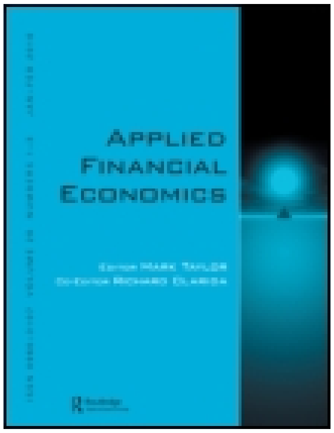

Applied Financial Economics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/rafe20

Pricing gold options under Markov-modulated jump-diffusion processes

Shih-Kuei Lin[a], Yu-Min Lian[a] & Szu-Lang Liao[a]
[a] Department of Money and Banking, National Chengchi University, Taipei City, Taiwan
Published online: 29 Apr 2014.

Applied Financial Economics, 2014
Vol. 24, No. 12, 825–836, http://dx.doi.org/10.1080/09603107.2014.914142

Routledge
Taylor & Francis Group

Pricing gold options under Markov-modulated jump-diffusion processes

Shih-Kuei Lin, Yu-Min Lian* and Szu-Lang Liao

Department of Money and Banking, National Chengchi University, Taipei City, Taiwan

In this study, we empirically investigate the properties of gold returns, and the European gold options are priced when the underlying gold price dynamics are driven by Markov-modulated jump-diffusion processes. Specifically, the jump events are captured by a compound Poisson process with a log-normal jump size, and the regime-switching intensity rate is governed by a continuous-time finite-state Markov chain. Under an incomplete market setting, we study the valuation of European gold options using the method of Esscher transform. The estimated results and numerical examples are provided.

Keywords: gold price; European gold option; Markov-modulated jump-diffusion process; Esscher transform

JEL Classification: C51; G12

I. Introduction

In a period where global financial markets crash and the global economy is in recession, investors are seeking trusted sources of security for their portfolios. Because the price of gold does not correlate highly with changes in most mineral commodities and other financial assets, this precious metal can play a saving role as a consistent portfolio diversifier – managing risk and mitigating potential losses in the portfolios of investors, an imperative in the prevailing environment (Capie *et al.*, 2005; Baur and Lucey, 2010; Baur and McDermott, 2010; Reboredo, 2013; Zagaglia and Marzo, 2013).

A variety of gold-linked instruments (e.g. gold options) have been invented for hedging the fluctuating risk in the gold market. Beckers (1984) and Ball *et al.* (1985) study the gold options market under the Black–Scholes framework. Ogden *et al.* (1990) investigate gold spot and futures options. However, the presence of jumps in the time series of commodities or equities can have serious implications on pricing-related derivatives. Over the last couple of decades, the increasing number of jump events, particularly the global financial crisis in the late 2000s, has created large fluctuations in the gold market. Therefore, it is critical to model the dynamic jump process adequately and price gold options corresponding to the changing gold price according to actual market developments.

Figure 1 depicts several significant jumps of the gold price within the last four decades. This empirical evidence reveals that the geometric Brownian motion (GBM) is not completely consistent with the reality. As a consequence, incorporating the sudden random shocks into a dynamic model is necessary and significant. The bottom panel of Fig. 1 exhibits different frequencies for jump events through time. Specifically, it shows larger jump activities in some time periods. In the energy crisis of the 1970s and global financial crisis of 2008, for instance, gold prices had larger jumps over the past four decades (based on the daily prices of gold from 1971 to 2010). In addition, there

*Corresponding author. E-mail: 95352506@nccu.edu.tw

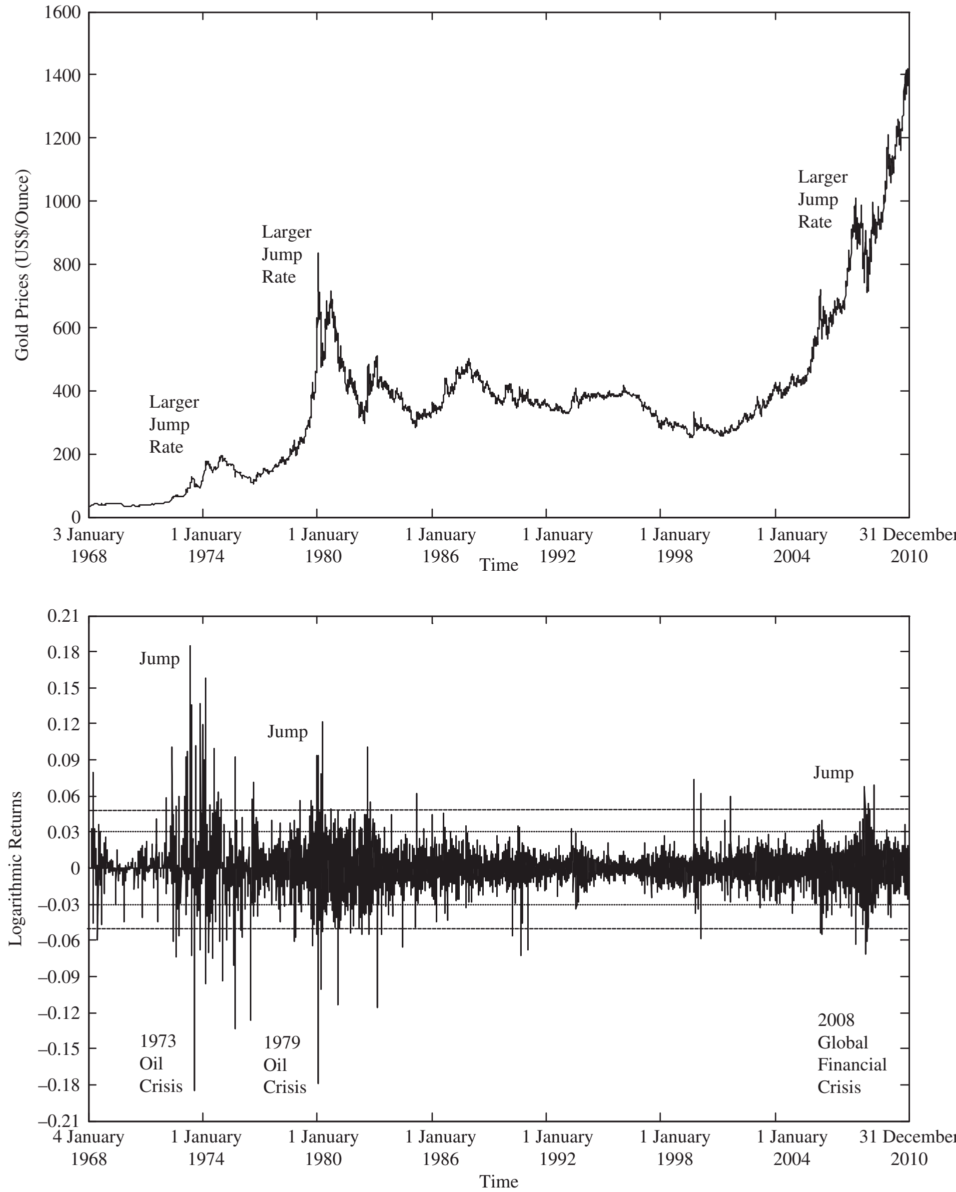

Fig. 1. Daily data for the gold prices (top panel) and logarithmic returns (bottom panel)
Notes: The bottom panel supposes that the daily logarithmic returns over ±3% and ±5% in magnitude are jumps. The spot data are from Datastream and cover the period from 3 January 1968 to 31 December 2010.

exists the so-called jump and volatility clustering in the logarithmic return series of gold prices, which means that periods of large (small) changes tend to be followed by periods of continued large (small) changes. Roughly speaking, a stochastic process is said to have a long memory if it has an autocorrelation function that is not integrable. This happens, for instance, when the autocorrelation function decays asymptotically as a power law. Cheung and Lai (1993) investigate the long memory behaviour in gold returns. Nevertheless, the existing jump-diffusion processes, such as in Merton (1976) and Kou (2002), are unable to address the phenomena of volatility clustering and long memory.

According to the changing prices of gold in the top panel of Fig. 1, we could identify two regimes of the gold market. The first state is the relatively low-volatility regime and can be viewed as the ordinary state. The second state is the relatively high-volatility regime and can be regarded as the volatile state. Analysing returns from different periods allows us to investigate the potential effect of different jumps over time (switching regimes). The data set used in the descriptive analysis

Table 1. Descriptive statistics of the gold logarithmic returns

Classification	2005	2006	2007	2008	2009	2010	Total
Trading days	251	252	253	254	256	261	1527
Mean	0.0007	0.0009	0.0011	0.0001	0.0010	0.0009	0.0008
SD	0.0082	0.0152	0.0098	0.0215	0.0131	0.0103	0.0137
Skewness	0.0451	−0.4077	−0.5152	−0.0852	−0.0476	−0.3808	−0.2500
Kurtosis	3.9677	4.1217	4.4910	5.6810	3.9999	4.6083	7.7529
Days exceeding $\pm 3\%$	0	19	3	31	8	2	63
Days exceeding $\pm 5\%$	0	1	0	9	0	0	10

Notes: The descriptive statistics are reported for the gold logarithmic returns from 2005 to 2010. This table shows the number of days each year in which the gold yields a logarithmic return series over $\pm 3\%$ and $\pm 5\%$ in magnitude.

consists of the daily gold prices,[1] and the descriptive statistics of the logarithmic return series are presented in Table 1. The skewness and kurtosis suggest a leptokurtic distribution with negatively skewed returns in the gold market. Particularly, in 2006 and 2008, we can observe extreme movements and jumps in the daily returns, respectively. These can be regarded as the volatile state (state 2). In contrast, other periods can be viewed as the ordinary state (state 1). Suppose that the gold logarithmic returns that are over $\pm 3\%$ in magnitude are jumps. Table 1 shows that the mean frequency of the jumps in the entire period is 10.5, where the mean frequencies of the jumps in state 1 and state 2 are 3.25 and 25, respectively. State 1, therefore, is below the long-term average jump activity, whereas state 2 is above the long-term average jump activity. As shown in Table 1, the periods 2005 and 2007 are in state 1, but the regime transitions to state 2 in the periods 2006 and 2008, and then transitions back to state 1 in the periods 2009 and 2010. A further implication of the figures in Table 1 is that during the sample period, jump events could happen over time and the economy switches between two states of jump rates. As shown in Fig. 1 and the resulting statistics in Table 1, the changing gold price is likely to exert a slow shift in jump intensity from one state to another, and thus, the arrival intensity of jump events is significantly different under different regimes of the gold price. According to these previously observed characteristics in the gold market returns, such as unanticipated jump events, jump and volatility clustering, leptokurtic and asymmetric features, and the existence of regime-switching intensity rates, we are motivated to develop a Markov specification of jump intensity rates. Thus, allowing for different jump rates under different regimes, our dynamic process of jump activity is set to be a Markov-modulated Poisson process (Last and Brandt, 1995; Chang *et al.*, 2013) instead of a single Poisson process used in the Merton-type jump-diffusion model (JDM; Merton, 1976).

The jump-diffusion class of models is applied to capture the asset price dynamics and is used for option valuation. Amin (1993) develops a simple, discrete time model to price options when the underlying process follows a jump-diffusion process, but the model does not explain volatility clustering. Duan *et al.* (2006) evaluate options when there are jumps in the pricing kernel and correlated jumps in asset prices and volatilities. The models capture leptokurtosis and volatility clustering but do not show the regime-switching phenomenon. Under the regime-switching environment, Elliott *et al.* (2005) and Elliott and Osakwe (2006) investigate the option prices for pure diffusion dynamics and for pure jump processes, respectively. Chan and Maheu (2002) propose a time-varying Poisson jump model to describe the jump dynamics in stock market returns. Because these models are developed in the discrete time circumstance, they do not provide the closed-form solutions for option prices, and most of them do not test the empirical characteristics nor assess the models.

In this study, we incorporate both jump events and regime-switching intensity rates for the gold price by identifying a Markov-modulated jump-diffusion model (MMJM). Under such a model, the jump events are described as a compound Poisson process with the log-normal jump size setting used by Merton (1976), and the regime-switching intensity rate is captured by a continuous-time finite-state Markov chain whose states represent the hidden states of an economy. Under such gold price dynamics, which is an incomplete market, we employ the Esscher transform technique developed by Gerber and Shiu (1994) to determine the pricing kernel and the Esscher parameters (risk premiums), and then derive the pricing formulas of European gold options. Because jump arrivals and market states are hidden variables, we estimate the model parameters using the expectation maximization (EM) algorithm (Lange, 1995a, b) and obtain the SEs of parameter estimators using the supplemented expectation

[1] The tested data are from Datastream and cover the period from 4 January 2005 to 31 December 2010.

maximization (SEM) algorithm (Meng and Rubin, 1991). From the empirically estimated parameters in the dynamic model and the derived option prices, we show that the model is more accurate than competing models in pricing European gold call options.

This study makes three major contributions. First, we analyse the behaviour of gold prices to understand the operation of the gold market and the risks involved. Our findings are valuable for the valuation of other gold derivative assets for which the gold price dynamics are expected to follow the Markov-modulated jump-diffusion process. Second, we derive the generalized gold option pricing formula via Esscher transform and illustrate that the derived formula can be reduced to the pricing formulas of Merton (1976) and the Black–Scholes model (BSM). Finally, we use actual market data to investigate the pricing performance of the gold option pricing. The pricing results are significant for investors and for the organization of the gold market.

The remainder of this article is organized as follows. The next section illustrates the continuous-time dynamic model. Section III presents the change of measures and the pricing formulas of gold options. Section IV discusses the empirical and numerical results. The final section presents the conclusions of this study.

II. Model Setting

Let (Ω, F, P) be a complete probability space, where P is the physical probability measure. For each $t \in [0, T]$, the MMJM is used to model the gold price dynamics as follows:

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2 - \Lambda\kappa\right)t + \sigma W(t) + \sum_{k=1}^{\Phi(t)} Y_k\right\} \tag{1}$$

where the appreciation rate μ and the volatility σ are constants and $W(t)$ is a Wiener process under P. $\{Y_k : k = 1, 2, ...\}$ are the jump sizes, which are assumed to be independently identically distributed nonnegative random variables with the density function $f_Y(y)$. If a jump event occurs at time k, the jump size Y_k is normally distributed with mean μ_J and variance σ_J^2. Therefore, the mean percentage jump size of the gold price is $\kappa = E[\exp(Y_k) - 1] = \exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1$. The Markov-modulated Poisson process $\Phi(t)$ is a Poisson process whose stochastic jump intensity $\lambda_{X(t)}$ changes according to a hidden Markov chain $X(t)$, with the transition function $P_{ij}(t)$ on the finite state space $\mathrm{X} = \{1, 2, ..., I\}$. For $i, j \in \mathrm{X}$, we denote the transition rate $\Psi(i,j)$ from state $X(0) = i$ to state $X(t) = j$ of $\Phi(t)$ as follows:

$$\Psi(i,j) = \begin{cases} \psi(i,j), & i \neq j \\ -\sum\limits_{j, j \neq i} \psi(i,j), & \text{otherwise} \end{cases} \tag{2}$$

The notation $\Psi = (\Psi(i,j))_{I\times I}$ represents the $I \times I$ matrix of the transition rate with diagonal elements $\psi_{ii} = -\sum_{j=1, j\neq i}^{I} \psi_{ij} = -v_i$. v_i is the departure rate at which the process leaves state i. Since the Markov chain has a finite number of states, the Poisson arrival intensity takes discrete values corresponding to each state. Last and Brandt (1995) give the moment-generating function for the joint distribution function of $X(t)$ and $\Phi(t)$ via the Laplace inverse transform as follows:

$$P^{\Theta}(\zeta, t) = \sum_{n=0}^{\infty} P(n,t)\zeta^n, \quad 0 \leq \zeta \leq 1 \tag{3}$$

where $P(n,t) = (P_{ij}(n,t))_{I\times I}$ represents the $I \times I$ transition probability matrix and $P_{ij}(n,t) = P(X(0) = i, X(t) = j,\ \Phi(t) = n)$ denotes the transition probability with n jump times from state $X(0) = i$ to state $X(t) = j$. Here, $P(n, 0) = (1_{\{n=0\}} D_{ij})$, where $D_{ij} = 1$ if $i = j$ and 0, otherwise. Using the Kolmogorov forward equation, the derivative of $P(n,t)$ becomes:

$$\frac{d}{dt}P(n,t) = (\Psi - \Lambda)P(n,t) + 1_{\{n\geq 1\}}\Lambda P(n-1, t) \tag{4}$$

where $\Lambda = (\lambda_{X(t)})_{I\times I}$ denotes the $I \times I$ diagonal matrix of the intensity rate with diagonal elements λ_i. Thus, the unique solution of $P^{\Theta}(\zeta, t)$ can be obtained as

$$P^{\Theta}(\zeta, t) = \exp((\Psi - (1-\zeta)\Lambda)t) \tag{5}$$

where the exponential power series are given as $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ for any $I \times I$ matrix A and $A^0 = (D_{ij})$. Applying the Laplace inverse transform of Equation 3 and the unique solution of Equation 5, we have the joint distribution of $X(t)$ and $\Phi(t)$ at time t with the following equation:

$$P(n,t) = \frac{\partial^n}{\partial \zeta^n n!} P^{\Theta}(\zeta, t)\bigg|_{\zeta=0} \tag{6}$$

In Equation 1, we also assume that all random shock processes $W(t)$, $\Phi(t)$, $X(t)$ and Y_k are mutually independent.

III. Measure Change and Option Pricing

Esscher transform for Markov-modulated jump-diffusion model

There are infinitely equivalent martingale measures to price options since the security economy described by the MMJM is incomplete. In this circumstance, we relax the assumption of a diversifiable jump risk made by Merton (1976) and then apply the Esscher transform used by Elliott *et al.* (2005) and Elliott and Osakwe (2006) for the MMJM to determine a risk-neutral pricing measure. By decomposing $Z(t) = \log(S(t)/S(0)) = C(t) + J(t)$, we then get a continuous diffusion part $C(t) = \left(\mu - \frac{1}{2}\sigma^2 - \Lambda\kappa\right)t + \sigma W(t)$ and a jump part $J(t) = \sum_{k=1}^{\Phi(t)} Y_k$, for all $t \in [0, T]$. Now, write F_t^Z and F_t^X for the P-augmentation of the natural filtrations generated by $Z(t)$ and $X(t)$, respectively. For each $t \in [0, T]$, we define $F_t = F_t^Z \vee F_t^X$ as the σ- algebra. Then, the Esscher measure Q^{θ^h} equivalent to P on F_t, with respect to $\theta^h \in R$ for $h = \{C, J\}$, is given by the following:

$$\begin{aligned}\xi^{\theta^h}(t) &= \left.\frac{dQ^{\theta^h}}{dP}\right|_{F_t} \\ &= \frac{\exp\left(\theta^C\sigma W(t)\right)}{E^P\left[\exp\left(\theta^C\sigma W(t)\right)\big|_{F_0^Z}\right]} \cdot \frac{\exp\left(\theta^J \sum_{k=1}^{\Phi(t)} Y_k\right)}{E^P\left[\exp\left(\theta^J \sum_{k=1}^{\Phi(t)} Y_k\right)\bigg|_{F_0^X}\right]} \\ &= \exp\left(\theta^C\sigma W(t) - \frac{1}{2}\left(\theta^C\sigma\right)^2 t\right) \cdot \exp\left(\theta^J \sum_{k=1}^{\Phi(t)} Y_k - \Lambda_\kappa{}^{\theta^J} t\right)\end{aligned} \tag{7}$$

where θ^C and θ^J are the Esscher parameters of the continuous diffusion part and the jump part, respectively. The mean percentage jump size of the gold price becomes $\kappa^{\theta^J} = E\left[\exp\left(\theta^J Y_k\right) - 1\right] = \exp\left(\theta^J\mu_J + \frac{1}{2}\left(\theta^J\sigma_J\right)^2\right) - 1$. In addition, the concrete form of the Esscher transform density $\xi^{\theta^h}(t)$ is an exponential F_t-martingale.

To facilitate the no-arbitrage pricing with respect to the risk-neutral pricing measure, it is important to construct the gold price process so as to satisfy the martingale condition, that is, the existence of a risk-neutral pricing measure, under which the Markov-modulated jump-diffusion process for the gold price is an F_t-martingale. Let the Esscher transform be defined by Equation 7. Then, the martingale condition is satisfied if and only if:

$$\theta^C = \frac{r - \mu + \Lambda\kappa}{\sigma^2} \tag{8}$$

and

$$\theta^J = \frac{-\mu_J - \frac{1}{2}\sigma_J^2}{\sigma_J^2} \tag{9}$$

Appendix A presents the detailed proof.

An equivalent martingale measure can be treated as the Esscher measure Q^{θ^h} with respect to the measure P. We begin with identifying the gold price dynamics under the risk-neutral pricing measure Q^{θ^h}. Let θ^C and θ^J be the Esscher parameters of the risk-neutral Esscher measure. Then, under Q^{θ^h} and conditional on F_t^Z,

$$W^{\theta^h}(t) = W(t) - \theta^C\sigma t \tag{10}$$

is a Wiener process. Furthermore, under Q^{θ^h}, the transition probability matrix $P^{\theta^h}(n, t)$, the stochastic jump intensity Λ^{θ^h} of the Markov-modulated Poisson process $\Phi^{\theta^h}(t)$, and the jump size $Y_k^{\theta^h}$ are, respectively, given by the following:

$$P^{\theta^h}(n, t) = P(n, t)\left(\kappa^{\theta^J} + 1\right)^n \exp\left(-\Lambda\kappa^{\theta^J} t\right) \tag{11}$$

$$\Lambda^{\theta^h} = \Lambda\left(\kappa^{\theta^J} + 1\right) = \Lambda\exp\left(\theta^J\mu_J + \frac{1}{2}\left(\theta^J\sigma_J\right)^2\right) \tag{12}$$

and

$$Y_k^{\theta^h} \overset{i.i.d.}{\sim} N\left(\mu_J + \theta^J\sigma_J^2, \sigma_J^2\right) \tag{13}$$

which means that the investors receive the premiums $-\theta^C\sigma$ and $\left(\kappa^{\theta^J} + 1\right)$ for the continuous diffusion risk and jump risk at time t, respectively. Therefore, the Wiener process and jump intensity are affected by the measure change. Under Q^{θ^h}, the risk-neutral transition probability matrix becomes $P^{\theta^h}(n, t)$ with transition rate matrix Ψ and stochastic jump intensity Λ^{θ^h}. Through the change of measures, the jump size $Y_k^{\theta^h} \overset{i.i.d.}{\sim} N(\mu_J + \theta^J\sigma_J^2, \sigma_J^2)$. Appendix B shows the detailed proof.

By determining the Esscher parameters, we then get:

$$W^{\theta^h}(t) = W(t) + \left(\frac{\mu - r - \Lambda\kappa}{\sigma}\right)t \tag{14}$$

$$\Lambda^{\theta^h} = \Lambda\exp\left(-\frac{\mu_J^2}{2\sigma_J^2} + \frac{\sigma_J^2}{8}\right) \tag{15}$$

and

$$Y_k^{\theta^h} \overset{i.i.d.}{\sim} N\left(-\frac{1}{2}\sigma_J^2, \sigma_J^2\right) \tag{16}$$

where $\left(\frac{\mu - r - \Lambda\kappa}{\sigma}\right)$ and $\exp\left(-\frac{\mu_J^2}{2\sigma_J^2} + \frac{\sigma_J^2}{8}\right)$ are the market prices of the continuous diffusion risk and jump risk at time t, respectively. In addition, under Q^{θ^h}, the jump size $Y_k^{\theta^h}$ is normally distributed with mean $-\frac{1}{2}\sigma_J^2$ and variance σ_J^2. Consequently, the gold price dynamics under Q^{θ^h} is the following:

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W^{\theta^h}(t) + \sum_{k=1}^{\Phi^{\theta^h}(t)} Y_k^{\theta^h}\right\} \tag{17}$$

where $r = \mu - \Lambda\kappa + \theta^C\sigma^2 + \Lambda^{\theta^h}\kappa^{\theta^J}$, for all $t \in [0, T]$.

Pricing European gold options

Under the assumption that there are no arbitrage opportunities in the market, we price gold options under the risk-neutral pricing measure Q^{θ^h}. For the European gold call options with strike price K and time to expiration T, its price at time zero is given by the following:

$$\begin{aligned} C_{MMJM}^{\Lambda^{\theta^h}}(0) &= \sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}^{\theta^h}(n,T)\left(S(0)N(d_{1,n}) - K\exp(-rT)N(d_{2,n})\right) \\ &= \sum_{n=0}^{\infty} P^{\theta^h}(n,T)C_{BSM}\left(S(0), K, r, T, \sqrt{\frac{\sigma^2 T + \sigma_J^2 n}{T}}\right) \end{aligned} \tag{18}$$

where Λ^{θ^h} denotes the stochastic jump intensity of the Markov-modulated Poisson process $\Phi^{\theta^h}(T)$, π_i denotes the stationary distribution in state i, $P^{\theta^h}(n,T)$ denotes the transition probability matrix and n denotes the number of jumps in the time interval $[0, T]$. In addition, $N(\cdot)$ denotes the cumulative distribution function of a standard normal random variable, and

$$d_{1,n} = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma_J^2 n}{\sqrt{\sigma^2 T + \sigma_J^2 n}} \tag{19}$$

and

$$d_{2,n} = d_{1,n} - \sqrt{\sigma^2 T + \sigma_J^2 n} \tag{20}$$

Equation 18 can be viewed as a weighted sum of the expected European gold call option with weights being the transition probabilities. Appendix C gives the detailed proof.

To further illustrate the property of the generalized gold option pricing formula, we consider several special cases and show their specific formulas in the following examples. In a general setting with I states, if $\lambda_1 = \lambda_2 = \ldots = \lambda_I = \lambda$, then the Markov-modulated Poisson process reduces to the single Poisson process with intensity λ. Therefore, the solution of European gold call options can be derived as follows:

$$C_{MMJM}^{\lambda^{\theta^h}}(0) = \sum_{n=0}^{\infty}\frac{\exp(-\lambda^{\theta^h}T)(\lambda^{\theta^h}T)^n}{n!} C_{BSM}\left(S(0), K, r, T, \sqrt{\frac{\sigma^2 T + \sigma_J^2 n}{T}}\right) \tag{21}$$

where $\lambda^{\theta^h} = \lambda\left(\kappa^{\theta^J} + 1\right)$ denotes the jump intensity of the Poisson process under Q^{θ^h}. If $\kappa^{\theta^J} = 0$, this equation reduces to the pricing formula of Merton (1976), and the jump intensity and distribution are not altered by the measure change. If $\lambda = 0, \sigma_J = 0$ and $n = 0$, this equation reduces to the standard Black–Scholes option pricing formula:

$$C_{BSM}(S(0), K, r, T, \sigma) = S(0)N(d_1) - K\exp(-rT)N(d_2) \tag{22}$$

where $d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

IV. Empirical and Numerical Analyses

Empirical results

The previous discussion of Table 1 sheds doubts on the validity of the GBM assumption. Motivated by these findings, we examine the ability of various continuous-time processes in capturing the gold price dynamics. Uncovering the underlying gold price is a necessary step for choosing the appropriate option pricing model. In the empirical analyses, we employ the BSM as a benchmark for the actual data analysed. For simplicity, we consider that the economy shifts between the ordinary state (state 1) and volatile state (state 2). The transition probability matrix of the two-state Markov chain $X(t)$ is given by $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & 1-P_{11} \\ 1-P_{22} & P_{22} \end{bmatrix}$. Because the market state is hidden at time zero, the stationary distribution can be evaluated by the transition probability. Thus, the stationary distributions of state 1 and state 2 are $\pi_1 = {}^{v_2}\!/_{v_1+v_2}$ and $\pi_2 = {}^{v_1}\!/_{v_1+v_2}$, respectively. Given the gold price dynamics defined in Equation 1, we express the logarithmic return in discrete time as follows:

$$R(t) = \tilde{\mu} + \tilde{\sigma} Z + \begin{cases} \sum_{k=1}^{N_1(\Delta t)} Y_k & \text{if} \quad X(t) = 1 \\ \sum_{k=1}^{N_2(\Delta t)} Y_k & \text{if} \quad X(t) = 2 \end{cases} \tag{23}$$

where $\tilde{\mu} = \left(\mu - \frac{1}{2}\sigma^2 - \Lambda\kappa\right)\Delta t$ with $\Lambda\kappa = (p_{11}\lambda_1 + p_{22}\lambda_2)(\phi_Y(1)-1))$, $\tilde{\sigma} = \sigma\sqrt{\Delta t}$, $Z \sim N(0,1)$, $Y_k \sim N(\mu_J, \sigma_J^2)$ and $N_i(\Delta t)$ is a Poisson process with the intensity rate λ_i in the interval time Δt when the Markov chain $X(t)$ remains in state i. The states and the jump arrivals are unobserved. We apply the EM algorithm (Lange, 1995a, b) to calculate the maximum likelihood (ML) estimations. In the first step, given the observed return data $\boldsymbol{R}$ and the former one-period parameters $\Theta^{(k-1)}$, we compute the conditional expectation of the log complete-data likelihood function as $\Gamma\left(\Theta, \Theta^{(k-1)}\right) = E\left[\log \Pr(\boldsymbol{R}, \boldsymbol{X}, \boldsymbol{N}|\Theta)|\boldsymbol{R}, \Theta^{(k-1)}\right]$. Then, for the second step, we maximize the Γ-function to use the parameter set as $\Theta^{(k)} = \arg\max \Gamma\left(\Theta, \Theta^{(k-1)}\right)$. By the iteration and recursive computation of these two steps, the parameters converge the Γ-function to the local maximum in the incomplete-data likelihood function. Applying the code of the EM algorithm and the complete-data information matrix, we get the SEs of parameter estimators by the SEM algorithm (Meng and Rubin, 1991). Khalaf *et al.* (2003) combine bounds and Monte Carlo simulation techniques to test the generalized autoregressive conditional heteroscedasticity (GARCH) class of models with nuisance parameters. To determine whether the JDM outperforms the BSM, and whether the data fit the MMJM better than the JDM, we apply the likelihood ratio test as follows:

$$\Upsilon = 2(\ln L_1(\Theta) - \ln L_0(\Theta)) \overset{asy}{\rightarrow} \chi^2_{d,1-\alpha} \tag{24}$$

where $L_m(\Theta)$ represents the likelihood function under the hypothesis H_m for $m = \{0, 1\}$ and d denotes the difference of the parameters between the H_0 and H_1 constraints. If $\Upsilon > \chi^2_{d,1-\alpha}$, H_0 is rejected. The respective null hypotheses are that the BSM and JDM hold.

Table 2 presents several interesting results. First, in the case of the JDM versus MMJM, we can determine the value of d by the difference in the number of

Table 2. Estimated results of continuous-time models

Parameter	BSM	JDM	MMJM
p_{11}			0.9939 (0.0057)
p_{22}			0.9885 (0.0080)
μ	0.0008 (0.0004)	0.0014 (0.0004)	0.0013 (0.0003)
μ_J		−0.0009 (0.0007)	−0.0004 (0.0004)
σ	0.0137 (0.0002)	0.0068 (0.0002)	0.0066 (0.0006)
σ_J		0.0144 (0.0003)	0.0107 (0.0012)
λ_1		0.6593 (0.1766)	0.3942 (0.1991)
λ_2			2.7658 (0.7103)
Υ_1		244.20	
Υ_2			173.66

Notes: This table presents the empirical results of dynamic models, reporting the estimated parameters and corresponding SEs. The estimation settings for the BSM and JDM/MMJM are determined via the maximum likelihood (ML) approach and EM algorithm, respectively. The SEs of the parameter estimators for the BSM and JDM/MMJM obtained by the ML approach and SEM algorithm, respectively, are reported in parentheses. Υ_1 represents the likelihood ratio test for ML functions with the null hypothesis that there is no jump event; that is, the dynamic model is a BSM. Υ_2 shows the likelihood ratio test for ML functions with the null hypothesis that there is no switching regime, that is, the dynamic model is a JDM. Performance is evaluated in terms of both the likelihood ratio test and statistical accuracy.

parameters between these two models. At the confidence level of $1 - \alpha = 0.95$, the critical value for the aforementioned test is $\chi^2_{4,0.95} = 9.49$. In the gold market, we can observe that the MMJM is better than the JDM by the likelihood ratio Υ_2 of $173.66 > 9.49$ at the 0.05 statistical significance level, as shown in Table 2, which implies that the addition of the Markov-modulated Poisson process clearly dominates the single Poisson process. Through a similar procedure, we find that the JDM clearly dominates the BSM by the likelihood ratio Υ_1 of $244.20 > \chi^2_{3,0.95} = 7.82$ at the 0.05 statistical significance level, as shown in Table 2. Second, these two transition probabilities of the gold returns for the MMJM are almost 1, implying that the economy stays in each state for a period of time and then transitions to the other. The mean and SD of the gold logarithmic returns for the MMJM are 0.0013 and 0.0066, respectively. The additional jump component prescribes a drift of −0.0004 and a volatility of 0.0107. The jump intensity is found to be 0.3942 in state 1 and 2.7658 in state 2, which clearly shows different intensity rates in different states. The findings indicate that the gold price has a GBM structure with Markov-modulated Poisson processes; that is, they are subject to regime-switching movements that cannot be explained by standard jump-diffusion processes. Furthermore, they are consistent with the findings in the descriptive analysis of Table 1, namely the non-normality of returns and the existence of different jumps over time.

Table 3 shows the same mean and variance for the original data and continuous-time models. Comparing with the original data, our MMJM characterizes the skewness and kurtosis of the logarithmic return series.

Table 3. Distributional statistics for data and continuous-time models

Classification	Data	BSM	JDM	MMJM
Mean	0.0008	0.0008	0.0008	0.0008
Variance	0.0002	0.0002	0.0002	0.0002
Skewness	−0.2500	0	−0.1461	−0.1462
Kurtosis	7.7529	3	5.5581	5.9224

Notes: This table reports the distributional statistics for the original data and continuous-time models.

Figure 2 clearly shows the different frequencies of price jumps over the sample period. The time-varying volatility is captured by the MMJM in the form of regime-switching behaviour, that is, the oscillating periods of higher and lower intensity rates. As shown in Fig. 2, our sample period ends with two relatively consistent periods of higher intensity rates, corresponding to the Iran nuclear crisis of 2006 and US subprime mortgage financial crisis of 2008.

Concerning the volatility clustering, Fig. 3 exhibits a substantially positive autocorrelation in the squared logarithmic returns of gold in which the trend steadily declines as the lag length increases. The MMJM then captures not only the existence of volatility clustering but also the magnitude and decay of this phenomenon. Overall, these empirical results suggest that the MMJM provides an adequate description for the logarithmic returns of gold. It overcomes the shortcomings of the GBM and jump-

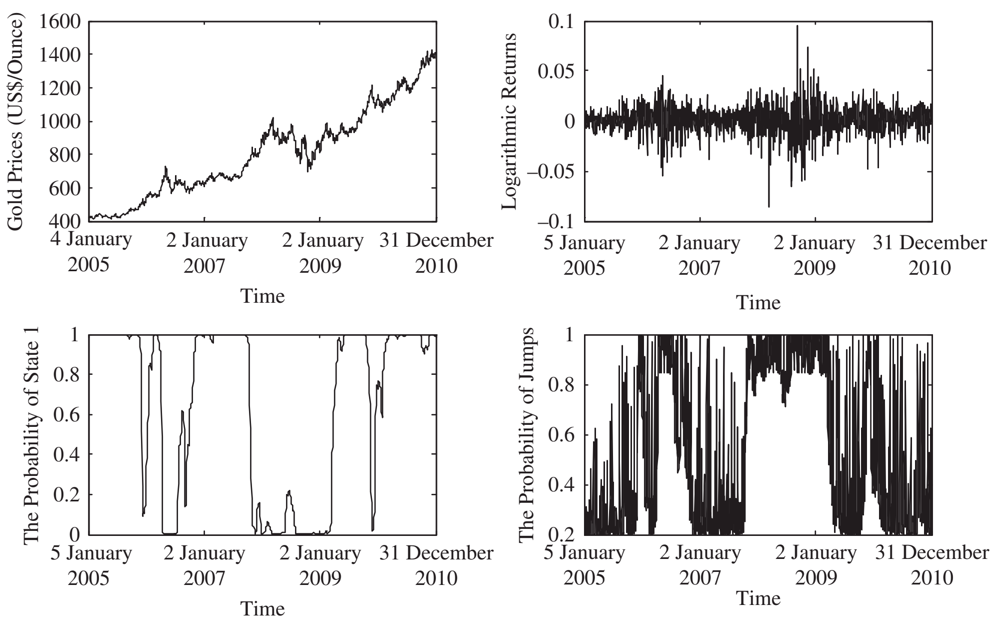

Fig. 2. Time series and jump characteristics for gold

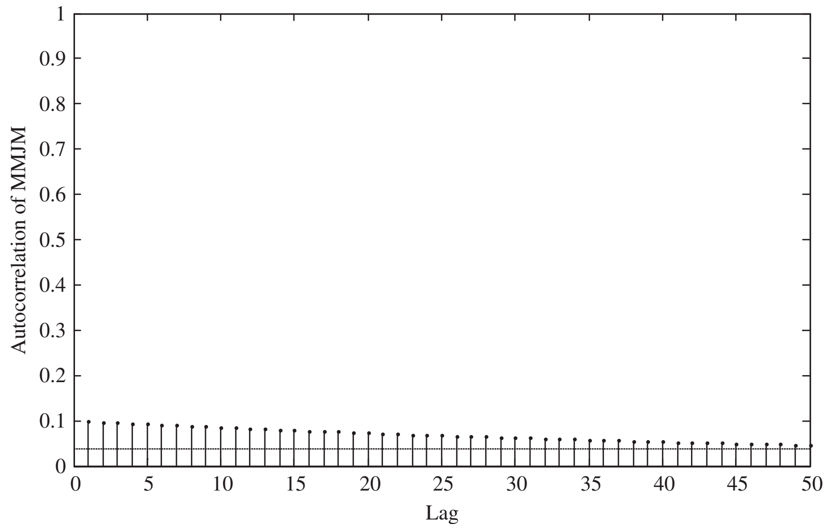

Fig. 3. Model autocorrelation of the squared logarithmic returns for gold

diffusion processes (Merton, 1976; Kou, 2002) in asset price modelling.

Pricing performance

In order to assess the empirical validity of the MMJM, we evaluate the out-of-sample pricing performance using actual option market data[2] from the European Exchange (Eurex). As a benchmark, we employ the BSM for pricing European gold options. The 1-year US treasury bill rate is used as a proxy for the risk-free rate. We apply the relative mean square errors (RMSEs) for model evaluation. An in-sample analysis and an analysis across moneyness levels are not possible because of the limited actual option price data. Table 4 reports the RMSE (pricing error) of each model using the out-of-sample data. With different strike prices K, it shows that pricing errors under the MMJM are all smaller than those of competing models in terms of RMSEs. Taking the strike price $K = 1400$ as an example, the largest improvement offered by the MMJM over the benchmark model and JDM varies between 0.0848 and 0.0538 in the reduction of RMSEs. The reduction of the RMSEs between the JDM and MMJM is more substantial than that of the RMSEs between the BSM and JDM. One can infer that for this reason, the Markov component contributes more to the superior pricing performance rather than the pure jump process. The numerical results show that the MMJM is more accurate than the BSM and JDM in pricing gold call options. As a consequence, the evidence presented suggests that it is worth accounting for regime-switching jump risks when pricing European gold call options.

Table 4. Out-of-sample pricing errors of European gold call option pricing models

K	BSM	JDM	MMJM
1300	0.0970	0.0906	0.0810
1320	0.1309	0.1217	0.1064
1340	0.1776	0.1649	0.1422
1360	0.2447	0.2274	0.1957
1380	0.3335	0.3104	0.2683
1400	0.4530	0.4220	0.3682

Notes: This table presents the RMSEs of various option pricing models. The parameters of each model are calibrated in the period from 5 January 2005 to 31 December 2010 and then the estimated parameters are used to evaluate the out-of-sample performance in the period from 24 January 2011 to 15 April 2011. The RMSEs are estimated by minimizing the sum of the squared pricing errors between model-determined prices and market prices (divided by the market prices). Pricing performance is evaluated for the aggregate sample on the basis of the RMSEs.

V. Conclusion

By analysing the gold logarithmic returns, this study finds different jump behaviours of gold prices in different time periods. The empirical results indicate that the gold price is better approximated by a GBM with Markov-modulated Poisson processes. This dynamic model also perfectly characterizes the leptokurtosis and volatility clustering of the logarithmic return series. Because the gold market described by the MMJM is incomplete, this study uses the Esscher transform to identify a risk-neutral pricing measure. After determining the risk premiums, we further derive the generalized pricing formula for

[2] The option data are from Bloomberg. These data correspond to the gold prices and cover the period between 24 January 2011 and 15 April 2011 (expiration date). There are a total of 60 observations for each call option contract.

gold options and demonstrate that the pricing formulas of the BSM and JDM are special cases of the generalized pricing formula. This study goes further to investigate the pricing performance of gold options under the BSM, JDM and MMJM in our sample data. The results show that the MMJM generates lower pricing errors than competing models, and pricing errors can be reduced with different strike prices depending on the RMSEs. In other words, we find that jump risks implied by our MMJM have a more significant impact on the gold option prices. As a consequence, the ability of market stakeholders to speculate and hedge their positions in the gold market is of utmost importance for considering regime-switching jump risks and for ensuring market efficiency.

References

Amin, K. I. (1993) Jump diffusion option valuation in discrete time, *The Journal of Finance*, **48**, 1833–63. doi:10.1111/j.1540-6261.1993.tb05130.x

Ball, C. A., Torous, W. N. and Tschoegl, A. E. (1985) An empirical investigation of the EOE gold options market, *Journal of Banking and Finance*, **9**, 101–13. doi:10.1016/0378-4266(85)90064-0

Baur, D. G. and Lucey, B. M. (2010) Is gold a hedge or a safe haven? An analysis of stocks, bonds and gold, *Financial Review*, **45**, 217–29. doi:10.1111/j.1540-6288.2010.00244.x

Baur, D. G. and McDermott, T. K. (2010) Is gold a safe haven? International evidence, *Journal of Banking and Finance*, **34**, 1886–98. doi:10.1016/j.jbankfin.2009.12.008

Beckers, S. (1984) On the efficiency of the gold options market, *Journal of Banking and Finance*, **8**, 459–70. doi:10.1016/S0378-4266(84)80064-3

Capie, F., Mills, T. C. and Wood, G. (2005) Gold as a hedge against the dollar, *Journal of International Financial Markets, Institutions and Money*, **15**, 343–52. doi:10.1016/j.intfin.2004.07.002

Chan, W. H. and Maheu, J. M. (2002) Conditional jump dynamics in stock market returns, *Journal of Business & Economic Statistics*, **20**, 377–89. doi:10.1198/073500102288618513

Chang, C., Fuh, C. D. and Lin, S. K. (2013) A tale of two regimes: theory and empirical evidence for a Markov-modulated jump diffusion model of equity returns and derivative pricing implications, *Journal of Banking and Finance*, **37**, 3204–17. doi:10.1016/j.jbankfin.2013.03.009

Cheung, Y. W. and Lai, K. S. (1993) Do gold market returns have long memory?, *Financial Review*, **28**, 181–202. doi:10.1111/j.1540-6288.1993.tb01344.x

Duan, J. C., Ritchken, P. and Sun, Z. (2006) Approximating GARCH-jump models, jump-diffusion processes, and option pricing, *Mathematical Finance*, **16**, 21–52. doi:10.1111/j.1467-9965.2006.00259.x

Elliott, R. J., Chan, L. and Siu, T. K. (2005) Option pricing and Esscher transform under regime switching, *Annals of Finance*, **1**, 423–32. doi:10.1007/s10436-005-0013-z

Elliott, R. J. and Osakwe, C.-J. U. (2006) Option pricing for pure jump processes with Markov switching compensators, *Finance and Stochastics*, **10**, 250–75. doi:10.1007/s00780-006-0004-6

Gerber, H. U. and Shiu, E. S. W. (1994) Option pricing by Esscher transforms (with discussions), *Transactions of Society of Actuaries*, **46**, 99–191.

Khalaf, L., Saphores, J.-D. and Bilodeau, J.-F. (2003) Simulation-based exact jump tests in models with conditional heteroskedasticity, *Journal of Economic Dynamics and Control*, **28**, 531–53. doi:10.1016/S0165-1889(03)00034-4

Kou, S. G. (2002) A jump-diffusion model for option pricing, *Management Science*, **48**, 1086–101. doi:10.1287/mnsc.48.8.1086.166

Lange, K. (1995a) A gradient algorithm locally equivalent to the EM algorithm, *Journal of the Royal Statistical Society*, **57**, 425–37.

Lange, K. (1995b) A quasi-Newton acceleration of the EM algorithm, *Statistica Sinica*, **5**, 1–18.

Last, G. and Brandt, A. (1995) *Marked Point Processes on the Real Line: the Dynamic Approach*, Springer-Verlag, New York.

Meng, X. L. and Rubin, D. B. (1991) Using EM to obtain asymptotic variance-covariance matrices: the SEM algorithm, *Journal of the American Statistical Association*, **86**, 899–909. doi:10.1080/01621459.1991.10475130

Merton, R. C. (1976) Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, **3**, 125–44. doi:10.1016/0304-405X(76)90022-2

Ogden, J. P., Tucker, A. L. and Vines, T. W. (1990) Arbitraging American gold spot and futures options, *Financial Review*, **25**, 577–92. doi:10.1111/j.1540-6288.1990.tb01299.x

Reboredo, J. C. (2013) Is gold a safe haven or a hedge for the US dollar? Implications for risk management, *Journal of Banking and Finance*, **37**, 2665–76. doi:10.1016/j.jbankfin.2013.03.020

Zagaglia, P. and Marzo, M. (2013) Gold and the U.S. dollar: tales from the turmoil, *Quantitative Finance*, **13**, 571–82. doi:10.1080/14697688.2012.708431

Appendix A

Proof of the martingale condition: Let E^{θ^h} denote the mathematical expectation operator with respect to the Esscher measure Q^{θ^h} equivalent to P. Using Equation 7, we obtain:

$$\begin{aligned} S(0) &= \exp(-rt)E^{\theta^h}\left[S(t)\middle|F_0\right] = \exp(-rt)E\left[\frac{dQ^{\theta^h}}{dP}S(t)\middle|F_0\right] \\ &= S(0)E\left[\exp\left(\left(\mu - r - \frac{1}{2}\sigma^2 - \Lambda\kappa\right)t + \sigma W(t) + \sum_{k=1}^{\Phi(t)} Y_k\right)\frac{dQ^{\theta^h}}{dP}\right] \\ &= S(0)\exp\left(\left(\mu - r - \frac{1}{2}\sigma^2 - \Lambda\kappa\right)t + \frac{1}{2}\left((1+\theta^C)\sigma\right)^2 t - \frac{1}{2}\left(\theta^C\sigma\right)^2 t\right)\cdot\exp\left(\Lambda\left(\kappa^{(1+\theta^J)} - \kappa^{\theta^J}\right)t\right) \end{aligned} \tag{A1}$$

From the mutual independence of random shocks $W(t)$, $\Phi(t)$ and Y_k, and then the martingale condition $E^{\theta^h}[\exp(-rt)S(t)|F_0] = S(0)$ holds if and only if the Esscher parameters θ^C and θ^J satisfy:

$$\mu - r - \Lambda\kappa + \theta^C\sigma^2 = 0 \tag{A2}$$

and

$$\mu_J + \frac{1}{2}\sigma_J^2 + \theta^J\sigma_J^2 = 0 \tag{A3}$$

for all $t \in [0, T]$. Therefore, we can define a pair of solutions of Esscher parameters for the martingale condition by Equations 8 and 9.

Appendix B

Proof of the Esscher transforms under the MMJM: Based on Equation 7, we apply the Girsanov theorem, and by the mutual independence of random shocks $W(t)$, $\Phi(t)$ and Y_k, we find that $W^{\theta^h}(t) = W(t) - \theta^C\sigma t$ is a Wiener process under Q^{θ^h}. Next we denote the moment-generating function of the random variable Y_k by $\phi_Y(\theta^J) = E\left[\exp\left(\theta^J Y_k\right)\right] = \kappa^{\theta^J} + 1$. This does not depend on the index k because $\{Y_k : k = 1, 2, \ldots\}$ all have the same distribution. Then, we have:

$$\begin{aligned} E\left[\exp\left(\theta^J\sum_{k=1}^{\Phi(t)} Y_k\right)\right] &= P(\Phi(t)=0) + \sum_{n=1}^{\infty} E\left[\exp\left(\theta^J\sum_{k=1}^{n} Y_k\right)\middle|\Phi(t)=n\right]P(\Phi(t)=n) \\ &= \sum_{n=0}^{\infty}\left(E\left[\exp\left(\theta^J Y_k\right)\right]\right)^n P(n,t) = \sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i\left(\phi_Y(\theta^J)\right)^n P_{ij}(n,t) = \exp\left(\Lambda\kappa^{\theta^J}t\right) \end{aligned} \tag{B1}$$

where π_i denotes the stationary distribution in state i. This limiting distribution can be computed by $\psi_{jj}\pi_i = \sum_{k=1,k\neq j}^{I}\psi_{kj}\pi_j$ along with the constraint $\sum_{j=1}^{I}\pi_j = 1$. Specifically, we note that $\theta^J\sum_{k=1}^{\Phi(t)} Y_k - \Lambda\kappa^{\theta^J}t$ is a martingale at time t. Given $\Phi(t) = n$, the Radon–Nikodym derivative of the transition probability can be set as follows:

$$\left.\frac{dQ_{prob.}^{\theta^h}}{dP_{prob.}}\right|_{\Phi(t)=n} = \left(\kappa^{\theta^J}+1\right)^n\exp\left(-\Lambda\kappa^{\theta^J}t\right) \tag{B2}$$

Then, we get $dP^{\theta^h}(n,t) = dP(n,t)\left(\kappa^{\theta^J}+1\right)^n\exp\left(-\Lambda\kappa^{\theta^J}t\right)$, where $P^{\theta^h}(n,t)$ denotes the transition probability matrix under Q^{θ^h}. Also, we use Equation 3 and its unique solution given by Equation 5. Letting $P^{\theta^h}(n,t) = P(n,t)\left(\kappa^{\theta^J}+1\right)^n\exp\left(-\Lambda\kappa^{\theta^J}t\right)$, we get:

$$\begin{aligned} P^{\theta^h\Theta}(\zeta,t) &= \sum_{n=0}^{\infty}P(n,t)\left(\kappa^{\theta^J}+1\right)^n\exp\left(-\Lambda\kappa^{\theta^J}t\right)\zeta^n \\ &= \sum_{n=0}^{\infty}P(n,t)\left(\zeta\left(\kappa^{\theta^J}+1\right)\right)^n\exp\left(-\Lambda\kappa^{\theta^J}t\right) \\ &= \exp\left(\left(\Psi - (1-\zeta)\Lambda\left(\kappa^{\theta^J}+1\right)\right)t\right) \end{aligned} \tag{B3}$$

Therefore, under Q^{θ^h}, the jump risk can be formulated by the Esscher transform intensity of the Markov-modulated Poisson process. The stochastic jump intensity $\Lambda^{\theta^h} = \Lambda\left(\kappa^{\theta^J}+1\right) = \Lambda\exp\left(\theta^J\mu_J + \frac{1}{2}\left(\theta^J\sigma_J\right)^2\right)$ is

$$
\begin{aligned}
B &= K\exp(-rT)E^{\theta^h}\left[1_{\{S(T)\geq K\}}\right] \\
&= K\exp(-rT)\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}^{\theta^h}(n,T)Q^{\theta^h}\left(-\frac{N(0,\sigma^2T+\sigma_J^2 n)}{\sqrt{\sigma^2T+\sigma_J^2 n}}\leq d_{2,n}\middle|\Phi^{\theta^h}(T)=n\right) \\
&= K\exp(-rT)\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}^{\theta^h}(n,T)N(d_{2,n})
\end{aligned}
\quad (C2)
$$

altered by the measure change. Finally, we investigate the jump size, where $\{Y_1, Y_2, \ldots, Y_n\}$ are independently identically distributed random variables. Hence the Radon–Nikodym derivative of each specific jump size can be written as:

$$
\left.\frac{dQ_Y^{\theta^h}}{dP_Y}\right|_{F_t^X} = \frac{\exp(\theta^J Y_k)}{E\left[\exp(\theta^J Y_k)\middle|_{F_0^X}\right]} \quad (B4)
$$

Then, we obtain $dQ_Y^{\theta^h} = \frac{1}{\sqrt{2\pi\sigma_J^2}}\exp\left(-\frac{\left(Y_k-\left(\mu_J+\theta^J\sigma_J^2\right)\right)^2}{2\sigma_J^2}\right)$. Furthermore, under the physical probability measure P, the density function of each specific jump size Y_k is $f_Y(y)$. Through the change of measures, under Q^{θ^h}, the density function of each specific jump size $Y_k^{\theta^h}$ is $f_Y^{\theta^h}(y) = f_Y(y)\cdot\left.\frac{dQ_Y^{\theta^h}}{dP_Y}\right|_{F_t^X}$.

Appendix C

Proof of the derivation of the generalized gold option pricing formula: Let $C_{MMJM}^{\Lambda^{\theta^h}}(0)$ represent the value of the option at time zero with strike price K and matured at time T, and we have the following equation:

$$
C_{MMJM}^{\Lambda^{\theta^h}}(0) = \exp(-rT)E^{\theta^h}\left[(S(T)-K)^+|F_0\right] = A - B \quad (C1)
$$

First, we calculate B as follows:

where $d_{2,n} = \frac{\ln(\frac{S(0)}{K})+(r-\frac{1}{2}\sigma^2)T-\frac{1}{2}\sigma_J^2 n}{\sqrt{\sigma^2T+\sigma_J^2 n}}$. Second, we calculate A:

$$
A = \exp(-rT)E^{\theta^h}\left[S(T)1_{\{S(T)\geq K\}}\right] \quad (C3)
$$

We denote the Radon–Nikodym derivative for the continuous diffusion part by the following formula:

$$
\left.\frac{dQ^{(1+\theta^h)}}{dQ^{\theta^h}}\right|_{F_t} = \exp\left(\sigma W^{\theta^h}(t) - \frac{1}{2}\sigma^2 t\right) \quad (C4)
$$

For each $t\in[0,T]$, under the risk-neutral pricing measure $Q^{(1+\theta^h)}$, we have the gold price dynamics as:

$$
S(t) = S(0)\exp\left\{\left(r+\frac{1}{2}\sigma^2\right)t + \sigma W^{(1+\theta^h)}(t) + \sum_{k=1}^{\Phi^{(1+\theta^h)}(t)} Y_k^{(1+\theta^h)}\right\} \quad (C5)
$$

where $W^{(1+\theta^h)}(t)$ is a Wiener process, $\Phi^{(1+\theta^h)}(t)$ is a Markov-modulated Poisson process with the transition probability matrix $P^{(1+\theta^h)}(n,t) = P^{\theta^h}(n,t)$, and the jump size $Y_k^{(1+\theta^h)} \overset{i.i.d.}{\sim} N\left(\frac{1}{2}\sigma_J^2, \sigma_J^2\right)$. Hence, Equation C3 can be rewritten as:

$$
\begin{aligned}
& S(0)E^{(1+\theta^h)}\left[1_{\{S(T)\geq K\}}\right] \\
&= S(0)\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}^{(1+\theta^h)}(n,T)Q^{(1+\theta^h)}\left(-\frac{N(0,\sigma^2T+\sigma_J^2 n)}{\sqrt{\sigma^2T+\sigma_J^2 n}}\leq d_{1,n}\middle|\Phi^{(1+\theta^h)}(T)=n\right) \\
&= S(0)\sum_{n=0}^{\infty}\sum_{i=1}^{I}\sum_{j=1}^{I}\pi_i P_{ij}^{(1+\theta^h)}(n,T)N(d_{1,n})
\end{aligned}
\quad (C6)
$$

where $d_{1,n} = \frac{\ln(\frac{S(0)}{K})+(r+\frac{1}{2}\sigma^2)T+\frac{1}{2}\sigma_J^2 n}{\sqrt{\sigma^2T+\sigma_J^2 n}}$. Combining Equations C2 and C6, the generalized gold option pricing formula is obtained.