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Resolving the Asset Allocation Puzzle with Intertemporal Hedging and Nontraded Assets in the Stochastic Environment

Simon H. Yen
Professor, Department of Finance,
National Chengchi University

Yuan-Hung Hsu Ku *
Ph.D. Candidate, Department of Finance,
National Chengchi University

ABSTRACT

Canner, Mankiw and Weil (1997) point out that the popular financial advisors on portfolio allocation among cash, bonds and stocks appear not to follow the mutual-fund separation theorem and call the inconsistency between separation theorem and popular financial advice “an asset allocation puzzle.” For solving the asset allocation puzzle, we provide an analysis of the optimal dynamic asset allocation strategy for a long-horizon investor who has nontraded assets under an economic environment with stochastic investment opportunities and incomplete financial markets. We propose another distinguishing hedging component of the dynamic asset allocation for the stock index fund: the human capital hedging component, the hedging demand which characterizes the demand arising from the desire to hedge against changes in the labor income in contrast to Merton (1973). When we incorporate nontraded assets with intertemporal hedging, we can solve the asset allocation puzzle successfully.

JEL classification: G12
Keywords: Asset allocation puzzle, stochastic opportunities, incomplete markets, intertemporal model, perturbation method

Ⅰ. Introduction

Asset allocation strategy is a key component of the investment process. The textbook mutual-fund separation theorem states that all rational investors should divide their assets between a riskless asset and a single mutual-fund of risky assets which is the same for all investors; more conservative investors should only increase the ratio of riskless asset to the single risky mutual-fund, but the relative holdings of risky assets should not be changed according to investors' risk aversion.

Recently, Canner, Mankiw and Weil (1997) (CMW hereafter) point out that popular financial advisors on portfolio allocation among cash, bonds and stocks

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appear not to follow the mutual-fund separation theorem. In contrast to the mutual-fund separation theorem, these popular financial advisors recommend that more risk-averse investors should hold a higher ratio of bonds to stocks. These asset allocation recommendations are contrary to the mutual-fund separation theorem, which states that the relative holdings of risky assets should be independent from investors' risk aversion and all investors should hold the same composition of risky assets. CMW call the inconsistency between separation theorem and popular financial advice “an asset allocation puzzle”. They explore various possible explanations for this puzzle and conclude that explaining popular financial advice is difficult using models of fully rational investors. “Although we have not been able to explain popular advice within a rational model, it is possible that others will succeed where we have failed” (CMW, 1997).

Recently, several research efforts have attempted to explain the asset allocation puzzle by theoretical models or empirical data. Elton and Gruber (2000) examine the data of the recommendations of some leading financial experts to see if they are consistent with modern portfolio theory for the asset allocation decision. They point out that although CMW argue that the advisors' recommendation was inconsistent with modern portfolio theory and, hence, was irrational, but they show that CMW’s conclusions are wrong. They find out whether or not short sale constraint, and different data sets employed can lead to very different relationships between the ratios of bonds to stocks with risk aversion attitude. Under reasonable assumptions, they show that the asset allocation recommendations employing the data used in CMW’s study are consistent with modern portfolio theory.

Brennan and Xia (2000) develop a dynamic portfolio optimization model to resolve the asset allocation puzzle. In their model, the long-horizon investor is concerned with maximizing the expected value of an iso-elastic utility from terminal wealth. In their model, the bond price covary negatively with the stochastic interest rate process, and thus it causes the investor to face the risk that the investment opportunity set as measured by the stochastic interest rate will move adversely. Their model proposes that hedging demand on the instantaneous riskless rate and changes in the central tendency of the rate rationalize the asset allocation puzzle by supporting the financial advice. However, in this model they admit that “the model does not provide general predictions about the relation between risk aversion and cash holdings”, though this model can identify the positive relation between optimal bond-stock ratio and risk aversion.

Bajeux-Besnainou, Jordan and Portait (2001) provide a continuous-time model of stochastic interest rates and stock prices involving one state variable to support the rationality of popular financial advice about asset allocations of stocks, bonds and cash. In their Vasicek (1977) type model for investors displaying hyperbolic absolute risk aversion (HARA) utility function, the two key assumptions of complete markets and investor's horizon exceeding the maturity of cash explain the asset allocation puzzle.

In these recent papers that attempted to solve the asset allocation puzzle, all assume that investors invest in traded assets only. In the real world, not all the compositions of personal wealth are traded as readily as stocks and bonds. However, one of the most important assumptions in the mutual-fund separation theorem is that all assets are traded. In this paper, we assume there are exiting traded and nontraded assets. In the composition of personal wealth, human capital—the present value of future labor earning－is probably the most important nontraded asset (CMW, 1997). In addition, human capital also forms a substantial part of the total capital in an economy whereas stocks form only a small part of the aggregated wealth (Mayers, 1972). Jagannathan and Wang (1996) show that the monthly per capita income in the United States from salaries and wages was about 63 percent of the monthly personal income from all sources whereas from dividends there was less than 3 percent during the period from 1959 to 1992. Human capital is quite important when explaining average return, and there is a significant link between the return on financial assets and that on human capital. If investors hold nontraded assets such as human capital with a heavy weight, and they are concerned about their total return, the optimal quantities of traded assets will reflect their covariances with nontraded assets.

In this paper, we provide a dynamic asset allocation strategy model for investors with time-varying stochastic expected return and volatility in incomplete markets. We incorporate the human capital and the intertemporal hedging components to resolve the asset allocation puzzle. CMW (1997) suggest that considerations of intertemporal hedging and nontraded assets might help explain the asset allocation puzzle. However, we do not specifically assume the instantaneous riskless interest rate to be in a Vasicek (1977) type market as Bajeux-Besnainou et al. (2001) do. Nor do the bonds covary negatively with expectations about future stochastic interest rate as Brennan and Xia (2000) in our model. In contrast to Bajeux-Besnainou et al. (2001) and Brennan and Xia (2000) who employ the power utility function, we adopt Duffie and Epstein's (1992b) recursive utility function which is a generalization of the standard power utility function. Furthermore, our model also extends theirs in two different ways. First, we set the risky asset returns with time-varying conditional variance and mean-reverting expected returns. More recently, empirical research about asset returns have show with time-varying volatility, i.e. the conditional variance of asset returns is not constant over time (Campbell 1987, Harvey 1989, 1991, Glosten, Jagannathan, and Runkle 1993), and asset returns are mean-reverting (Fama and French 1988, Poterba and Summers 1988, Cecchetti, Lam and Mark, 1990, Chen and Jeon 1998, Balvers, Wu, and Gilliland 2000). They all induce time variation in the investment opportunity.

Secondly, the setting is incomplete market. There are some papers to explain the problem of consumption and portfolio choice in an incomplete market setting. Duffie et al. (1997) show the existence of optimal policies in an incomplete market setting using the stochastic dynamic programming approach for CRRA utility functions. Cuoco (1997) explores the problem of the existence of nontraded stochastic labor income and portfolio constraints, and discuss the optimal consumption and portfolio choice in that situation. Fleming and Zariphopoulou (1991), and Vila and Zariphopoulou (1997) investigate optimal consumption and portfolio choice with borrowing constraints (Campbell, 2000). In this paper, the incomplete market setting is from we assume that the state variable is not perfectly instantaneously correlated with the risky asset return.

In this generalized intertemporal asset allocation model with nontraded assets under stochastic environment, Merton's approach (1971, 1973) could not be used to drive a closed-form solution by solving a nonlinear differential equation on the hedging portfolio. Recently, some have begun to work on it. Liu (2001) characterizes the closed-form solution of optimal portfolio under the setting of short rate following the square root diffusion process and stock returns exhibiting stochastic volatility or predictability. He shows that Merton's optimal dynamic portfolio selection problem expressed by a nonlinear partial differential equation can be reduced to the system of ordinary differential equations. Chacko and Viceira (2002) derive the optimal portfolio with stochastic volatility in incomplete market by an approximate solution method finding a solution around the unconditional mean of the log consumption-wealth ratio. Campbell and Viceira (2001) use log-linear approximation to characterize the portfolio demand under a stochastic opportunity set in a discrete-time model. And they find that the ratio of bonds to stocks in the optimal portfolio increases with risk aversion. In this way, their model also helps to resolve the asset allocation puzzle.

In this paper, we will follow the stochastic opportunity setting of Chacko and Viceira (2002) to derive the optimal asset allocation strategy for an infinitely lived investor. However, the only one risky tradable asset model of Chacko and Viceira (2002) mainly derive the explicit solution on a log-linear expansion of the consumption-wealth ratio around its unconditional mean. Their solution is obtained around a particular point in the state space—the unconditional mean of the log consumption-wealth ratio. In contrast to them, we will derive the exact solution by the perturbation method of approximation around a particular point in the preference space. This paper is organized as follows. Section 2 describes the model and the environment used in this paper. Section 3 develops the optimal dynamic asset allocation strategy for the long-horizon investor and resolves the asset allocation puzzle. Section 4 provides some analyses on the optimal dynamic asset allocation strategy and its effects for the solving of asset allocation puzzle. Finally, section 5 presents some conclusions.

Ⅱ. The model

2.1 Asset price and human capital dynamics

There are three kinds of assets available for trading in the economy. The first kind of assets is the stock index fund, if P_t denotes the price of the stock index fund at time t, then the return dynamics for this risky asset is given by

$$\frac{dP_t}{P_t} = (r + « _{P_t})dt + \pm_{P_t} dZ_P, \tag{1}$$

where $«_{P_t}$ is the time-varying instantaneous expected risk premium on the stock index fund; $\pm_{P_t}$ is the time-varying instantaneous standard deviation of the return on the stock index fund; and dZ_P is a Wiener process. The second kind of asset is the bond fund. B_t denotes the price of the bond fund at time t, and its stochastic process is given by

$$\frac{dB_t}{B_t} = (r + «_B)dt + \pm_{B_t} dZ_B, \tag{2}$$

where $«_B$ is the instantaneous expected risk premium on the bond fund, $\pm_{P_t}$ is the time-varying instantaneous standard deviation of the return on the bond fund, and dZ_B is a Wiener process independent with dZ_P. The last asset is an instantaneous risk-free money-market fund, which pays the interest rate r. We assume that the short rate is constant to focus on stochastic volatility.

The long-horizon investor is not assumed to invest in traded assets only. Human capital is probably the most important nontraded asset of the greater part of investors. We assume the labor supply of this long-horizon investor is inelastically and the labor income is exogenous. Under the above setting, the value of human capital will be perfectly correlated with labor income, meaning that the return of human capital can be expressed as the proportional change in labor income (Fama and Schwert, 1977; Viceira, 2001).

Next, we assume that the dynamic return on labor income (human capital) for the long-horizon investor is described by the following stochastic differential equation,

$$\frac{dH_t}{H_t} = «_H dt + \pm_{H_t} dZ_H, \tag{3}$$

where $«_H$ is the instantaneous expected return of the labor income; $\pm_{H_t}$ is the time-varying instantaneous standard deviation of the return on the investor's labor income; and dZ_H is a Wiener process. We assume the unexpected returns on the stock index fund are instantaneously and positively correlated with human capital return, and this instantaneous correlation is given by $°_{PH}$.

2.2 Volatility process and the investment opportunity set

Our convention is to denote stochastic variables with a subscript "t"; thus, we let the expected excess return and the conditional variance of the risky assets vary stochastically over time. For the above setting, the investment opportunity is time-varying. We assume that the instantaneous variance process of return on the stock index fund is

$$\pm_{P_t}^2 = X_t^{-\frac{1}{}}, \tag{4}$$

and we assume X_t has the following mean-reverting process:

$$dX_t = ®(m - X_t)dt + \pm\sqrt{X_t}\,dZ_X, \tag{5}$$

where m is the long-term mean and $®$ is the reversion parameter of the mean-reverting process. We apply Ito's lemma to (5) and find that (5) is equivalent to directly assume that the mean-reverting process is as stated in the following stochastic differential equation:

$$\frac{d\pm^2_{P_t}}{\pm^2_{P_t}} = \left[\frac{1}{}® - \frac{1}{}®\pm^2_{P_t}\left(m - \frac{1}{2}\frac{1+}{}\frac{\pm^2}{®}\right)\right]dt - \frac{1}{}\pm(\pm_{P_t})\ \ dZ_X$$

$$= \left[\frac{1}{}®\left(m - \frac{1}{2}\frac{1+}{}\frac{\pm^2}{®}\right)\right]\cdot\left[\left(m - \frac{1}{2}\frac{1+}{}\frac{\pm^2}{®}\right)^{-1} - \pm^2_{P_t}\right]dt - \frac{1}{}\pm(\pm_{P_t})\ \ dZ_X$$

$$\equiv ®_{\pm^2_{P_t}}(m_{\pm^2_{P_t}} - \pm^2_{P_t})dt - \frac{1}{}\pm(\pm_{P_t})\ \ dZ_X, \tag{6}$$

where $®_{\pm^2_{P_t}} = \frac{1}{}®\cdot m^{-1}_{\pm^2_{P_t}}$; $m_{\pm^2_{P_t}} = \left(m - \frac{1}{2}\frac{1+}{}\frac{\pm^2}{®}\right)^{-1}$ and $\pm^2_{P_t}$ is well-defined provided $m \geq 0$. From the setting of X_t, volatility ($\pm^2_{P_t}$) is mean-reverting also. For simplifying, we assume the case of $=1$ in this article; this volatility dynamics specified is the same as proposed by Chacko and Viceira (2002). We also assume that $\pm_{B_t} = {}^{a}X_t^{-\frac{1}{2}}$ and $\pm_{H_t} = {}^{-}X_t^{\frac{1}{2}}$ for the time-varying instantaneous standard deviation of the return on the bond fund and the human capital.

2.3 Utility of the long-horizon investor

Following Campbell (1993), Campbell and Viceira (1999, 2001) and Chacko and Viceira (2002), we use the recursive utility to describe investor's preference. Epstein and Zin (1989, 1991) derive a parameterization of recursive utility in a discrete-time setting, while Duffie and Epstein (1992a, 1992b) offer a continuous-time setting. We adopt the Duffie and Epstein (1992b) describing investor preference by a recursive utility function which is a generalization of the standard and the time-separable power utility function that separates the elasticity of the intertemporal substitution parameter from relative risk aversion coefficient. It means that the power utility is just a special case of the recursive utility function when the elasticity of the intertemporal substitution parameter is just the inverse of the relative risk aversion coefficient.

$$J = E_t[\int_t^\infty f(C_2, J_2)\, d^2], \tag{7}$$

where $f(C_2, J_2)$ is a normalized aggregator of investor's current consumption (C_2) and utility has the following form:

$$f(C,J) = °(1-\frac{1}{¨})^{-1}(1-¥)J\left[\left(\frac{C}{((1-¥)J)^{\frac{1}{1-¥}}}\right)^{1-\frac{1}{¨}} - 1\right], \tag{8}$$

where ¥ is the coefficient of relative risk aversion, ° is the rate of time preference and ¨ is the elasticity of intertemporal substitution, they are all larger than zero.

We assume the wealth of an economic agent comes from only two sources: financial wealth and human capital wealth. We also further assume that the economic agent receives $(1-¦_t)$ of his total income from finance incomes and $¦_t$ from income related to human capital (or labor income). The investor's objective is to maximize the expected lifetime utility described above subject to the following intertemporal budget constraint,

$$dW_t = \left[(1-¦_t)\left(n_t^P «_{P_t} + n_t^B «_B + r\right)W_t - C_t\right] dt + (1-¦_t)\left(n_t^P ±_{P_t} W_t dZ_P + n_t^B ±_{B_t} W_t dZ_B\right) + ¦_t\left(«_H W_t dt + ±_{H_t} W_t dZ_H\right), \tag{9}$$

where W_t represents the investor's total wealth, n_t^P and n_t^B are the fractions of the investor's financial wealth allocated to the stock index fund and the bond fund at time t, respectively.

Ⅲ. Optimal dynamic asset allocation strategy for bonds, stocks and cash

3.1 Special case of the optimal dynamic asset allocation strategy

The value function of the problem (J) is to maximize the investor's expected lifetime utility. The principle of optimality leads to the following Bellman equation for the utility function. Under the above setting, the Bellman equation will satisfy

$$\begin{aligned} 0 = \sup_{n^P, n^B, C} \Big\{ & f(C_2, J_2) + J_W\left[(1-¦_t)\left(n_t^P «_{P_t} + n_t^B «_B + r\right)W_t - C_t + ¦_t «_H W_t\right] \\ & + J_X ®(m - X_t) + \frac{1}{2} J_{WW}\Big[(1-¦_t)^2\left((n_t^P)^2 ±_{P_t}^2 + (n_t^B)^2 ±_{B_t}^2\right) + ¦_t^2 ±_{H_t}^2 \\ & + 2¦_t(1-¦_t) n_t^P ±_{P_t} ±_{H_t} °_{PH}\Big] W_t^2 + \frac{1}{2} J_{XX} ±^2 X_t + J_{WX}\Big[(1-¦_t) \\ & \left(n_t^P ± \sqrt{X_t} ±_{P_t} °_{PX} + n_t^B ± \sqrt{X_t} ±_{B_t} °_{BX}\right) + ¦_t\left(± \sqrt{X_t} ±_{H_t} °_{HX}\right)\Big] W_t \Big\}, \end{aligned} \tag{10}$$

where J_W, J_X denote the derivatives of J with respect to W and X, respectively, and we use similar notation for higher derivatives. We also note that $°_{PX}$, $°_{BX}$ and $°_{HX}$ are instantaneous correlations of the unexpected return on stock index fund, bond fund and human capital with state variable X_t, respectively, and $°_{PH}$ is the instantaneous correlation of the unexpected return on stock with human capital returns.

The first order conditions for this equation are

$$C_t = J_W^{¨} J^{\frac{1-¨¥}{1-¥}} °^{¨} (1-¥)^{\frac{1-¨¥}{1-¥}}, \tag{11}$$

$$n_t^B = \frac{-J_W}{J_{WW} W_t} \frac{1}{1-¦_t} \frac{«_B}{±_{B_t}^2} + \frac{-J_{WX}}{J_{WW} W_t} \frac{1}{1-¦_t} \frac{± \sqrt{X_t} ±_{B_t} °_{BX}}{±_{B_t}^2}, \tag{12a}$$

$$n_t^P = \frac{-J_W}{J_{WW}W_t}\frac{1}{1-\omega_t}\frac{\mu_{P_t}}{\sigma^2_{P_t}} + \frac{-J_{WX}}{J_{WW}W_t}\frac{1}{1-\omega_t}\frac{\sigma\sqrt{X_t}\sigma_{P_t}\rho_{PX}}{\sigma^2_{P_t}} - \frac{\omega_t}{1-\omega_t}\frac{\sigma_{P_t}\sigma_{H_t}\rho_{PH}}{\sigma^2_{P_t}} \text{ .(12b)}$$

The optimal portfolio allocation in the stock index fund has three components, while the bond fund has only two. In both optimal portfolio allocation in the stock index fund and bond fund, their first terms are the mean-variance portfolio weight. They are for an investor who invests only in a single period horizon or under constant investment opportunity set, the myopic demands. Their second terms are the intertemporal hedging demands which characterize the demand arising from the desire to hedge against changes in the investment opportunity set. The hedging demands are determined by the product of the "beta" of the time-varying state variable with respect to the risky assets and the instantaneous rates of changes of the value function. In the optimal portfolio allocation of the stock index fund, its third term is the hedging demand which characterizes the demand arising from the desire to hedge against changes in the human capital return. The nontraded asset (human capital) hedging demand is determined by the product of the "beta" of the human capital return with respect to the stock index fund and the relative weight of labor income with financial income.

In fact, the first order conditions for our problem are not explicit solutions unless we know the complicated indirect utility function. Substituting the first-order solutions back into the Bellman equation and rearranging, we get

$$\begin{aligned}
0 = {} & f(C(J),J) - J_W C(J) + J_W \omega_t \mu_H W_t - \frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{\mu_{P_t}^2}{\sigma_{P_t}^2} + J_W r W(1-\omega_t) + J_X \kappa(m - X_t) \\
& + \frac{1}{2} J_{WW} W_t^2 \omega_t^2 \sigma_{H_t}^2 - J_W W \omega_t \sigma_{H_t}\rho_{PH}\frac{\mu_{P_t}}{\sigma_{P_t}} - \frac{1}{2} J_{WW} W_t^2 \omega_t^2 \sigma_{H_t}^2 \rho_{PH}^2 + \frac{1}{2} J_{XX}\sigma^2 X_t \\
& - \frac{J_W}{J_{WW}} J_{WX}\sigma\sqrt{X_t}\rho_{PX}\frac{\mu_{P_t}}{\sigma_{P_t}} - J_{WX}\omega_t\sigma_{H_t}\rho_{PH}\sigma\sqrt{X_t}\rho_{PX}W_t \\
& - \frac{1}{2}\frac{(J_{WX})^2\sigma^2\rho_{PX}^2}{J_{WW}}X_t + J_{WX}\omega_t\sigma_{H_t}\sigma\sqrt{X_t}\rho_{HX}W_t - \frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{\mu_B^2}{\sigma_{B_t}^2} \\
& - \frac{J_W}{J_{WW}} J_{WX}\sigma\sqrt{X_t}\rho_{BX}\frac{\mu_B}{\sigma_{B_t}} - \frac{1}{2}\frac{(J_{WX})^2\sigma^2\rho_{BX}^2}{J_{WW}}X_t .
\end{aligned} \qquad (13)$$

We conjecture taking a solution of the functional form $J(W_t, X_t) = I(X_t)\frac{W_t^{1-\gamma}}{1-\gamma}$ when $\psi = 1$, and substituting it into the above equation (13)

$$0=\left(\log ° -\frac{¥}{1-¥}\log(1-¥)-\frac{1}{1-¥}\log I-1\right)° I+\frac{1}{2}\frac{I«_{P_t}^2}{¥}X_t+(1-¦_t)Ir+¦_t«_H I$$
$$+I_X\frac{1}{1-¥}®(m-X_t)-\frac{1}{2}¦_t^{2-2}I¥X_t-¦_t^{-}«_{P_t}°_{PH}IX_t+\frac{1}{2}¦_t^{2-2}I¥°_{PH}^2X_t$$
$$+\frac{1}{2}I_{XX}\frac{1}{1-¥}±^2X_t+\frac{1}{¥}I_X°_{PX}±«_{P_t}X_t-I_X¦_t^{-}°_{PH}±°_{PX}X_t+\frac{1}{2}\frac{(I_X)^2}{I}\frac{±^2°_{PX}^2}{¥}X_t$$
$$+¦_t^{-}I_X°_{HX}±X_t+\frac{1}{2}\frac{1}{a^2}\frac{I«_B^2}{¥}X_t+\frac{1}{¥}\frac{1}{a}I_X°_{BX}±«_BX_t+\frac{1}{2}\frac{(I_X)^2}{I}\frac{±^2°_{BX}^2}{¥}X_t\,. \quad (14)$$

We guess that the above ordinary differential equation has a solution of the form $I=\exp(Q_0+Q_1X_t)$; we can express (14) as

$$0=\left(\log ° -\frac{¥}{1-¥}\log(1-¥)-\frac{1}{1-¥}(Q_0+Q_1X_t)-1\right)°+\frac{1}{2}\frac{«_{P_t}^2}{¥}X_t+(1-¦_t)r+¦_t«_H$$
$$+Q_1\frac{1}{1-¥}®(m-X_t)-\frac{1}{2}¦_t^{2-2}¥X_t-¦_t^{-}«_{P_t}°_{PH}X_t+\frac{1}{2}¦_t^{2-2}¥°_{PH}^2X_t$$
$$+\frac{1}{2}Q_1^2\frac{1}{1-¥}±^2X_t+\frac{1}{¥}Q_1°_{PX}±«_{P_t}X_t-Q_1¦_t^{-}°_{PH}±°_{PX}X_t+\frac{1}{2}Q_1^2\frac{±^2°_{PX}^2}{¥}X_t$$
$$+¦_t^{-}Q_1°_{HX}±X_t+\frac{1}{2}\frac{1}{a^2}\frac{«_B^2}{¥}X_t+\frac{1}{¥}\frac{1}{a}Q_1°_{BX}±«_BX_t+\frac{1}{2}Q_1^2\frac{±^2°_{BX}^2}{¥}X_t\,. \quad (15)$$

Rearranging the above equation, we have the following two equations for Q_0 and Q_1,

$$\left(\frac{1}{1-¥}®m\right)Q_1-\left(\frac{°}{1-¥}\right)Q_0+\left(\log ° -\frac{¥}{1-¥}\log(1-¥)-1\right)°+(1-¦_t)r+¦_t«_H=0 \quad (16a)$$

$$Q_1^2\left(\frac{1}{2}\frac{1}{1-¥}±^2+\frac{1}{2}\frac{±^2°_{PX}^2}{¥}+\frac{1}{2}\frac{±^2°_{BX}^2}{¥}\right)+Q_1\left(-\frac{°}{1-¥}-\frac{1}{1-¥}®+\frac{1}{¥}°_{PX}±«_{P_t}\right.$$
$$\left.-¦_t^{-}°_{PH}±°_{PX}+¦_t^{-}°_{HX}±+\frac{1}{¥}\frac{1}{a}°_{BX}±«_B\right)+\frac{1}{2}\frac{«_{P_t}^2}{¥}-\frac{1}{2}¦_t^{2-2}¥$$
$$-¦_t^{-}«_{P_t}°_{PH}+\frac{1}{2}¦_t^{2-2}¥°_{PH}^2+\frac{1}{2}\frac{1}{a^2}\frac{«_B^2}{¥}=0\,, \quad (16b)$$

and we have:

$$Q_1=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\,, \quad (17)$$

where

$$a=\frac{1}{2}\frac{1}{1-¥}±^2+\frac{1}{2}\frac{±^2°_{PX}^2}{¥}+\frac{1}{2}\frac{±^2°_{BX}^2}{¥}$$

$$b=-\frac{\text{°}+\text{®}}{1-\text{¥}}+\frac{1}{\text{¥}}\text{°}_{PX}\text{±«}_{P_t}-/_t^{\,-}\text{°}_{PH}\text{±°}_{PX}+/_t^{\,-}\text{°}_{HX}\text{±}+\frac{1}{\text{¥}}\frac{1}{a}\text{°}_{BX}\text{±«}_{B}$$

$$c=\frac{1}{2}\frac{\text{«}_{P_t}^{2}}{\text{¥}}-\frac{1}{2}/_t^{\,2-2}\text{¥}-/_t^{\,-}\text{«}_{P_t}\text{°}_{PH}+\frac{1}{2}/_t^{\,2-2}\text{¥}\,\text{°}_{PH}^{2}+\frac{1}{2}\frac{1}{a^2}\frac{\text{«}_{B}^{2}}{\text{¥}}$$

and

$$Q_0=\left(\frac{\text{®}\,m}{\text{°}}\right)Q_1+(1-\text{¥})\log\text{°}-\text{¥}\log(1-\text{¥})-1+\text{¥}+r(1-/_t)\frac{1-\text{¥}}{\text{°}}+/_t\text{«}_H\frac{1-\text{¥}}{\text{°}}. \quad (18)$$

Now we can get the indirect utility function and the investor's optimal dynamic asset allocation strategy in bond fund and stock index fund.

The indirect utility function is

$$J(W_t,X_t)=I(X_t)\frac{W_t^{1-\text{¥}}}{1-\text{¥}}=\exp(Q_0+Q_1X_t)\frac{W_t^{1-\text{¥}}}{1-\text{¥}}, \quad (19)$$

and the investor's optimal dynamic asset allocation strategy in the bond fund and the stock index fund are

$$n_t^{B}=\frac{1}{\text{¥}}\frac{1}{1-/_t}\frac{\text{«}_B}{\text{±}_{B_t}^{2}}+\frac{Q_1}{\text{¥}}\frac{1}{1-/_t}\frac{a\,\text{±}\,\text{°}_{BX}}{\text{±}_{B_t}^{2}} \quad (20a)$$

$$n_t^{P}=\frac{1}{\text{¥}}\frac{1}{1-/_t}\frac{\text{«}_{P_t}}{\text{±}_{P_t}^{2}}+\frac{Q_1}{\text{¥}}\frac{1}{1-/_t}\frac{\text{±}\,\text{°}_{PX}}{\text{±}_{P_t}^{2}}-\frac{/_t}{1-/_t}\frac{\text{±}_{P_t}\text{±}_{H_t}\text{°}_{PH}}{\text{±}_{P_t}^{2}}. \quad (20b)$$

However, for the time being, we cannot really solve this model for two reasons. First, we do not yet consider the setting of the time-varying instantaneous expected risk premium on the stock index fund, «_{P_t}. Secondly, the above solution is just a special case for our model setting when $\text{¨}=1$. In the next section, we will use the perturbation method to find the generalized solutions to our model.

3.2 The generalized solutions by the perturbation method

The basic idea of the perturbation method is to formulate a general problem, but first we find a particular case that has a known solution, and then use that particular case and its solution as a starting point for computing approximate solutions to nearby problems. This approach is the application of implicit function theorems, Taylor series expansions, and techniques from bifurcation theory and singularity theory. This method is widely used in mathematical physics, particular in quantum mechanics and general relativity theory, with much success (Judd, 1998).

In many financial economics models, determination of the unknown function is a key point of economic analysis under the assumption of function form. However, for the more generalized of the models, especially the intertemporal consumption and portfolio choice problem with stochastic nonlinear differential equations, the desire for a closed-form solution often restricts the analysis. Very recently this situation has begun to change as a result of several related developments. One of these developments is the use of the perturbation method in some special cases with closed-form solutions for computing approximate solutions. This helps us to make explicit economic analyses. They offer analytical insights into investor behavior in models that fall outside the still limited class that can be solved exactly (Campbell, 2000).

Campbell (1993), Judd and Guu (1997, 2000), Kogan and Uppal (2000), Chacko and Viceira (2002) and Campbell and Viceira (1999, 2001) have ever used perturbation approaches to solve dynamic economic or financial models. In the rest of the paper, we will apply the perturbation method to solve our model. In the context of our problem, the insight is that the solution for the recursive utility function when $¨=1$ provides a convenient starting point for the expansion. We can apply the previous section for $¨=1$, which can be used as a starting point around this solution for computing our model.

For the setting of the time-varying instantaneous expected risk premium on the stock index fund $«_{P_t}$, we assume that $«_{P_t} = «_0 + «_1 \pm^2_{P_t}$, and use the perturbation method around the results of the previous section for computing our model. The Bellman equation can be expressed as

$$0 = -\frac{°¨}{1-¨} I^{1+\frac{1-¨}{1-¥}} + \frac{¨}{1-¨} ° I + \frac{1}{2}\frac{I}{¥}\left(«_0^2 X_t + «_1^2 \frac{1}{X_t} + 2«_0«_1 \right) + (1-¦_t) I r + ¦_t «_H I$$
$$+ I_X \frac{1}{1-¥} ®(m - X_t) - \frac{1}{2} ¦_t^{2-2} I¥ X_t - ¦_t^{-}(«_0 X_t + «_1) °_{PH} I + \frac{1}{2} ¦_t^{2-2} I¥ °^2_{PH} X_t$$
$$+ \frac{1}{2} I_{XX} \frac{1}{1-¥} \pm^2 X_t + \frac{1}{¥} I_X °_{PX} \pm («_0 X_t + «_1) - I_X ¦_t^{-} °_{PH} \pm °_{PX} X_t + \frac{1}{2}\frac{(I_X)^2}{I}\frac{\pm^2 °^2_{PX}}{¥} X_t$$
$$+ ¦_t^{-} I_X °_{HX} \pm X_t + \frac{1}{2}\frac{1}{a^2}\frac{I «_B^2}{¥} X_t + \frac{1}{¥}\frac{1}{a} I_X °_{BX} \pm «_B X_t + \frac{1}{2}\frac{(I_X)^2}{I}\frac{\pm^2 °^2_{BX}}{¥} X_t \,. \quad (21)$$

In general, the above equation cannot be computed in closed-form. Our approach is to obtain an asymptotic approximation to equation (21), where the expansion is by taking a first-order expansion of it around the elasticity of intertemporal substitution $¨=1$. We can write

$$I^{1+\frac{1-¨}{1-¥}} \approx I + I^{1+\frac{1-¨}{1-¥}} \log\left(I \cdot \frac{(-1)}{1-¥} \right)\Bigg|_{¨=1} (¨-1) = I + \frac{1-¨}{1-¥} I \log I \,. \quad (22)$$

Substituting (22) into equation (21) and guessing that this equation has a solution of the form

$$I = \exp(Q_0 + Q_1 \log X_t + Q_2 X_t)$$
$$\approx \exp\left\{ Q_0 + Q_1\left[-\log ® + \log\left(® m - \pm^2\right) + 1 - \frac{® m - \pm^2}{® X_t} \right] + Q_2 X_t \right\}, \quad (23)$$

where the second approximate equality is from equation (6) by taking first-order Taylor expansion around the long-term mean $m_{\pm^2_{P_t}} = \left(m - \frac{1}{2}\frac{1+ \quad \pm^2}{®} \right)^{-1}$ of the process of the stochastic volatility, we can express equation (21) as

$$
\begin{aligned}
0 = & -\frac{\text{°¨}}{1-\text{¨}} - \frac{\text{°¨}}{1-\text{¥}}\left\{Q_0 + Q_1\left[-\log \text{®} + \log\left(\text{®}\, m - \pm^2\right) + 1 - \frac{\text{®}\, m - \pm^2}{\text{®}\, X_t}\right] + Q_2 X_t\right\} \\
& + \frac{\text{¨}}{1-\text{¨}}\text{°} + X_t\, [Q_2{}^2 (\frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\, \text{°}^2_{PX} + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\, \text{°}^2_{BX}) + Q_2 (-\frac{1}{1-\text{¥}}\text{®} \\
& + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_0 - \text{¦}_t^{-}\, \text{°}_{PH}\pm\text{°}_{PX} + \text{¦}_t^{-}\pm\text{°}_{HX} + \frac{1}{\text{¥}}\frac{1}{a}\text{°}_{BX}\pm\text{«}_B) + \frac{1}{2}\frac{1}{\text{¥}}\text{«}_0^2 - \frac{1}{2}\text{¦}_t{}^{2-2}\text{¥} \\
& + \frac{1}{2}\text{°}^2_{PH}\text{¦}_t{}^{2-2}\text{¥} + \frac{1}{2}\frac{1}{\text{¥}}\frac{1}{a^2}\text{«}_B^2 - \text{¦}_t^{-}\text{«}_0\text{°}_{PH}] + \frac{1}{X_t}[Q_1{}^2 (\frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{PX} \\
& + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{BX}) + Q_1 (\frac{1}{1-\text{¥}}\text{®}\, m - \frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_1) + \frac{1}{2}\frac{1}{\text{¥}}\text{«}_1^2] + Q_1 (-\frac{1}{1-\text{¥}}\text{®} \\
& + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_0 - \text{¦}_t^{-}\,\text{°}_{PH}\pm\text{°}_{PX} + \text{¦}_t^{-}\,\text{°}_{HX}\pm + \frac{1}{\text{¥}}\frac{1}{a}\text{°}_{BX}\pm\text{«}_B) + Q_2 (\frac{1}{1-\text{¥}}\text{®}\, m \\
& + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_1) + Q_1 Q_2 (\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{\text{¥}}\pm^2\,\text{°}^2_{PX} + \frac{1}{\text{¥}}\pm^2\,\text{°}^2_{BX}) \\
& + (1-\text{¦}_t)\, r + \text{¦}_t\text{«}_H + \frac{1}{\text{¥}}\text{«}_0\text{«}_1 - \text{¦}_t^{-}\text{«}_1\text{°}_{PH}\,.
\end{aligned} \tag{24}
$$

Rearranging the above equation, we have the following three equations for Q_0, Q_1 and Q_2,

$$
\begin{aligned}
& Q_2{}^2 (\frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{PX} + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{BX}) + Q_2 (-\frac{1}{1-\text{¥}}\text{®} + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_0 \\
& - \frac{\text{°¨}}{1-\text{¥}}) + \frac{1}{2}\frac{1}{\text{¥}}\text{«}_0^2 = 0
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
& Q_1{}^2 (\frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{PX} + \frac{1}{2}\frac{1}{\text{¥}}\pm^2\,\text{°}^2_{BX}) + Q_1 (\frac{1}{1-\text{¥}}\text{®}\, m - \frac{1}{2}\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_1 \\
& + \frac{\text{°¨}}{1-\text{¥}}\frac{\text{®}\, m - \pm^2}{\text{®}} - \text{¦}_t^{-}\,\text{°}_{PH}\pm\text{°}_{PX} + \text{¦}_t^{-}\pm\text{°}_{HX} + \frac{1}{\text{¥}}\frac{1}{a}\text{°}_{BX}\pm\text{«}_B) + \frac{1}{2}\frac{1}{\text{¥}}\text{«}_1^2 \\
& - \frac{1}{2}\text{¦}_t{}^{2-2}\text{¥} + \frac{1}{2}\text{°}^2_{PH}\text{¦}_t{}^{2-2}\text{¥} + \frac{1}{2}\frac{1}{\text{¥}}\frac{1}{a^2}\text{«}_B^2 - \text{¦}_t^{-}\text{«}_1\text{°}_{PH} = 0
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
& Q_0 (-\frac{\text{°¨}}{1-\text{¥}}) + Q_1 [-\frac{\text{°¨}}{1-\text{¥}}\left(-\log \text{®} + \log(\text{®}\, m - \pm^2) + 1\right) - \frac{1}{1-\text{¥}}\text{®} + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_0] \\
& + Q_2 (\frac{1}{1-\text{¥}}\text{®}\, m - \text{¦}_t^{-}\,\text{°}_{PH}\pm\text{°}_{PX} + \text{¦}_t^{-}\,\text{°}_{HX}\pm + \frac{1}{\text{¥}}\frac{1}{a}\text{°}_{BX}\pm\text{«}_B + \frac{1}{\text{¥}}\text{°}_{PX}\pm\text{«}_1) \\
& + Q_1 Q_2 (\frac{1}{1-\text{¥}}\pm^2 + \frac{1}{\text{¥}}\pm^2\,\text{°}^2_{PX} + \frac{1}{\text{¥}}\pm^2\,\text{°}^2_{BX}) - \frac{\text{°¨}}{1-\text{¨}} + \frac{\text{¨}}{1-\text{¨}}\text{°} + (1-\text{¦}_t)\, r \\
& + \text{¦}_t\text{«}_H + \frac{1}{\text{¥}}\text{«}_0\text{«}_1 - \text{¦}_t^{-}\text{«}_0\text{°}_{PH} = 0\,,
\end{aligned} \tag{27}
$$

where Q_2 and Q_1 can be solved to the quadratic equation (25) and (26), and Q_0 can be solved to the linear equation (27), given Q_1 and Q_2.

Now, we can really get the indirect utility function and the investor's optimal dynamic asset allocation strategy on bonds, stocks and cash with human capital in the stochastic environment.

The indirect utility function is

$$J(W_t, X_t) = I(X_t)\frac{W_t^{1-¥}}{1-¥} = \left[\exp\left(Q_0 + Q_1 \log X_t + Q_2 X_t\right)\right]\frac{W_t^{1-¥}}{1-¥}\,. \tag{28}$$

The investor's optimal instantaneous consumption is

$$\frac{C_t}{W_t} = °^{¨} \exp\left\{\left(\frac{1-¨}{1-¥}\right)\left(Q_0 + Q_1 \log X_t + Q_2 X_t\right)\right\}, \tag{29}$$

and the investor's optimal dynamic asset allocation strategy in the stock index fund, the bond fund and the money-market fund are

$$n_t^B = \frac{1}{¥}\frac{1}{1-¦_t}\left(«_B \frac{1}{±_{B_t}^2}\right) + \frac{1}{¥}\frac{1}{1-¦_t} ± \; °_{BX}\left(Q_1 + Q_2\, a^2 \frac{1}{±_{B_t}^2}\right) \tag{30a}$$

$$n_t^P = \frac{1}{¥}\frac{1}{1-¦_t}\left(«_1 + «_0 \frac{1}{±_{P_t}^2}\right) + \frac{1}{¥}\frac{1}{1-¦_t} ± \; °_{PX}\left(Q_1 + Q_2 \frac{1}{±_{P_t}^2}\right) - \frac{¦_t}{1-¦_t}\, °_{PH}\, ¯\frac{1}{±_{P_t}^2} \tag{30b}$$

$$\begin{aligned} n_t^M &= 1 - n_t^B - n_t^P \\ &= 1 - \frac{1}{¥}\frac{1}{1-¦_t}\left\{\left(«_B \frac{1}{±_{B_t}^2} + «_1 + «_0 \frac{1}{±_{P_t}^2}\right) + ±\left[°_{BX}\left(Q_1 + Q_2\, a^2 \frac{1}{±_{B_t}^2}\right)\right.\right. \\ &\quad \left.\left. + \; °_{PX}\left(Q_1 + Q_2 \frac{1}{±_{P_t}^2}\right)\right]\right\} + \frac{¦_t}{1-¦_t}\, °_{PH}\, ¯\frac{1}{±_{P_t}^2}\,. \end{aligned} \tag{30c}$$

The myopic components and intertemporal hedging demands of the stock index fund and the bond fund go to zero as investors' risk aversion approach infinity. This follows from the fact that $\lim_{¥\to\infty}\frac{1}{¥}Q_1 = 0$ and $\lim_{¥\to\infty}\frac{1}{¥}Q_2 = 0$.

The ratios of bond to stock and cash to stock under the optimal asset allocation strategy are

$$\frac{n_t^B}{n_t^P} = \frac{(«_B \frac{1}{±_{B_t}^2}) + ± \; °_{BX}\,(Q_1 + Q_2\, a^2 \frac{1}{±_{B_t}^2})}{(«_1 + «_0 \frac{1}{±_{P_t}^2}) + ± \; °_{PX}\,(Q_1 + Q_2 \frac{1}{±_{P_t}^2}) - ¥\, ¦_t\, ¯\, °_{PH} \frac{1}{±_{P_t}^2}} \tag{31}$$

$$\frac{n_t^M}{n_t^P} = \frac{¥(1-¦_t) - \{(«_B \frac{1}{±_{B_t}^2} + «_1 + «_0 \frac{1}{±_{P_t}^2}) + ±[\,°_{BX}(Q_1 + Q_2\, a^2 \frac{1}{±_{B_t}^2}) + \,°_{PX}(Q_1 + Q_2 \frac{1}{±_{P_t}^2})]\} + ¥\, ¦_t\, ¯\, °_{PH} \frac{1}{±_{P_t}^2}}{(«_1 + «_0 \frac{1}{±_{P_t}^2}) + ± \; °_{PX}(Q_1 + Q_2 \frac{1}{±_{P_t}^2}) - ¥\, ¦_t\, ¯\, °_{PH} \frac{1}{±_{P_t}^2}} \tag{32}$$

These ratios are all increasing in investor's coefficient of relative risk aversion and thus are consistent with popular financial advice.

For the time being, we have solved our problem of the dynamic asset allocation strategy for long-horizon investors with the stochastic expected return and volatility in incomplete markets explicitly. And by this closed-form solution, we resolve the asset allocation puzzle directly. In the next section, we will provide some analyses of our results and their effects on holdings for the results of resolving the asset allocation puzzle.

Ⅳ. The analyses on the optimal dynamic asset allocation strategy and the solution of the asset allocation puzzle

The investor's optimal dynamic asset allocation strategy for the bond fund has two components, whereas the stock index fund has three. The first two components of stock and bond, the myopic components and the intertemporal hedging components are similar, and another human capital hedging component is provided in the optimal stock index fund portfolio. First, the dependence of their myopic components is simple. The myopic component of the stock index fund is affine functions of the reciprocal of the stochastic volatility, while in the bond fund it is linear function of the reciprocal of the stochastic volatility, and they both decrease with the coefficient of relative risk aversion. Since volatility is time varying, the myopic components are time varying, too. The position of these myopic components can be either positive or negative depending on α_0, α_1, α_B and the level of volatility.

The intertemporal hedging components of the optimal asset allocation for the two risky assets are affine functions of the reciprocal of the stochastic volatility with coefficient Q_1 and Q_2, and decrease with the coefficient of relative risk aversion. Q_1 and Q_2 are the solutions of the two independent quadratic equations (26) and (25), respectively, and Q_0 is the solution of (27), given Q_1 and Q_2. For the quadratic equation of Q_2, when $\gamma > 1$ and $\frac{\gamma}{\gamma - 1} > (\rho_{PX}^2 + \rho_{BX}^2)$, it ensures that the denominator of the solutions of the quadratic equation (25) is negative and its discriminant is always non-negative, so its roots are always real. In addition, under these conditions, it also ensures that the products of the roots of the solutions in the quadratic equations (25) are always negative. It causes these two roots also to have opposite signs, and only the solution associated to the positive root of the discriminant of the quadratic equation (25) (i.e. only the negative root of the equation (25)) maximizes the value function J. For the coefficient Q_1, its roots are always real provided that $\left| \frac{\alpha_1}{\gamma I_t} - \rho_{PH} \right| > 1$ when $\gamma > 1$ and $\frac{\gamma}{\gamma - 1} > (\rho_{PX}^2 + \rho_{BX}^2)$. And the equation (26) also has two real roots of opposite signs from the theory of quadratic equation described above. Only the solution associated with the negative root of the discriminant of the quadratic equation (26) (i.e. only the positive root of the equation (26)) maximizes the value function.

We assume that the sign of the instantaneous correlation of unexpected return on the stock index fund with state variable (ρ_{PX}) is opposite to the instantaneous correlation of unexpected return on the bond fund with state variable (ρ_{BX}) to each other. Since $Q_1 > 0$, the sign of the intercept of the intertemporal hedging component for the optimal stock index fund is positive when $\gamma > 1$ and $\rho_{PX} > 0$, these terms are independent of the level of the volatility, and are counteracted partially or totally by their hedging demand from the effect of stochastic volatility. The intertemporal hedging component of the optimal asset allocation for stock index fund is affected by the instantaneous correlation of unexpected return on the risky asset with state variable (ρ_{PX}) which is equal to $-\rho_{P\sigma^2_{P_t}}$, where $\rho_{P\sigma^2_{P_t}}$ is the instantaneous correlation of unexpected return with proportional change in stochastic volatility on the stock index fund. If $\rho_{PX} > 0$ i.e. $\rho_{P\sigma^2_{P_t}} < 0$, it means that when the unexpected return on the stock index fund is low (the market situation is bad), the states of the market uncertainty will be high. Since $Q_2 < 0$ when $\gamma > 1$, the positive instantaneous correlation of unexpected return on the stock index fund with state variable (ρ_{PX}) implies that the more conservative investor will hold a negative hedging demand coming from pure changes in stock volatility on the stock index fund; this is because it lacks hedging ability against an increase in stock volatility. Similarly, in the optimal bond holding, its intertemporal hedging component is affected by the instantaneous correlation of unexpected return on the bond fund with state variable (ρ_{BX}). Since we have assumed the sign of ρ_{BX} is opposite to ρ_{PX} , in the situation of $\rho_{PX} > 0$, the ρ_{BX} will be negative, $\rho_{BX} < 0$. Since $Q_2 < 0$ when $\gamma > 1$, the more conservative investor will hold a positive hedging demand coming from pure changes in bond price volatility because it provides hedging ability against an increase in volatility. It is because $\rho_{BX} < 0$ ($\rho_{B\sigma^2_{B_t}} > 0$, it is the instantaneous correlation of unexpected return with proportional change in stochastic volatility on the bond fund) implies that bonds tend to be better in those states of the world where uncertainty is high.

There is another distinguishing hedging component in the stock index fund for the optimal dynamic asset allocation in this paper: the human capital hedging component is affected by the human capital return directly from the assumption of the returns of the stock being correlated with the returns of human capital. The investor looking for a hedge against changes in labor income should choose another hedging position on the stock index fund. The human capital hedging component of the optimal asset allocation for the stock index fund is a linear function of the reciprocal of the stochastic volatility. This human capital hedging component also depends on the relative weight of the wealth in the human capital with financial wealth and the instantaneous correlation of the unexpected return on the stock index fund with the human capital (ρ_{PH}).

Most importantly, in our model the human capital hedging component doesn’t like the myopic component or the intertemporal hedging component which decrease with the coefficient of relative risk aversion. In fact, it is independent of the attitude of risk aversion or the preference. The human capital hedging component also depends on the instantaneous correlation of the unexpected return on the stock index fund with the human capital. The absolute size of the human capital hedging component is increasing with the absolute value of this instantaneous correlation. When we assume the instantaneous correlation is positive, the investor will hold a negative position on the stock index fund for the human capital hedging component which provides partial hedging against changes in the human capital return. From the previous section, the ratios of bond to stock and cash to stock under the closed-form solution of the optimal asset allocation strategy are increasing in investor's risk aversion. The realized results solve the asset allocation puzzle by integrating human capital into the intertemporal asset allocation model. In contrast to Merton (1973) has shown that the investor's asset allocation demands have two components which satisfy the Separation Theorem, our model shows that investors looking for a hedge against changes in the human capital should choose another composition in the stock index fund.

V. Conclusion

Recently, Canner, Mankiw and Weil (1997) point out that the popular financial advisors on portfolio allocation among cash, bonds and stocks appear not to follow the mutual-fund separation theorem. They explore various possible explanations of this puzzle and conclude that explaining popular financial advice is difficult using models of fully rational investors. However, they suggest that considerations of intertemporal hedging and nontraded assets might help to solve the puzzle. In this paper, we have provided an analysis of the optimal dynamic asset allocation strategy for a long-horizon investor with the nontraded asset under an economic environment of stochastic investment opportunities and incomplete financial markets to solve the asset allocation puzzle.

Some recent studies highlight the key role of the human capital in asset pricing model and asset allocation strategy. Therefore, we choose to incorporate the human capital and the intertemporal hedging components to resolve the asset allocation puzzle. Our closed-form solutions rely on the perturbation method. We derive the exact solutions by the perturbation method of approximation around a particular point in the preference space. We show that the optimal dynamic asset allocation strategy of the stock index fund has three components: the myopic component, the intertemporal hedging component which constitutes the best protection against stochastic states variable, and the human capital hedging component which constitutes the best protection against changes in labor income; whereas the bond fund has only the myopic component and the intertemporal hedging component. Their myopic components and the intertemporal hedging components are all affine functions of the reciprocal of the stochastic volatility on the risky assets and decrease with the coefficient of relative risk aversion.

In this paper, we propose another distinguishing hedging component of the dynamic asset allocation for stock index fund: the human capital hedging component, the hedging demand which characterizes the demand arising from the desire to hedge against changes in the labor income in contrast to Merton (1973). The human capital hedging component of the optimal asset allocation for the stock is a linear function of the reciprocal of the stochastic volatility on the stock index fund, so this component is time-varying. It also depends on the relative weight of the wealth of the human capital with financial wealth and the instantaneous correlation of the unexpected return on the stock index fund with the human capital

(ρ_{PH}). The human capital hedging component is independent of the investor's attitude of risk aversion or the preference. We show that the ratios of bond to stock and cash to stock under the optimal asset allocation strategy are all increasing in the investor's coefficient of relative risk aversion, and thus are consistent with the popular financial advice. In other words, when we incorporate nontraded assets with intertemporal hedging, we solve the asset allocation puzzle successfully.

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隨機環境下利用跨期避險與不可交易資產求解資產配置的迷思

顏錫銘
政治大學財務管理學系教授

徐辜元宏
政治大學財務管理學系博士候選人

摘　要

Canner, Mankiw and Weil (1997) **指出，一般財務顧問公司對於投資者風險態度的差異所提出之投資建議與財務理論間存在著嚴重的不一致性，其將之稱為「資產配置的迷思**(An asset allocation puzzle)**」。本文乃提出一理性的長期投資者模型，在考量投資者之不可交易資產與隨機投資機會下，提出最適動態資產配置策略，並解決了此資產配置的迷思，與現今之一般財務顧問公司對於投資者之投資建議相一致。**

關鍵字：資產配置迷思、隨機投資機會、不完全市場、跨期模型、擾動法