-varying investment opportunity set is derived from the state variable, which is an economic variable and has extensive and profound effects on the investment opportunities over time. The volatility of the inflation rate or other macroeconomic variables are examples of state variables. The investment opportunity set is described by the state variable X_t in our model. By applying Ito's lemma to (6), we find that this is equivalent to directly assuming the mean-reverting process to be the following stochastic differential equation:

$$\begin{aligned}
\frac{d\sigma^2_{P_t}}{\sigma^2_{P_t}} &= \left[\frac{1}{\beta}\pi - \frac{1}{\beta}\pi\sigma^{2\beta}_{P_t}\left(m - \frac{1}{2}\frac{1+\beta}{\beta}\frac{\sigma^2}{\pi}\right)\right]dt - \frac{1}{\beta}\sigma(\sigma_{P_t})^{\beta}dZ_X \\
&= \left[\frac{1}{\beta}\pi\left(m - \frac{1}{2}\frac{1+\beta}{\beta}\frac{\sigma^2}{\pi}\right)\right]\cdot\left[\left(m - \frac{1}{2}\frac{1+\beta}{\beta}\frac{\sigma^2}{\pi}\right)^{-1} - \sigma^{2\beta}_{P_t}\right]dt \\
&\qquad - \frac{1}{\beta}\sigma(\sigma_{P_t})^{\beta}dZ_X \\
&\equiv \pi_{\sigma^2_{P_t}}(m_{\sigma^2_{P_t}} - \sigma^{2\beta}_{P_t})dt - \frac{1}{\beta}\sigma(\sigma_{P_t})^{\beta}dZ_X,
\end{aligned} \tag{7}$$

where $\pi_{\sigma^2_{P_t}} = (1/\beta)\pi \cdot m^{-1}_{\sigma^2_{P_t}}$; $m_{\sigma^2_{P_t}} = \{m - (1/2)[(1+\beta)/\beta](\sigma^2/\pi)\}^{-1}$ and $\sigma^2_{P_t}$ is well-defined provided $m \geq 0$. From the setting of X_t, volatility $(\sigma^2_{P_t})$ is mean-reverting also. For simplification, we assume that $\beta = 1$, implying that the volatility dynamics is the same as that specified by Chacko and Viceira (2002). We also assume that $\sigma_{L_t} = X_t^{1/2}$ in relation to the time-varying instantaneous standard deviation of the growth rate of the pension fund liability.[6]

In this paper, we assume that $\sigma^2_{p_t} = 1/X_t$ and $\sigma^2_{L_t} = X_t$. The basic idea behind this is that some studies on optimal monetary policy find evidence of a tradeoff

[6] In this paper, this assumption implies that this model can only be applied to positive macroeconomic variables. As the model is focused on the volatilities of risky assets and pension liabilities, we emphasize the second moments of macroeconomic variables, such as inflation volatility – the generalized definition of macroeconomic variables.

between inflation and output volatility. Therefore, aggregate shocks that move output and inflation in opposite directions give rise to a tradeoff between output and inflation volatility (Taylor, 1979; Fuhrer, 1997 and Lee, 2002, etc.). Many countries in the world including the United States still have explicit targets for inflation or long-term monetary policy goals that are aimed at maintaining price stability. If policy-makers keep inflation at a low target level over time, inflation will then be less variable, while output will inevitably exhibit larger fluctuations and vice versa (Lee, 2002). The switching of targets due to changes in monetary policy over time, and their sometimes being stricter and sometimes milder, will give rise to a mean-reverting phenomenon of inflation volatility. In Lee (2002), a bivariate GARCH model is established to capture the inverse relationship between the conditional variances of output and inflation. For this reason, as we assume the risky asset to be the stock of the output, and the inflation variance to be the state variable (X_t), we assume, at the same time, that $\sigma^2_{p_t} = 1/X_t$ in our paper.

We continue to set the inflation variance as the state variable based on the assumption of the relationship $\sigma^2_{L_t} = X_t$. We know that very often the pension liability, which mainly consists of payroll, is positively correlated with the inflation rate. Then, the increase in inflation volatility will often increase the uncertainty of the payroll of the pension, as it will increase the volatility of the pension liability. Consequently, if we assume that the state variable is the inflation variance, then we intuitively assume that there exists a positive relationship between the variance of the pension liability and the state variable.

2.4 Utility of the pension fund

Following Merton (1993), we also assume that there exists a preference ranking alternative intertemporal program.[7] We adopt Duffie and Epstein's (1992) description

[7] This paper and that of Merton (1993) share a similar problem in terms of treating the pension fund or university as an economic agent that maximizes expected utility. The pension fund can be viewed as a nexus of contracts among economic agents, who must be paid benefits. Just as Constantinides (1993) points out, a research through the academic literature on the objective of the university provides valuable insights but falls short of providing answers as to whose preference should be adopted. Universities are a diverse group of institutions with heterogeneous objectives. The pension funds are also a diverse group of the insured with heterogeneous objectives. Constantinides (1993) also concludes that it is an open question whether these heterogeneous objectives are reasonable across the whole spectrum of universities. This is a question that is also raised in the modeling of pension funds. For more concinnity in this paper, we must provide an additional assumption that all those people who are insured of the pension fund have homogeneous objectives as in the case of many microeconomic and macroeconomic models. Furthermore, we assume that the preference characterized by the recursive utility function is for the pension fund manager, who acts in conformity with this homogeneous objective.

of pension fund preference by means of a recursive utility function, which is a generalization of the standard and time-separable power utility function that separates the elasticity of intertemporal substitution parameter from the relative risk aversion coefficient:

$$J = E_t \left[\int_t^\infty f(Y_\tau, J_\tau) d\tau \right], \tag{8}$$

where $f(Y_\tau, J_\tau)$ is a normalized aggregator of insured benefit (Y_τ) and utility has the following form:

$$f(Y, J) = \rho \left(1 - \frac{1}{\varphi}\right)^{-1} (1-\gamma) J \left[\left(\frac{Y}{((1-\gamma)J)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\varphi}} - 1 \right], \tag{9}$$

were γ is the coefficient of relative risk aversion, ρ is the rate of time preference and φ is the elasticity of intertemporal substitution, all being larger than zero.

The objective is to maximize the expected life-time utility defined above under the following intertemporal budget constraint,

$$dS_t = \left[(n_t \mu_{P_t} + r) A_t - k \frac{1}{F} \mu_L A_t - Y_t \right] dt + n_t \sigma_{P_t} A_t dZ_P - k \frac{1}{F} \sigma_{L_t} A_t dZ_L, \tag{10}$$

where S_t represents the pension fund's surplus, and n_t is the fraction of the pension fund allocated to the risky asset at time t.[8]

[8] In this paper, we set the pension liability (L_t) as the value of the "legal benefits" for the insured. This kind of legal benefit corresponds roughly to the projected benefit obligation. We set the insured benefit (Y_t) as the dividend payment which is an endogenous variable, that is dependent upon the preference and investment returns of the pension fund's surplus (S_t) in the pension plan. Furthermore, the pension fund's surplus depends on the subtraction of the pension liability (L_t), and the insured benefit (Y_t) (characterized as dividend payments for the consumption needs of the insured) from the pension fund's asset (A_t). In this paper, our aim is to determine only the optimal dynamic asset allocation strategies while giving consideration to the connected relationships described above.

3. OPTIMAL DYNAMIC ASSET ALLOCATION STRATEGY FOR INTERTEMPORAL PENSION FUND MANAGEMENT

3.1 A special case of the optimal dynamic asset allocation strategy

The value function (J) seeks to maximize the expected life-time utility of the pension fund. The Bellman equation for the utility function can be derived in accordance with the principle of optimality. Based on the above, the Bellman equation will satisfy,

$$\begin{aligned} 0 = \sup_{n,Y} \ & \left\{ f(Y_\tau, J_\tau) + J_S \left[n\mu_{P_t} A_t + rA_t - k\frac{1}{F}\mu_L A_t - Y_t \right] + J_X \pi (m - X_t) \right. \\ & + \frac{1}{2} J_{SS} \left(n^2 \sigma_{P_t}^2 + k^2 \frac{1}{F^2} \sigma_{L_t}^2 - 2k\frac{1}{F} n \sigma_{P_t} \sigma_{L_t} \rho_{PL} \right) A_t^2 + \frac{1}{2} J_{XX} \sigma^2 X_t \\ & \left. + J_{SX} \left(n\sigma\sqrt{X_t}\sigma_{P_t}\rho_{PX} - k\frac{1}{F}\sigma_{L_t}\sigma\sqrt{X_t}\rho_{LX} \right) A_t \right\}, \quad (11) \end{aligned}$$

where J_S, J_X denote the derivatives of J with respect to S and X, respectively, with similar notations being used for higher derivatives. Meanwhile, ρ_{PX} is the instantaneous correlation between the unexpected return on risky assets and the state variable X_t; ρ_{LX} is the instantaneous correlation between the growth rate of the pension fund liability and X_t; and ρ_{PL} is the instantaneous correlation between the unexpected return on risky assets and the growth rate of the pension fund liability.

The first-order conditions for this equation are

$$Y_t = J_S^{-\varphi} J^{\frac{1-\varphi\gamma}{1-\gamma}} \rho^\varphi (1-\gamma)^{\frac{1-\varphi\gamma}{1-\gamma}}, \quad (12)$$

$$n_t = \frac{-J_S}{J_{SS} A_t} \frac{\mu_{P_t}}{\sigma_{P_t}^2} + \frac{-J_{SX}}{J_{SS} A_t} \frac{\sigma\sqrt{X_t}\sigma_{P_t}\rho_{PX}}{\sigma_{P_t}^2} + k\frac{1}{F}\frac{\sigma_{P_t}\sigma_{L_t}\rho_{PL}}{\sigma_{P_t}^2}. \quad (13)$$

The optimal dynamic asset allocation strategy has three components. The first is the mean-variance portfolio weight, which is based on the condition that an investor invests in a single-period horizon or invests in accordance with the constant investment opportunity set, i.e. the myopic demand. The second term is the intertemporal hedging demand that characterizes the demand arising from the desire to hedge against changes in the investment opportunity set. The hedging demand is determined by the product of the "beta" of the time-varying state variable with respect to the risky asset and the instantaneous rates of change in relation to the value function. The third term is the hedging demand that characterizes the demand arising from the desire to hedge against changes in the pension fund liability. The pension fund liability hedging demand is determined by the product of the "beta" within the pension fund liability with respect to the risky asset, the importance attached to the pension fund liability (k), and the inverse of the funding ratio.

In fact, the first-order conditions for our problem are not explicit solutions unless we know the complicated indirect utility function. By substituting the first-order solutions back into the Bellman equation, we obtain

$$\begin{aligned} 0 = {} & f(Y(J),J) - J_S Y(J) - \frac{1}{2}\frac{(J_S)^2\mu_{P_t}^2}{J_{SS}} X_t + J_S r A_t - J_S k \frac{1}{F}\mu_L A_t \\ & + J_X \pi(m - X_t) - \frac{1}{2}\frac{(J_{SX})^2\sigma^2\rho_{PX}^2}{J_{SS}} X_t - \frac{1}{2} J_{SS} k^2 \frac{1}{F^2} A_t^2 \rho_{PL}^2 X_t \\ & + \frac{1}{2} J_{SS} k^2 \frac{1}{F^2} A_t^2 X_t + J_S k \frac{1}{F} A_t \mu_{P_t} \rho_{PL} X_t + J_{SX} k \frac{1}{F} A_t \rho_{PX} \sigma \rho_{PL} X_t \\ & + \frac{1}{2} J_{XX} \sigma^2 X_t - \frac{J_S}{J_{SS}} J_{SX} \rho_{PX} \sigma \mu_{P_t} X_t - J_{SX} k \frac{1}{F} A_t \rho_{LX} \sigma X_t. \end{aligned} \tag{14}$$

If we conjecture that there exists a solution to the functional form $J(S_t, X_t) = I(X_t) \cdot (S_t^{1-\gamma}/1-\gamma)$ when $\varphi = 1$, and substitute it into equation (14), then the ordinary differential equation will have a solution of the form $I = \exp(Q_0 + Q_1 X_t)$. By rearranging that equation, we have two equations for Q_1 and Q_0 after collecting terms in X_t and 1. Full details are provided in Appendix.

We are now able to obtain the indirect utility function and the optimal dynamic asset allocation strategy for intertemporal pension fund management. The indirect utility function is

$$J(S_t, X_t) = I(X_t)\frac{S_t^{1-\gamma}}{1-\gamma} = \exp(Q_0 + Q_1 X_t)\frac{S_t^{1-\gamma}}{1-\gamma}, \tag{15}$$

while the optimal dynamic asset allocation strategy for intertemporal pension fund management is

$$n_t = \frac{1}{\gamma\iota}\frac{\mu_{P_t}}{\sigma^2_{P_t}} + \frac{Q_1}{\gamma\iota}\frac{\sigma\rho_{PX}}{\sigma^2_{P_t}} + k\frac{1}{F}\frac{\sigma_{P_t}\sigma_{L_t}\rho_{PL}}{\sigma^2_{P_t}}. \tag{16}$$

However, we have not really solved the model at this stage for two reasons. First, we have not considered the setting of the time-varying instantaneous expected risk premium in relation to the risky asset (μ_{P_t}). Second, the above solution is just a special case for our model setting when $\varphi = 1$. In the next section, we will use the perturbation method to find the general solution to our model.

3.2 Approximate results using perturbation methods

The basic idea behind the use of perturbation methods is that of formulating a general problem, on the condition that we find a particular case that has a known solution, and then use that particular case and its solution as a starting point for computing approximate solutions to nearby problems. This approach is an application of implicit function theorems, Taylor series expansions, and techniques from bifurcation theory and singularity theory. This method is widely used with much success in mathematical physics, particularly in quantum mechanics and general relativity theory (Judd, 1998).

In many financial economic models, determining the unknown function plays a key role in economic analysis under the assumption of a given functional form. However, the more generalized the model is, the more difficult it is to find a closed-form solution, especially in the case of an intertemporal consumption and portfolio choice problem with stochastic nonlinear partial differential equations. In spite of this, this situation has very recently begun to change as a result of several related developments. One of these developments has involved the use of perturbation methods in some special cases where closed-form solutions are derived for computing approximate solutions that will help make economic analysis more explicit. These methods offer analytical insights into investor behavior in models that fall outside the still-limited class that can be solved exactly (Campbell, 2000).

Judd and Guu (1997, 2000), Kogan and Uppal (2000), and Chacko and Viceira (2002), etc. have used this approach to solve dynamic economic or financial models. In the remainder of this paper, we will apply perturbation methods to solve our model. In the context of our problem, the insight we obtain is that the solution for the recursive utility function when $\varphi = 1$ provides a convenient starting point for performing the expansion. We apply the $\varphi = 1$ in the previous section as our starting point and compute our model around this solution.

For the setting of the time-varying instantaneous expected risk premium in relation to the risky asset, μ_{P_t}, we assume that $\mu_{P_t} = \mu_0 + \mu_1 \sigma^2_{P_t}$,[9] and use perturbation methods around the results of the previous section for computing our model. The Bellman equation can be expressed as:

$$
\begin{aligned}
0 = & -\frac{\rho^{\varphi}}{1-\varphi} I^{1+\frac{1-\varphi}{1-\gamma}} + \frac{\varphi}{1-\varphi}\rho I + \frac{1}{2}\frac{I}{\gamma}\left(\mu_0^2 X_t + \mu_1^2 \frac{1}{X_t} + 2\mu_0\mu_1\right) \\
& + Ir\iota - k\frac{1}{F}\iota\mu_L I + I_X \frac{1}{1-\gamma}\pi(m - X_t) + \frac{1}{2}\frac{(I_X)^2}{I}\frac{\sigma^2\rho^2_{PX}}{\gamma}X_t \\
& + \frac{1}{2}k^2\frac{1}{F^2}\iota^2 I\gamma\rho^2_{PL}X_t - \frac{1}{2}k^2\frac{1}{F^2}\iota^2 I\gamma X_t + k\frac{1}{F}\iota I\rho_{PL}(\mu_0 X_t + \mu_1) \\
& + k\frac{1}{F}\iota I_X\rho_{PX}\sigma\rho_{PL}X_t + \frac{1}{2}I_{XX}\frac{1}{1-\gamma}\sigma^2 X_t \\
& + \frac{1}{\gamma}I_X\rho_{PX}\sigma(\mu_0 X_t + \mu_1) - I_X k\frac{1}{F}\iota\rho_{LX}\sigma X_t.
\end{aligned} \tag{17}
$$

In general, the above equation cannot be computed in closed from. One approach is to obtain an asymptotic approximation to equation (17), where the expansion is performed by taking a first-order expansion of it around the elasticity of intertemporal substitution when $\varphi = 1$ (Campbell and Viceira (1999, 2001) and Chacko and Viceira (2002)),

$$
I^{1+\frac{1-\varphi}{1-\gamma}} \approx I + I^{1+\frac{1-\varphi}{1-\gamma}} \log\left(I \cdot \frac{(-1)}{1-\gamma}\right)\bigg|_{\varphi=1}(\varphi - 1) = I + \frac{1-\varphi}{1-\gamma} I \log I. \tag{18}
$$

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[9] Most of the recent studies on optimal asset allocation with time-varying volatility assume that expected excess returns on risky assets are constant. One of the reasons for this is that the expected excess return on the risky asset is often left unspecified as being constant in order to focus on the time-varying volatility. The other reason is to avoid blurring due to complicated computation. Only recently, this kind of constant specification has been improved with a natural extension by replacing the assumption of constant expected excess returns in order to allow the expected return on the risky asset to vary linearly with volatility or the standard deviation as in Liu (2001), and Chacko and Viceira (2002).
The basic idea behind the extension of the linear relationship specification with its conditional variance ($\sigma^2_{P_t}$) is similar to the spirit of the ARCH-in-mean model (Engle et al., 1987). The ARCH-in-mean model allows the mean of a sequence to depend linearly on its own conditional variance. The ARCH-M model' s original spirit is similar to the extension of the linear relationship specification of expected excess returns on the risky asset with its conditional variance.

By substituting (18) into equation (17), we guess that this equation has a solution of the form

$$\begin{aligned} I &= \exp(Q_0 + Q_1 \log X_t + Q_2 X_t) \\ &\approx \exp\left\{Q_0 + Q_1\left[-\log\pi + \log(\pi m - \sigma^2) + 1 - \frac{\pi m - \sigma^2}{\pi X_t}\right] + Q_2 X_t\right\}, \end{aligned} \tag{19}$$

where the second approximate equality is obtained from equation (7) by taking a first-order Taylor expansion around the long-term mean $m_{\sigma^2_{P_t}} = \{m - (1/2)[(1+\beta)/\beta](\sigma^2/\pi)\}^{-1}$ of the process of the stochastic volatility. From this we can derive the indirect utility function and the optimal dynamic asset allocation strategy for pension fund management without constraints when $\varphi = 1$. Full details of this are presented in Appendix.

The indirect utility function is

$$J(S_t, X_t) = I(X_t)\frac{S_t^{1-\gamma}}{1-\gamma} = \left[\exp(Q_0 + Q_1 \log X_t + Q_2 X_t)\right]\frac{S_t^{1-\gamma}}{1-\gamma}, \tag{20}$$

and the optimal dynamic asset allocation strategy for intertemporal pension fund management is

$$n_t = \frac{1}{\gamma\iota}\left(\mu_1 + \mu_0\frac{1}{\sigma^2_{P_t}}\right) + \frac{1}{\gamma\iota}\sigma\rho_{PX}\left(Q_1 + Q_2\frac{1}{\sigma^2_{P_t}}\right) + k\frac{1}{F}\frac{\sigma_{P_t}\sigma_{L_t}\rho_{PL}}{\sigma^2_{P_t}}. \tag{21}$$

Now we have explicitly solved the problem of the dynamic asset allocation strategy for intertemporal pension fund management with a time-varying expected return and volatility. In the next section, we provide analyses of our results.

4. ANALYSES OF THE OPTIMAL DYNAMIC ASSET ALLOCATION STRATEGY FOR INTERTEMPORAL PENSION FUND MANAGEMENT

The optimal dynamic asset allocation strategy for the pension fund in relation to the risky asset has three components: the myopic component, the intertemporal hedging component and the liability hedging component. First, the dependence of the myopic component is simple. It is an affine function of the reciprocal of the time-varying volatility and decreases with the coefficient of relative risk aversion. Since volatility is time varying, the myopic component is time varying, too. The position of the myopic component can be either positive or negative, depending on μ_0, μ_1 and the level of volatility.[10]

The intertemporal hedging component of the optimal asset allocation in relation to the risky asset is also an affine function of the reciprocal of the time-varying volatility with coefficients Q_1 and Q_2, and decreases with the coefficient of relative risk aversion. While Q_2 and Q_1 are the solutions to the two independent quadratic equations (A7) and (A8), respectively, Q_0 is the solution to (A9), given Q_1 and Q_2. When $\gamma > 1$ in the case of the coefficient Q_1, the equation (A8) has two real roots with opposite signs according to the quadratic equation theory. Furthermore, the value function J is maximized only by the solution associated with the negative root of the discriminant of the quadratic equation (A8), i.e. the positive root of equation (A8). The roots of the quadratic equation for Q_2 are always real provided that $|\mu_0(1/\gamma\iota)(F/k) + \rho_{PL}| > 1$ when $\gamma > 1$. These two roots also have opposite signs, and the value function is maximized only with the solution associated with the positive root of the discriminant of the quadratic equation (A7), i.e. the negative root of equation (A7).

Since $Q_1 > 0$, the sign of the intercept of the intertemporal hedging component is positive when both $\gamma > 1$ and $\rho_{PX} > 0$. This positive hedging component is independent of the level of the volatility, and is counteracted partially by the negative hedging demand from the effect of stochastic volatility. The intertemporal hedging component of the optimal asset allocation in relation to the risky asset is affected by the instantaneous correlation between the unexpected return on the risky asset and the

[10] A great many applications of ARCH-M (or GARCH-M) models to different stock index returns have been reported by numerous authors, such as French et al. (1987), Chou (1988), Friedman and Kuttner (1992) and Bollerslev et al. (1992), etc., just as most of these empirical studies have shown that the risk premium in relation to the risky asset is significantly an increasing function of its own conditional variance. From these results, we predict that the sign of μ_1 (the coefficient of $\sigma^2_{P_t}$) is positive. This prediction is intuitive because the greater the conditional variance of returns, the greater the compensation necessary to induce the agent to hold the risky asset (Enders, 1995). Some of these relevant empirical studies show that the intercept of the linear relationship between the expected excess returns on risky assets with their own conditional variances is positive, although not all of the empirical results show that the intercept is significantly different from zero. The positive sign of the intercept means that investors also need another kind of compensation to hold the risky asset apart from the compensation associated with the conditional variance risk measure. From this, we also predict that the sign of μ_0 is positive.

state variable (ρ_{PX}), which is also equal to $-\rho_{P\sigma^2_{P_t}}$, where $\rho_{P\sigma^2_{P_t}}$ is the instantaneous correlation between the unexpected return and the proportional change in the stochastic volatility of the risky asset. If $\rho_{PX} > 0$, i.e. $\rho_{P\sigma^2_{P_t}} < 0$, then the unexpected return on the risky asset will be low (the market situation is bad), and the state of the market uncertainty will be high. Since $Q_2 < 0$ when $\gamma > 1$, the positive instantaneous correlation between the unexpected return on the risky asset and the state variable (ρ_{PX}) implies that the pension fund will be characterized by a negative intertemporal hedging demand due to changes solely in the volatility of the risky asset, which lacks the ability to hedge against an increase in volatility.

Similar discussions are found in Liu (2001) and Chacko and Viceira (2002). However, in our generalized model, taking into consideration the pension fund liability also affects the intertemporal hedging component through the coefficient Q_2. In the previous section we assumed that the instantaneous correlation between the growth rate of the pension fund liability and the state variable is ρ_{LX} which is equal to $\rho_{L\sigma^2_{L_t}}$, given the setting of $\sigma_{L_t} = X_t^{1/2}$. $\rho_{L\sigma^2_{L_t}}$ is the instantaneous correlation between the growth rate of the pension fund liability and the proportional change in the stochastic volatility. The size of the negative hedging demand due to the effect of the stochastic volatility of the intertemporal hedging component is increasing with the instantaneous correlation between the growth rate of the pension fund liability and the proportional change in the stochastic volatility $(\rho_{L\sigma^2_{L_t}})$ when $\gamma > 1$. Since $Q_2 < 0$, we have $(\partial Q_2/\partial\rho_{LX}) = (\partial Q_2/\partial\rho_{L\sigma^2_{L_t}}) < 0$ (for a detailed analysis, please see Appendix). This implies that an increase in the instantaneous correlation $\rho_{L\sigma^2_{L_t}}$ increases the absolute value of Q_2 and hence the absolute value of the negative hedging demand due to the effect of the stochastic volatility of the intertemporal hedging component. This means that when high uncertainty results from the high growth rate of the pension fund liability, the pension fund will place more negative weight on the risky asset for hedging purposes.

In this paper, there is another distinct hedging component, the liability hedging component, which is directly affected by the pension fund liability. The liability hedging component of the optimal asset allocation in relation to a risky asset is a linear function of the reciprocal of the stochastic volatility. This liability hedging component also depends on the inverse of the funding ratio $(1/F)$, the importance attached to the liability (k), and the instantaneous correlation between the unexpected return on the risky asset and the growth rate of the pension fund liability (ρ_{PL}).

Most importantly, the liability hedging component is not like the myopic component, the intertemporal hedging component which decreases with the coefficient of relative risk aversion. It is in fact independent of either the risk aversion behavior or the preference. By contrast, it depends on the funding ratio. When the funding ratio is high, the liability hedging component will be low. This makes sense because the lower the relative magnitude of the pension fund liability, the less necessary it is to hedge against such risk.

The liability hedging component also depends on the instantaneous correlation between the unexpected return on the risky asset and the growth rate of the pension fund liability. The magnitude of the liability hedging component increases with the absolute value of this instantaneous correlation. When the instantaneous correlation is positive, the pension fund holds a positive position on the risky asset in relation to the liability hedging component, which provides partial hedging against liability changes and vice versa. If the instantaneous correlation is equal to zero, the return on the risky asset is uncorrelated with the growth rate of the pension fund liability. The risky asset thus provides no ability to hedge against an increase in liability, and the liability hedging component is unnecessary. This paper presents a generalized model of the importance to be attached to liability (k), without a specification regarding the value of k. When $k = 0$, it is a special case of asset-only optimization for a pension fund dynamic asset allocation strategy. However, in this model, the importance to be attached to the pension fund liability can vary between zero and one. The greater the desire to hedge the pension fund liability, the more importance (k) that is attached to liability, and the more weight that is assigned to the liability hedging component.

5. CONCLUSIONS

In this paper, we provide an analysis of the optimal dynamic asset allocation strategy for a pension fund under an economic environment with time-varying investment opportunities. We extend the static mean-variance optimization approach of Sharpe and Tint (1990), which includes liabilities in asset mix considerations for pension fund asset allocation strategies, to an intertemporal model.

Our closed-form solutions are obtained using perturbation methods. We derive the approximate analytical solution using perturbation methods of approximation around a particular point in the preference space. We show that the optimal dynamic asset allocation strategy for the pension fund in the risky asset has three components:

the myopic component, the intertemporal hedging component, and the liability hedging component. The myopic component and the intertemporal hedging component are all affine functions of the reciprocal of the time-varying volatility of the risky asset and are inversely correlated with the coefficient of relative risk aversion. We show that the intertemporal hedging demand of the pension fund is negative when it comes from changes that are solely in the risky asset's volatility, because in this case the risky asset lacks the ability to hedge against an increase in volatility. In addition, under the condition of the pension liability's high uncertainty and high growth rate, more negative weight will be assigned to the intertemporal component in hedging.

In this paper, we propose a new hedging component, the liability hedging component, of the dynamic asset allocation for pension fund management, in contrast to Merton (1993). The component is a hedging demand that arises from the desire to hedge against changes in the pension fund liability. The liability hedging component of the optimal asset allocation of the risky asset is time-varying, as it is a linear function of the reciprocal of the stochastic volatility of the risky asset. We find that a higher funding ratio is associated with a lower liability hedging component. This makes sense because the lower the relative magnitude of the pension fund liability, the less necessary it is to hedge against such risk. In addition, as the pension fund manager attaches greater importance to the liability, he increases the holding of the liability hedging component. The greater the desire to hedge the pension fund liability, the greater the importance to be attached to the liability (k), and the more weight that is assigned to the liability hedging component. The liability hedging component is independent of the attitude to risk aversion or the preference. Our intertemporal model can be extended to situations where the liability is not the item most necessarily involved, but where other assets beyond the decisions made in asset allocation form part of the beneficial owner's net worth as described by Sharpe and Tint (1990) using a static model. They are all based on the chosen concept of the net worth of the beneficial owner. More generally, after taking into consideration the relationships between assets included and excluded in the asset allocation field, or between liabilities and assets in the allocation field, we can apply this intertemporal model in discussing other assets that are normally excluded from asset allocation decisions.

APPENDIX

The derivation of the exact solution for the optimal dynamic asset allocation strategy when $\varphi = 1$

We conjecture that there exists a solution for the functional form $J(S_t, X_t) = I(X_t)$ $(S_t^{1-\gamma}/1-\gamma)$ when $\varphi = 1$, and substitute it into equation (14):

$$\begin{aligned}0 = &\left(\log\rho - \frac{\gamma}{1-\gamma}\log(1-\gamma) - \frac{1}{1-\gamma}\log I - 1\right)\rho I + \frac{1}{2}\frac{I\mu_{P_t}^2}{\gamma}X_t \\ &+ I\iota r - k\frac{1}{F}\mu_L I\iota + I_X\frac{1}{1-\gamma}\pi(m - X_t) + \frac{1}{2}\frac{(I_X)^2}{I}\frac{\sigma^2\rho_{PX}^2}{\gamma}X_t \\ &+ \frac{1}{2}k^2\frac{1}{F^2}I\gamma\iota^2\rho_{PL}^2X_t - \frac{1}{2}k^2\frac{1}{F^2}I\iota^2\gamma X_t + k\frac{1}{F}I\iota\mu_{P_t}\rho_{PL}X_t \\ &+ k\frac{1}{F}\iota I_X\rho_{PX}\sigma\rho_{PL}X_t + \frac{1}{2}I_{XX}\frac{1}{1-\gamma}\sigma^2X_t \\ &+ \frac{1}{\gamma}I_X\rho_{PX}\sigma\mu_{P_t}X_t - I_Xk\frac{1}{F}\iota\rho_{LX}\sigma X_t, \end{aligned} \tag{A1}$$

where $\iota \equiv 1/[1-(k/F)]$. The above ordinary differential equation has a solution that takes the form $I = \exp(Q_0 + Q_1X_t)$, so that (A1) can be expressed as

$$\begin{aligned}0 = &\left(\log\rho - \frac{\gamma}{1-\gamma}\log(1-\gamma) - \frac{1}{1-\gamma}(Q_0 + Q_1X_t) - 1\right)\rho \\ &+ \frac{1}{2}\frac{\mu_{P_t}^2}{\gamma}X_t + r\iota - k\frac{1}{F}\iota\mu_L + Q_1\frac{1}{1-\gamma}\pi(m - X_t) \\ &+ \frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma}Q_1^2X_t + \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2X_t - \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma X_t \\ &+ k\frac{1}{F}\iota\mu_{P_t}\rho_{PL}X_t + k\frac{1}{F}\iota Q_1\rho_{PX}\sigma\rho_{PL}X_t + \frac{1}{2}Q_1^2\frac{1}{1-\gamma}\sigma^2X_t \\ &+ \frac{1}{\gamma}Q_1\rho_{PX}\sigma\mu_{P_t}X_t - k\frac{1}{F}\iota Q_1\rho_{LX}\sigma X_t. \end{aligned} \tag{A2}$$

Rearranging the above equation, we have the following two equations for Q_0 and Q_1,

$$\left(\frac{1}{1-\gamma}\pi m\right) Q_1 - \left(\frac{1}{1-\gamma}\rho\right) Q_0 + \left(\log\rho - \frac{\gamma}{1-\gamma}\log(1-\gamma) - 1\right)\rho + r\iota - k\frac{1}{F}\iota\mu_L = 0, \quad \text{(A3)}$$

$$\left(\frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma} + \frac{1}{2}\frac{1}{1-\gamma}\sigma^2\right) Q_1^2 + \left(-\frac{1}{1-\gamma}\rho - \frac{1}{1-\gamma}\pi + k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL} + \frac{1}{\gamma}\rho_{PX}\sigma\mu_{P_t} - k\frac{1}{F}\iota\rho_{LX}\sigma\right) Q_1 + \frac{1}{2}\frac{\mu_{P_t}^2}{\gamma} + \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2 - \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma + k\frac{1}{F}\iota\mu_{P_t}\rho_{PL} = 0, \quad \text{(A4)}$$

and we also have:

$$Q_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where

$$\begin{aligned}
a &= \frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma} + \frac{1}{2}\frac{1}{1-\gamma}\sigma^2, \\
b &= -\frac{\rho+\pi}{1-\gamma} + k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL} + \frac{1}{\gamma}\rho_{PX}\sigma\mu_{P_t} - k\frac{1}{F}\iota\rho_{LX}\sigma, \\
c &= \frac{1}{2}\frac{\mu_{P_t}^2}{\gamma} + \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2 - \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma + k\frac{1}{F}\iota\mu_{P_t}\rho_{PL},
\end{aligned}$$

and

$$Q_0 = \left(\frac{\pi m}{\rho}\right) Q_1 + (1-\gamma)\log\rho - \gamma\log(1-\gamma) - 1 + \gamma + r\iota\frac{1-\gamma}{\rho} - k\frac{1}{F}\iota\mu_L\frac{1-\gamma}{\rho}. \quad \text{(A5)}$$

From this result, we can obtain the indirect utility function and the optimal dynamic asset allocation strategy for intertemporal pension fund management when $\varphi = 1$.

The derivation of the approximate results

After taking a first-order expansion of equation (17) around the elasticity of intertemporal substitution with $\varphi = 1$, we have

$$\begin{aligned} 0 = & -\frac{\rho^{\varphi}}{1-\varphi} - \frac{\rho^{\varphi}}{1-\gamma}\left\{Q_0 + Q_1\left[-\log\pi + \log(\pi m - \sigma^2) + 1 - \frac{\pi m - \sigma^2}{\pi X_t}\right]\right. \\ & \left. + Q_2 X_t\right\} + \frac{\varphi}{1-\varphi}\,\rho + \frac{1}{2\gamma}\left(\mu_0^2 X_t + \mu_1^2\frac{1}{X_t} + 2\mu_0\mu_1\right) + r\iota - k\frac{1}{F}\iota\mu_L \\ & + \frac{1}{1-\gamma}\pi(m - X_t)\left(Q_1\frac{1}{X_t} + Q_2\right) + \frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma}\left(Q_1\frac{1}{X_t} + Q_2\right)^2 X_t \\ & + \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2 X_t - \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma X_t + k\frac{1}{F}\iota\rho_{PL}(\mu_0 X_t + \mu_1) \\ & + k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL}\left(Q_1\frac{1}{X_t} + Q_2\right)X_t + \frac{1}{2}\frac{1}{1-\gamma}\sigma^2\left[\left(Q_1\frac{1}{X_t} + Q_2\right)^2\right. \\ & \left. - Q_1\frac{1}{X_t^2}\right]X_t + \frac{1}{\gamma}\rho_{PX}\sigma\,(\mu_0 X_t + \mu_1)\left(Q_1\frac{1}{X_t} + Q_2\right) \\ & - k\frac{1}{F}\iota\rho_{LX}\sigma\left(Q_1\frac{1}{X_t} + Q_2\right)X_t. \end{aligned} \tag{A6}$$

Rearranging the above equation, we have the following three equations for Q_2, Q_1 and Q_0:

$$\begin{aligned} & Q_2^2\left(\frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma} + \frac{1}{2}\frac{1}{1-\gamma}\sigma^2\right) + Q_2\left(-\frac{\rho^{\varphi}}{1-\gamma} - \frac{1}{1-\gamma}\pi + k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL}\right. \\ & \left. + \frac{1}{\gamma}\rho_{PX}\sigma\mu_0 - k\frac{1}{F}\iota\rho_{LX}\sigma\right) + \frac{1}{2\gamma}\mu_0^2 + \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2 \\ & + k\frac{1}{F}\iota\mu_0\rho_{PL} - \frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma = 0, \end{aligned} \tag{A7}$$

$$\begin{aligned} & Q_1^2\left(\frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma} + \frac{1}{2}\frac{1}{1-\gamma}\sigma^2\right) + Q_1\left(\frac{\rho^{\varphi}}{1-\gamma}\frac{\pi m - \sigma^2}{\pi} + \frac{\pi}{1-\gamma}m\right. \\ & \left. - \frac{1}{2}\frac{1}{1-\gamma}\sigma^2 + \frac{1}{\gamma}\rho_{PX}\sigma\mu_1\right) + \frac{1}{2\gamma}\mu_1^2 = 0, \end{aligned} \tag{A8}$$

$$
\begin{aligned}
Q_2\left(\frac{\pi}{1-\gamma}m+\frac{1}{\gamma}\rho_{PX}\sigma\mu_1\right)+Q_1\Bigg[\frac{\rho^{\varphi}}{1-\gamma}\log\pi-\frac{\rho^{\varphi}}{1-\gamma}\log(\pi m-\sigma^2)\\
-\frac{\rho^{\varphi}}{1-\gamma}-\frac{\pi}{1-\gamma}+k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL}+\frac{1}{\gamma}\rho_{PX}\sigma\mu_0-k\frac{1}{F}\iota\rho_{LX}\sigma\Bigg]\\
-Q_0\left(\frac{\rho^{\varphi}}{1-\gamma}\right)+Q_1Q_2\left(\frac{1}{\gamma}\sigma^2\rho_{PX}^2+\frac{1}{1-\gamma}\sigma^2\right)-\frac{\rho^{\varphi}}{1-\varphi}+\frac{\varphi}{1-\varphi}\rho\\
+\frac{1}{\gamma}\mu_0\mu_1+r\iota-k\frac{1}{F}\iota\mu_L+k\frac{1}{F}\iota\rho_{PL}\mu_1=0, \quad \text{(A9)}
\end{aligned}
$$

where Q_2 and Q_1 can be solved from quadratic equations (A7) and (A8), respectively, and Q_0 can be solved from the linear equation (A9), given Q_1 and Q_2.

Analysis of $(\partial Q_2/\partial\rho_{LX})$

In equation (A7),

$$
\begin{aligned}
Q_2^2\left(\frac{1}{2}\frac{\sigma^2\rho_{PX}^2}{\gamma}+\frac{1}{2}\frac{1}{1-\gamma}\sigma^2\right)+Q_2\Bigg(-\frac{\rho^{\varphi}}{1-\gamma}-\frac{1}{1-\gamma}\pi+k\frac{1}{F}\iota\rho_{PX}\sigma\rho_{PL}\\
+\frac{1}{\gamma}\rho_{PX}\sigma\mu_0-k\frac{1}{F}\iota\rho_{LX}\sigma\Bigg)+\frac{1}{2\gamma}\mu_0^2+\frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma\rho_{PL}^2\\
+k\frac{1}{F}\iota\mu_0\rho_{PL}-\frac{1}{2}k^2\frac{1}{F^2}\iota^2\gamma=0. \qquad \text{(A7)}
\end{aligned}
$$

By letting A, B, and C refer to the coefficients associated with Q_2^2, Q_2^1, and Q_2^0, we obtain

$$
\phi(Q_2)=AQ_2^2+BQ_2+C. \qquad \text{(A7')}
$$

Totally differentiating (A7) with respect to ρ_{LX} yields

$$
\frac{\partial Q_2}{\partial\rho_{LX}}=\frac{\dfrac{\partial\phi(Q_2)}{\partial\rho_{LX}}}{\dfrac{-\partial\phi(Q_2)}{\partial Q_2}}<0,
$$

since $\partial\phi(Q_2)/\partial\rho_{LX} = -(k\iota\sigma/F)Q_2 > 0$, $(\because Q_2 < 0)$ and

$$\frac{-\partial\rho(Q_2)}{\partial Q_2} = -(2AQ_2 + B) = -\left[2A\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) + B\right]$$
$$= -\sqrt{B^2-4AC} < 0.$$

(This result is obtained by choosing the solution associated with the negative root of equation (A7), i.e. choosing the positive root of the discriminant of the quadratic equation (A7) as discussed earlier.)

Examples and economic intuition of this paper

We give two numerical examples in this appendix. In the first example, we set $\pi = 0.2493$, $m = 27.1174$, $\sigma = 0.8063$, $\rho_{PX} = 0.8775$, $\rho = 0.0400$, $k = 0.9000$, $1/F = 0.0900$, $\rho_{PL} = 0.9000$, $\mu_0 = 0.0822$, $\rho_{LX} = 0.6500$, $\mu_1 = 0.0025$, $\varphi = 0.100$, and $\sigma_{P_t} = 0.2500$. In the second example, instead of setting a positive value, we set a negative instantaneous correlation $\rho_{PL} = -0.1$ between the unexpected return on the risky asset and the growth rate of the pension liability.

(1) With these two examples, we show that the optimal dynamic asset allocation strategy for the pension fund in relation to the risky asset is the summation of the myopic component, the intertemporal hedging component and the liability hedging component.

(2) In Figures A1 to A5, the myopic component and the intertemporal hedging component are decreasing with the coefficient of relative risk aversion, as is the optimal dynamic asset allocation strategy for the pension fund in terms of the risky asset. In addition, the myopic component, the intertemporal hedging component and the liability hedging component are all decreasing with the time-varying volatility, as is the optimal dynamic asset allocation strategy for the pension fund in terms of the risky asset.

Case 1:

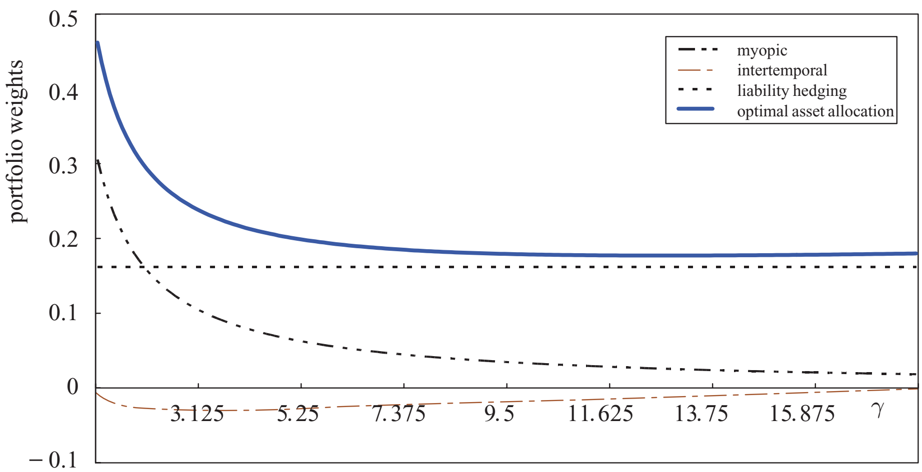

Figure A1 The Optimal Dynamic Asset Allocation Strategy for Pension Fund Management in Terms of the Risky Asset and Its Components in Relation to γ

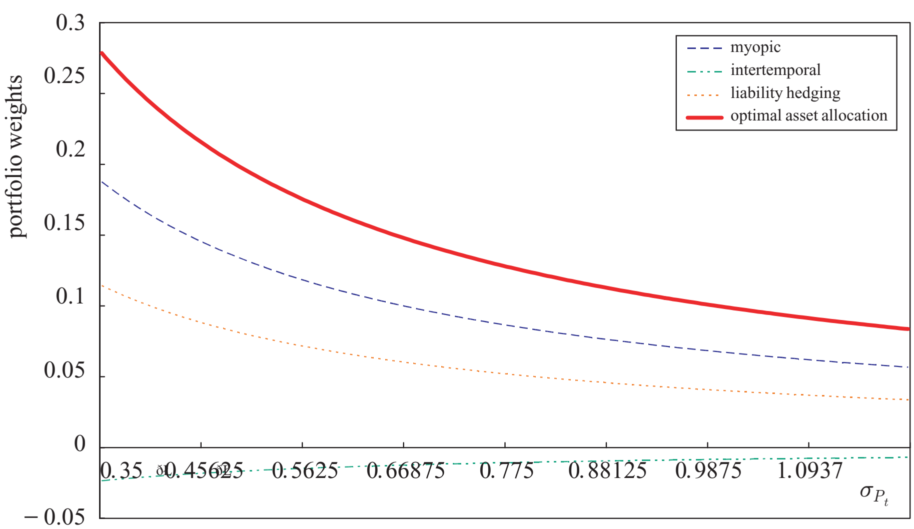

Figure A2 The Optimal Dynamic Asset Allocation Strategy for Pension Fund Management in Terms of the Risky Asset and Its Components in Relation to σ_{P_t}

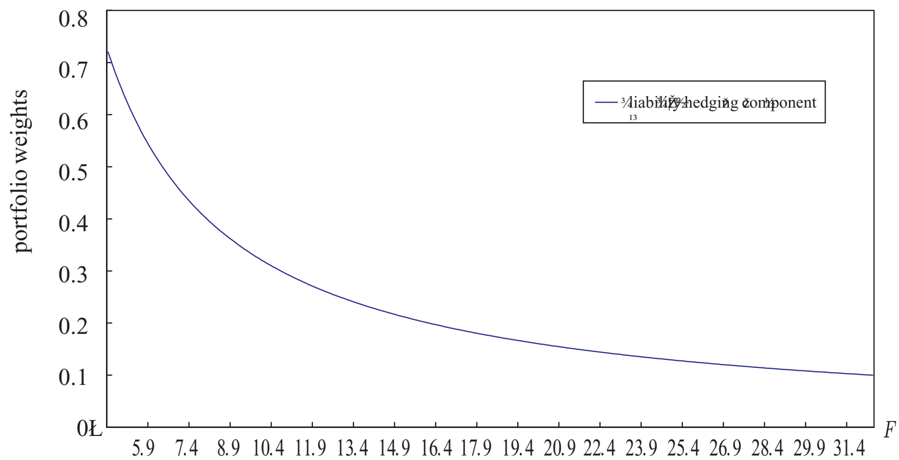

Figure A3 The Optimal Liability Hedging Component for Pension Fund Management in Relation to the Funding Ratio F

Case 2:

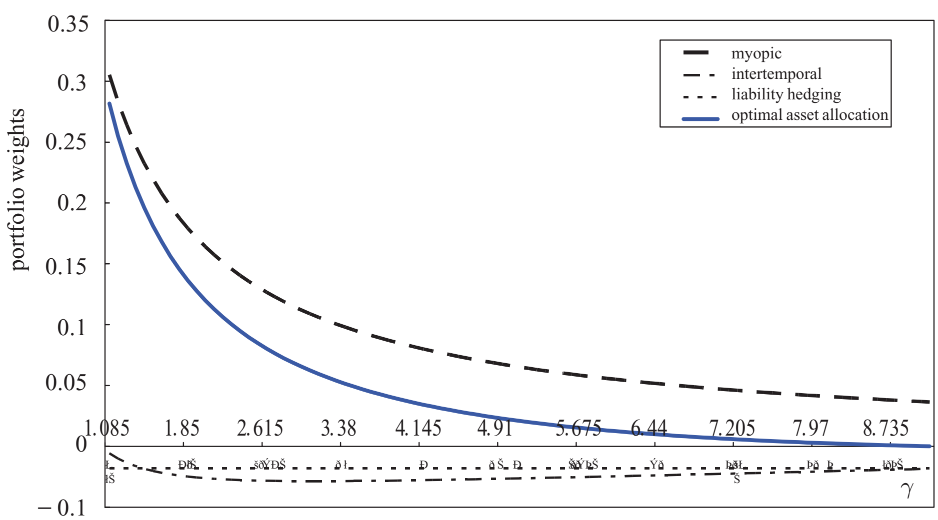

Figure A4 The Optimal Dynamic Asset Allocation Strategy for Pension Fund Management in Terms of the Risky Asset and Its Components in Relation to γ

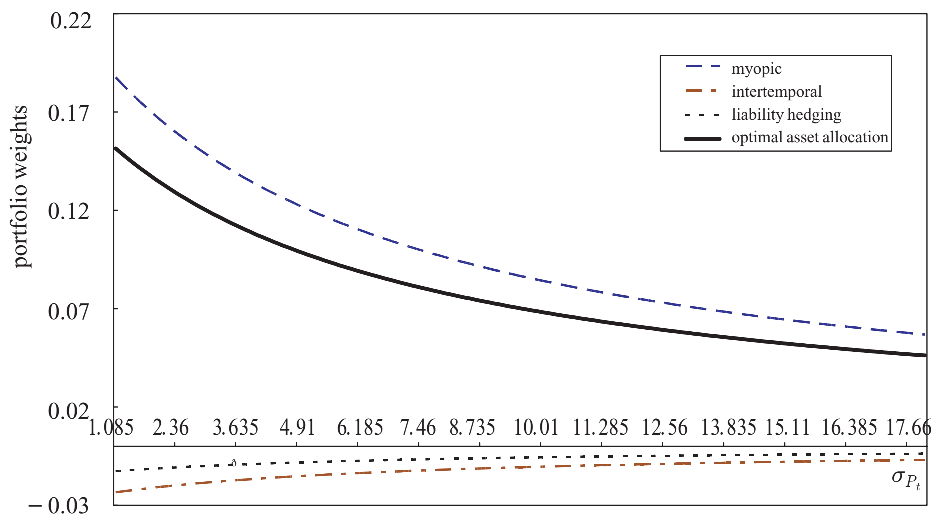

Figure A5 The Optimal Dynamic Asset Allocation Strategy for Pension Fund Management in Terms of the Risky Asset and Its Components in Relation to σ_{P_t}

(3) We set $\rho_{PX} = 0.8775$, i.e. $\rho_{PX} > 0$, i.e. $\rho_{P\sigma^2_{P_t}} < 0$, meaning that when the unexpected return on the risky asset is low (the market situation is bad), market uncertainty will be high. Since $Q_2 < 0$, the positive instantaneous correlation between the unexpected return on the risky asset and the state variable (ρ_{PX}) implies that the pension fund will hold a negative intertemporal hedging demand coming from changes solely in the volatility of the risky asset because it lacks the hedging ability against an increase in volatility. The magnitude of the negative intertemporal hedging demand increases with the extent of the pension fund manager's risk aversion.

(4) However, when the coefficient of relative risk aversion is increasing more and more, the intertemporal hedging demand will be dominated by this factor. As in the case of the intertemporal hedging demand in equation (21), it goes to zero when the coefficient of relative risk aversion approaches infinity.

(5) The liability hedging component depends on the funding ratio. When the funding ratio is high, the liability hedging component will be low. This makes sense because the lower the relative magnitude of the pension liability, the less necessary it is to hedge against such risk.

The acceptability of the approximation error in this paper

The perturbation method for finding the approximate solution is applied only to certain special cases. In this appendix, we provide numerical evidence to support the view that the approximation error is acceptable in this model.

Table A1 displays the average percentages of approximation errors by simulating 1,000 samples for each condition. These average percentages are calculated with different combinations of coefficients of relative risk aversion γ and an elasticity of intertemporal substitution departing from one, i.e. when $\varphi = 0.1, 0.5, 5.0$ and 10 $(\varphi > 0)$. We find that the average percentages of approximation errors in our model do not differ from zero until the second (even the fourth) digit after the decimal point. This result supports the view that the approximation error is on average acceptable in this model.

Table A1 Average Percentages of Approximation Errors of the Optimal Asset Allocation in Terms of the Risky Asset

γ	φ				
	10	5	1	0.5	0.1
1.085	0.0016	0.0016	0.0000	0.0035	0.0070
2.955	0.0111	0.0111	0.0000	0.0268	0.0563
4.825	0.0114	0.0113	0.0000	0.0280	0.0599
6.695	0.0101	0.0100	0.0000	0.0252	0.0545
8.565	0.0083	0.0083	0.0000	0.0211	0.0462
10.435	0.0066	0.0065	0.0000	0.0168	0.0370
12.305	0.0048	0.0048	0.0000	0.0124	0.0277
14.175	0.0033	0.0032	0.0000	0.0083	0.0185
16.045	0.0017	0.0017	0.0000	0.0044	0.0098
18.000	0.0002	0.0002	0.0000	0.0005	0.0012

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時變波動度下跨期退休基金管理之動態資產配置策略

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摘　要

本文主要在探討時變環境下,跨期退休基金管理之最適動態資產配置策略。在納入退休基金管理中二項重要的特性: 多期投資與退休基金負債的考量後, 提出退休基金的最適跨期資產配置策略, 本文同時利用 Sharpe and Tint (1990) 所提出的彈性考量退休基金負債的觀念, 納入本文之模型設定當中, 使本文能同時涵蓋不同退休基金管理者之不同負債考量進入資產配置之最適配置策略。本文所提出之跨期退休基金管理之最適動態資產配置策略,除了包含單期與面對時變環境下之跨期避險成分外, 更提出退休基金管理者如何依據其退休基金之特性, 建構其資產配置中之退休基金負債避險成分。