Pricing generalized capped exchange options

Chou-Wen Wang\textsuperscript{a}, Szu-Lang Liaob and Ting-Yi Wuc,\textsuperscript{c,*}

\textsuperscript{a}Department of Risk Management and Insurance, National Kaohsiung First University of Science and Technology, Taiwan
\textsuperscript{b}Department of Money and Banking, National Chengchi University, Taiwan
\textsuperscript{c}National Kaohsiung First University of Science and Technology and Kao Yuan University, Taiwan

The article makes two contributions to the literature. The first contribution is to derive a closed-form solution of Taiwanese capped options. We also provide the properties of Taiwanese capped options and the phenomenon of delta jump at monitoring dates. When the interest rate changes dramatically, instead of deriving the pricing formulas for derivatives separately, the second contribution is to provide the closed-form solution of generalized capped exchange options with stochastic barriers under the Hull and White framework. Special cases of generalized capped exchange options with stochastic barriers are abundant. They include capped (floored) options, capped (floored) options with exponential barriers, capped (floored) options with related assets or indices as triggers and capped (floored) options with related assets or indices as triggers and other related assets as barriers.

I. Introduction

Path-dependent options, whose payoffs are influenced by the path of the prices of underlying assets, have become increasingly popular in recent years. One of the path-dependent options is the capped (floored) option. The capped (floored) option would be early exercised if the stock price at a pre-specified date is higher (lower) than the barrier. In practice, the capped options have been traded for many years. CBOE first issued the CAPS option with the S&P index in 1991. In 1999, Polaris Securities in Taiwan issued a capped warrant that the option will be early exercised if the closing prices of 2303, United Microelectronics Corporation, on the Taiwan Stock Exchange (TSE) are above 150\% of the initial strike price $90 during the whole period after the issue of the warrant. We show the six Taiwanese capped options listed on the Taiwan Stock Exchange from 1998 to 1999 in Table 1.

Options similar to Taiwanese capped (floored) options are reset options and barrier options. Their common factor is that the barrier will influence the option’s payoff. Their differences are when the stock price touches the barrier at some monitoring date, the strike price of the reset option will be adjusted to a new level, but the capped (floored) option will be automatically early exercised and the barrier option will become a plain vanilla option or become nullified.

For reset options, Gray and Whaley (1997) examined the pricing of the put warrant with periodic reset and the warrant’s risk characteristics. Gray and Whaley (1999) also provided a closed-form solution for reset options with a single reset date. Cheng and Zhang (2000) studied the reset options for which the

*Corresponding author. E-mail: wty001@yahoo.com.tw
II. Pricing Taiwanese Capped Options

In this section we first introduce the trading economy and then derive the closed-form valuation of Taiwanese capped options. We assume that the dynamics of the domestic underlying asset price $S(t)$ are described by the following linear stochastic differential equation:

$$dS(t) = (u - \delta)S(t)dt + \sigma S(t)dW_t$$  \hspace{1cm} (1)$$

where $u$ and $\sigma > 0$ are constants, $\delta$ is the continuous dividend yield and $W_t$ is a one-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, F, P)$. The money market account, $B(t)$, corresponds to the wealth accumulated from an initial $1$ investment at spot interest rate $r$ in each subsequent period. Therefore,

$$dB(t) = rB(t)dt$$  \hspace{1cm} (2)$$

or equivalently,

$$B(T) = B(t)e^{(T-t)}$$  \hspace{1cm} (3)$$

Assume that there exists a unique risk-neutral probability measure $P^*$ on $(\Omega, F)$, which is given by the Radon–Nikodym derivative

$$\frac{dP^*}{dP} = \exp\left(\frac{r - u}{\sigma} W_T - \frac{1}{2} \left(\frac{r - u}{\sigma}\right)^2 T\right), \hspace{1cm} 0 \leq t < T$$  \hspace{1cm} (4)$$

where $T$ is the maturity date of options. Under the spot martingale measure or risk neutral probability
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measure $P^*$, the dynamics of the underlying asset price $S(t)$ become
\[
dS(t) = (r - \delta)S(t)dt + \sigma S(t)dW_t^\ast
\]
where the process $W_t^\ast$ is defined by
\[
dW_t^\ast = dW_t - \frac{r - u}{\sigma}dt
\]
and, by Girsanov's theorem, $W_t^\ast$ is a standard Brownian motion under $P^*$.

Now, consider Taiwanese capped options with discrete monitoring dates at time $T_j, j = 1, \ldots, m$ and $t < T_1 < T_2 < \ldots < T_n \leq T$. The barrier level $B$ is higher than the strike price $K$. The payoff of the option is determined by the following conditions: 1. If the stock price is below the barrier level $B$ at all monitoring dates $T_j, j = 1, \ldots, n$, the terminal payoff is
\[
\max(S(T) - K, 0) \equiv (S(T) - K)^+
\]
If the stock price at some monitoring date $T_k$ touch the barrier for the first time, the option will terminate at time $T_k$ and the payoff of the Taiwanese capped option at time $T_k$ is
\[
C^T_{Capped} = (S(T_k) - K)^+ = S(T_k) - K
\]
Accordingly, the valuation of a Taiwanese capped option with discrete monitoring dates at time $t$ is as follows:
\[
C^T_{Capped} = \exp[-(r(T_1 - t))E^*_\mu\{[S(T_1) - K]I(S(T_1) \geq B)\}F_t] + \sum_{j=2}^{n}\exp[-(r(T_j - t))E^*_\mu\{[S(T_j) - K]I(S(T_j) \geq B, S(T_h) < B, h < j)\}F_t] \geq B, S(T_h) < B, h < j]F_t + \exp[-(r(T - t))E^*_\mu\{[S(T) - K]I(S(T_h) \geq B, h = 1, \ldots, n)\}F_t] < B, h = 1, \ldots, n]F_t
\]
where $I(\cdot)$ is the indicator function and $E^*_\mu(\cdot)$ is the expectation under the risk-neutral probability measure $P^*$. We present the closed-form solution of (9) in the following theorem.

**Theorem 1**: Under assumptions (1) and (2), the closed-form solution of the Taiwanese capped option $C^T_{Capped}$ is as follows:
\[
C^T_{Capped} = S(t)\left\{\sum_{j=1}^{n+1}\exp(-\delta(T_j - t))Y_{ij}\right\} - K\left\{\sum_{j=1}^{n+1}\exp(-r(T_j - t))Y_{ij}\right\}
\]
where
\[
Y_{ij} = \begin{cases} 
N[g_1(B)] & \text{for } j = 1 \\
N_j[-g_1(B), \ldots, -g_{j-1}(B), g_j(B), \Psi_j] & \text{for } j = 2, \ldots, n \\
N_{n+1}[-g_1(B), \ldots, -g_n(B), g_{n+1}(K), \Psi_{n+1}] & \text{for } j = n + 1 
\end{cases}
\]
\[
g_{i}(x) = \frac{\ln(S(t)/x) + (r - \delta + (1/2)\sigma^2)(T_h - t)}{\sigma\sqrt{T_h - t}}
\]
\[
q_{i}(x) = \frac{\ln(S(t)/x) + (r - \delta - (1/2)\sigma^2)(T_h - t)}{\sigma\sqrt{T_h - t}}
\]
\[
\Psi_j = \rho_{i,j}\}_{i=1}^{j-1}, k = 1, \ldots, j, i = 2, \ldots, n + 1
\]
where $N(\cdot, \cdot)$ is the cumulative multivariate normal distribution with covariance matrix $\sum$ and $\rho_{i,k}$ is given by
\[
\rho_{i,k} = \rho_{i,i} = 1 = \frac{\min(T_i, T_k) - t}{\max(T_i, T_k) - t}
\]
\[
\rho_{i,k} = \frac{T_i - t}{T_k - t}
\]

The proof of Theorem 1 is shown in Appendix A.

If $T_n = T_{n+1} = T$, it means that the last day of the monitoring period is time $T$. Based on the information set $\{S(T_h) < B, h = 1, \ldots, n - 1\}$, no matter whether the price of $S(T)$ is less than, equal to or larger than $B$, the terminal payoff is $C^T_{Capped} = (S(T) - K)^+$. Therefore, the closed-form solution of Taiwanese capped options is the same as the case of $n - 1$ monitoring dates with $t < T_1 < T_2 < \ldots, < T_{n-1} < T$. 

Theorem 1
Similar to the closed-form valuations of exotic options, such as options on the maximum or minimum of several assets (Johnson, 1987), discrete partial barrier options (Heynen and Kat, 1996), reset options (Cheng and Zhang, 2000), Liao and Wang, 2003), or economic models with limited dependent variables, including multinomial probit, panel studies, spatial analysis and time series analysis, the closed-form solutions for Taiwanese capped options involve the calculation of multivariate normal distributions.

Among the methods of evaluating multivariate normal cumulative probabilities, as pointed out by Gollwitzer and Rackwitz (1987), Deák (1988) and Vijverberg (1997), Monte Carlo simulator methods seem to be the most promising methods for higher-order probabilities, preferable over analytical approximations or numerical integration methods. Hajivassiliou et al. (1996) surveyed eleven Monte Carlo techniques for evaluating multivariate normal probabilities and they found that the Geweke- Hajivassiliou-Keane (GHK) simulator is overall the most reliable method. Consequently, for the closed-form solution for Taiwanese capped options with a large number of discrete monitoring dates, we suggest using the GHK simulator to compute the multivariate normal cumulative probabilities.

However, the results from the GHK simulator may change with different trials. Fortunately, the correlation matrix of the cumulative multivariate normal distribution in (10) satisfies the conditions: \( \rho_{ij} = \gamma_i / \gamma_j \) and \( |\gamma_i| < |\gamma_j| \) for \( i < j \). Curnow and Dunnett (1961) showed that an \( n \)-dimensional integral of multivariate normal distribution with the above restriction can be reduced to either a \( n/2 \)-dimensional integral or a \( (n-1)/2 \)-dimensional integral depending on whether \( n \) is even or odd. Hence, if the number of monitoring dates is not too large, we suggest using the dimension-reduction method proposed by Curnow and Dunnett (1961) with a suitable numerical integral such as Simpson’s Rule to compute the numerical values of Taiwanese capped options more accurately.

### III. Numerical Analysis of Taiwanese Capped Options

#### Characteristics of Taiwanese capped options

We now consider some important properties of Taiwanese capped options using a numerical example. Assume that \( \delta = 0 \), \( K = 100 \), \( t = 0 \) and \( T = 1 \) (one year to maturity). We summarize the numerical results in Table 2. There are some properties possessed by Taiwanese capped options: (1) The values of Taiwanese capped options decrease with the number of monitoring dates and increase with the current price of the underlying asset, risk-free interest rate and volatility of the underlying asset. (2) Due to the higher probability that the price of the underlying asset will touch the barrier \( B \), a lower barrier level \( B \) has lower values for Taiwanese capped options. (3) Because the values of Taiwanese capped options are decreasing functions of the number of monitoring dates, the values of these options are always smaller than the values of plain vanilla European call options. As a result, under the same terms, a Taiwanese capped option with more intensive monitoring dates has a lower value.

#### Delta jump of Taiwanese capped options

In addition, Taiwanese capped options also exhibit the phenomenon of delta jump. In order to describe the phenomenon of delta jump without loss of generality, let us examine a scenario with only one monitoring date and \( T_1 < T_2 = T \). The delta under \( n = 1 \) is as follows:

\[
\Delta(t, S(t)) = \left\{ \sum_{j=1}^{2} \exp(-\delta(T_j - t)) Y_j \right\}
\]

\[
+ \frac{(B - K) \exp(-r(T_1 - t) - 0.5q_1^2(B))}{S(t) \sigma \sqrt{2\pi(T_1 - t)}}
\]

\[
+ \exp(-\delta(T - t)) \left[ \frac{\exp(-0.5q_2^2(K))}{\sigma \sqrt{2\pi(T - t)}} \right]
\]

\[
\times N \left( \frac{-g_4(B) - \rho g_2(K)}{\sqrt{1 - \rho^2}} \right)
\]

\[
+ \frac{\exp(-0.5q_1^2(B))}{\sigma \sqrt{2\pi(T_1 - t)}} N \left( \frac{g_2(K) + \rho g_1(B)}{\sqrt{1 - \rho^2}} \right)
\]

\[
- \frac{K}{S(t)} \left[ \exp(-r(T - t)) \frac{\exp(-0.5q_1^2(K))}{\sigma \sqrt{2\pi(T - t)}} \right]
\]

\[
\times N \left( \frac{-g_1(B) - \rho q_2(K)}{\sqrt{1 - \rho^2}} \right)
\]

\[
+ \frac{\exp(-0.5q_2^2(B))}{\sigma \sqrt{2\pi(T_1 - t)}} N \left( \frac{q_2(K) + \rho q_1(B)}{\sqrt{1 - \rho^2}} \right)
\]
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When \( t \to T_1 \), \( g_1(B) \to \infty \) and \( q_1(B) \to \infty \), consequently, as \( t \to T_1 \), we have
\[
\lim_{t \to T_1} \text{Delta}(t, S(t)) = 1, \text{ for } S(t) \geq B
\] (18)

For time \( t > T_1 \), if the price of the underlying asset at time \( T_1 \) was larger than the barrier level \( S(T_1) > B \), the option has been automatically exercised so that the delta at time \( t \) is 0. Otherwise, the delta at time \( t > T_1 \) is the same as the delta of plain vanilla European call options. In sum, the delta at time \( t > T_1 \) is given by the following expression:
\[
\text{Delta}(T_1, S(T_1)) = \begin{cases} 
1 & \text{if } S(T_1) < B \\
& \text{or } q_1(B) \to \infty \\
0 & \text{otherwise}
\end{cases}
\]

From (18) and (19), we can see that the delta at \( T_1 \) is discontinuous in the underlying asset price. Such a discontinuity does increase the difficulty of risk management.

IV. Pricin Generalized Capped Exchange Options Under Stochastic Interest Rates

Since Taiwanese capped options are short-term securities, we assume the short-term risk-free rate is constant in the above sections. However, in some circumstances, the short-term rate may fluctuate dramatically. For instance, the financial markets may face uncertain prospects caused by unexpected change of monetary policy or other financial shocks. When these events happen, the short-term rate would change significantly. Thus, instead of assuming a constant interest rate, we also derive the pricing formulas under stochastic interest rates by using extended Vasicek’s model. Moreover, in the following subsection, we provide the closed-form solutions of generalized capped exchange options with stochastic barriers and other assets as triggers. Taiwanese capped options are thus one of the special cases.

Security market economy under the extended vasicek framework

Assume that there are no transaction costs, taxes or restrictions on short selling or other market imperfections in the trading economy. In the trading interval \([0, T^*] \), let \( W_t = [W_1(t), \ldots, W_d(t); t \in [0, T]] \) be a \( d \)-dimensional standard Brownian motion given on a filtered probability space \((\Omega, F, P)\). Under the Hull and White (extended Vasicek) model, the instantaneous spot rate process is given by the integrated version of (20):
\[
dr(t) = [\theta(t) - \beta(t)r(t)]dt + \sigma_r(t) \cdot dW_t, \quad 0 \leq t \leq T
\] (20)

where \( \sigma_r(t) \) is a \( d \)-factor time-varying volatility. We also assume the dynamics of the \( y \)th primary security, \( S_y \), are given by the following expression
\[
\frac{dS_y(t)}{S_y(t)} = [\mu_y(t) - \delta_y(t)]dt + \sigma_y(t) \cdot dW_t
\] (21)

where \( \sigma_y(t) \in \mathbb{R}^d \) is a \( d \)-factor time-varying volatility; \( \mu_y(t) \) is the expected rate of return and \( \delta_y(t) \) represents the continuous time-varying dividend payout rate at time \( t \). Hence, under the setup of (21), the time-varying instantaneous correlation coefficient between the spot rate and the price of primary security \( S_y \) is given by \( \rho_{rt} = \frac{\sigma_r(t) \cdot \sigma_y(t) / \sqrt{\sigma_r(t)} \cdot \sqrt{\sigma_y(t)}^2}{|\sigma_r(t)|} \), where \(|\cdot|\) is the Euclidean norm in \( \mathbb{R}^d \).

Moreover, we assume that \( B(t, T) \) is the price at time \( t \) of a zero coupon bond that pays one dollar at time \( T \). From no-arbitrage opportunity condition, the forward price \( F^*_y(t, T) \) of a primary security \( S_y \) at time \( t \) for settlement date \( T \) equals \( \exp(-\int_t^T \delta_y(s)ds)S_y(t) / B(t, T) \). Under the forward neutral probability measure \( P_T \), the forward price of stock and contingent claims at time \( t \), with settlement date \( T \), follows a \( P_T \)-martingale. Hence, by Ito’s Lemma, the forward price \( F^*_y(t, T) \) satisfies
\[
d\ln[F^*_y(t, T)] = \left[ \frac{1}{2} \gamma^2_y(t, T) \right] dt + \gamma_y(t, T) \cdot dW_t^T
\] (22)

where \( \gamma_y(t, T) = \sigma_y(t) - b(t, T) \) is the \( d \)-factor volatility of the forward price of the primary security; \( b(t, T) = -C(t, T) \sigma_y(t) \), \( C(t, T) = \int_t^T \exp(-[b(s)-b(t)]s)ds \) and \( b(t) = \int_0^t \beta(s)ds \). \( \gamma_y(t, T) = [1 / B(t, T)] \times \partial B(t, T) / \partial r \) is the semielasticity of the zero coupon bond price \( B(t, T) \) with respect to the spot rate \( r \). In addition, one or more of the assets indexed by \( y = 1, 2, 3, 4 \) can easily be zero coupon bonds with maturities \( T_y \leq T \) and \( \sigma_y(t) = -C(t, T_y) \sigma_y(t) \).

Closed-form solution for generalized capped exchange options with a stochastic barrier and another asset as trigger

We now consider the generalized capped options with discrete monitoring dates at time \( T_j, j = 1, \ldots, n \) and \( t < T_1 < T_2 \cdots T_n \leq T \). Let \( S_1 \) be the trigger asset and \( S_2 \) be the stochastic barrier. The payoff of a generalized capped exchange option is...
determined by the following conditions: If the price of $S_1$ is less than the price of $S_2$ at each monitoring date $T_j$ for $j = 1, \ldots, n$, the terminal payoff is

$$\max(S_3(T) - S_4(T), 0) = (S_3(T) - S_4(T))^+$$

(23)

Otherwise, if the price of $S_1$ at some monitoring date $T_k$ is greater than or equal to the price of $S_2$ for the first time, the option will be terminated at time $T_k$ with payoff at time $T_k$:

$$C_{T_k}^{\text{Capped}} = (S_3(T_k) - S_4(T_k))^+$$

(24)

Accordingly, the valuation of a capped exchange option with discrete monitoring dates and stochastic barrier at time $t$ is as follows:

$$C_t^{\text{Capped}} = B(t, T_1)E_1 \left[ [S_3(T_1) - S_4(T_1)]^+ \times I(S_1(T_1) \geq S_2(T_1)) | F_t \right] + \sum_{j=2}^{n} B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \times I(S_1(T_j) \geq S_2(T_j), S_1(T_k) < S_2(T_k), h < j) | F_t \right] + B(t, T)E_{n+1} \left[ [S_3(T) - S_4(T)]^+ \times I(S_1(T_k) < S_2(T_k), h = 1, \ldots, n) | F_t \right]$$

(25)

where $E_j(\cdot)$ is the expectation under the forward neutral probability measure $P_{T_j}$ for the settlement date $T_j$, $j = 1, \ldots, n + 1$. We show the closed-form solution of (25) in the following theorem.

**Theorem 2:** Under the assumptions of (20) and (21), the closed-form solution of a generalized capped exchange option is as follows:

$$C_t^{\text{Capped}} = S_3(t) \sum_{j=1}^{n+1} \exp \left( - \int_{t}^{T_j} \delta_3(u) du \right) M_{ij} - S_4(t) \sum_{j=1}^{n+1} \exp \left( - \int_{t}^{T_j} \delta_4(u) du \right) M_{2j}$$

(26)

where

$$M_{1j} = \begin{cases} N_2 \left[ d_4(T_1, T_1), e_1(T_1), \sum_{j=1}^{2} \right] & \text{for } j = 1 \\ N_{j+1} \left[ -d_3(T_1, T_j), \ldots, -d_3(T_{j-1}, T_j), d_3(T_j, T_j), e_1(T_j), \sum_{j=1}^{2} \right] & \text{for } j = 2, \ldots, n \\ N_{n+1} \left[ -d_3(T_1, T), \ldots, -d_3(T_n, T), e_1(T), \sum_{j=1}^{2} \right] & \text{for } j = n + 1 \end{cases}$$

$$M_{2j} = \begin{cases} N_2 \left[ d_4(T_1, T_1), e_2(T_1), \sum_{j=1}^{2} \right] & \text{for } j = 1 \\ N_{j+1} \left[ -d_4(T_1, T_j), \ldots, -d_4(T_{j-1}, T_j), d_4(T_j, T_j), e_2(T_j), \sum_{j=1}^{2} \right] & \text{for } j = 2, \ldots, n \\ N_{n+1} \left[ -d_4(T_1, T), \ldots, -d_4(T_n, T), e_2(T), \sum_{j=1}^{2} \right] & \text{for } j = n + 1 \end{cases}$$

(27)

where $N(i, \sum)$ is the cumulative multivariate normal distribution with mean vector $0$ and covariance matrix $\sum$ and

$$d_n(T_q, T_k) = \frac{\ln(F_{S_3}(t, T_j)/F_{S_4}(t, T_j)) - G_{d}(T_q, T_k)}{v_{1,2}(T_q, T_k)}$$

(29)

$$e_{1,2}(T_k) = \frac{\ln(F_{S_3}(t, T_j)/F_{S_4}(t, T_j)) \pm (1/2)v_{1,2}^2(T_k, T_k)}{v_{1,2}(T_q, T_k)}$$

(30)

$$G_{d}(T_q, T_k) = \int_{t}^{T_q} \frac{1}{2} \gamma_1(u, T_k)^2 - \frac{1}{2} \gamma_2(u, T_k)^2 - \gamma_3(u, T_k) \cdot (\gamma_1(u, T_k) - \gamma_2(u, T_k)) du$$

(31)

$$v_{1,2}^2(T_q, T_k) = \int_{t}^{T_q} \gamma_1(u, T_k)^2 du$$

(32)

$$\sum_{j} = (\hat{\rho}_{i,k})_{(n+1) \times (n+1)}, i, k = 1, \ldots, n + 1$$

(33)

$$\sum_{j} = (\rho_{i,k})_{(n+1) \times (n+1)}, i, k = 1, \ldots, j + 1$$

(34)

where $\hat{\rho}_{i,k}$ and $\rho_{i,k}$ are given by

$$\hat{\rho}_{i,k} = \rho_{i,k} = \begin{cases} \frac{\int_{T_i}^{T_{j+1}} (\gamma_1(u, T_k) - \gamma_2(u, T_k)) du}{v_{1,2}(T_i, T) v_{1,2}(T, T)} & \text{for } i = 1, \ldots, n, k = n + 1 \\ 1 & \text{for } i = k \end{cases}$$

(35)
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The proof of Theorem 2 is shown in Appendix B.

Similarly, for the special case of $T_n = T_{n+1} = T$, based on the information set $\{S_1(T_h) < S_2(T_h), h = 1, \ldots, n - 1\}$, no matter what the price of $S_i(T)$ or $S_2(T)$ is, the terminal payoff is $C_{27}^{\text{Capped}} = (S_3(T) - S_4(T))^+$. Consequently, the closed-form solution of the generalized capped exchange options is the same as the case of $n - 1$ monitoring dates with $t < T_1 < T_2 < \cdots < T_{n-1} < T$.

V. Special Cases of Generalized Capped Exchange Options

In this section, we discuss some special cases of the generalized capped exchange options and the corresponding properties.

Taiwanese capped options

Consider the simplest case of $B(t, T) = \exp(-r(T - t))$, $S_2 = B$, $S_1 = S_3 = S$, $\delta_1(t) = \delta$, $\sigma_1(t) = \sigma$, $S_4 = K$ and $K < B$. Under this setup, the generalized capped exchange option is the Taiwanese capped option with constant barrier $B$, constant interest rate $r$ and constant dividend yield $\delta$. Then, the closed-form solution for the Taiwanese capped option $C_7^{\text{Capped}}$ at time $t$ is the same as (10).

Taiwanese floored options

In this subsection, we consider another special case of generalized capped exchange options with $B(t, T) = \exp(-r(T - t))$, $S_1 = F$, $S_3 = K$, $\delta_2(t) = \delta$, $\sigma_2(t) = \sigma$, $S_2 = S_4 = S$ and $K > F$. Under this setup, the generalized capped exchange option is the Taiwanese floored option with constant barrier $F$, constant interest rate $r$ and constant dividend yield $\delta$. Consequently, we can rewrite Equation 26 as follows:

$$C_7^{\text{Floored}} = K \left\{ \sum_{j=1}^{n+1} \exp(-r(T_j - t))X_{2j} \right\} - S \left\{ \sum_{j=1}^{n+1} \exp(-\delta(T_j - t))X_{1j} \right\}$$

where

$$X_{1j} = \begin{cases} N[-q_1(F)] & \text{for } j = 1 \\ N[g_1(F), \ldots, g_{j-1}(F), -q_j(F), \Psi_j] & \text{for } j = 2, \ldots, n \\ N_{n+1}[g_1(F), \ldots, g_n(F), -q_{n+1}(K), \Psi_{n+1}] & \text{for } j = n + 1 \end{cases}$$

$$X_{2j} = \begin{cases} N[-q_1(F)] & \text{for } j = 1 \\ N[g_1(F), \ldots, g_{j-1}(F), -q_j(F), \Psi_j] & \text{for } j = 2, \ldots, n \\ N_{n+1}[g_1(F), \ldots, g_n(F), -q_{n+1}(K), \Psi_{n+1}] & \text{for } j = n + 1 \end{cases}$$

Similarly, for the special case of $T_n = T_{n+1} = T$, the closed-form solution of the Taiwanese floored option is the same as the case of $n - 1$ monitoring dates with $t < T_1 < T_2 < \cdots < T_{n-1} < T$.

To show the properties of Taiwanese floored options, we also assume that $\delta = 0, K = 100, t = 0$ and $T = 1$ (one year to maturity). We summarize the numerical results in Table 3. Similar to plain vanilla
Table 3. Prices of plain vanilla put option and Taiwanese floored option

<table>
<thead>
<tr>
<th>σ</th>
<th>$S(t)$ $F$</th>
<th>Monitoring dates (end of month)</th>
<th>$S(t)$ $B$</th>
<th>Monitoring dates (end of month)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>16.6570</td>
<td>$[6, 12]$</td>
<td>16.7396</td>
<td>$[6, 12]$</td>
</tr>
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Notes: Let $\delta = 0, K = 100, t = 0, T = 1$. $[6, 12]$ means that the monitoring dates are the last days of the 6th month and the 12th month. Some properties exist in Taiwanese floored options: (1) Their values decrease with the current price of the underlying asset, risk-free interest rate $r$ and volatility of the underlying asset $\sigma$. (2) Due to the higher probability that the price of the underlying asset will touch the barrier level $B$, a lower barrier level $B$ has lower values of Taiwanese floored options. (3) Because the values are decreasing functions of the number of monitoring dates, the values of these options are always lower than the values of plain vanilla European call options.
put options, the values of Taiwanese floored options are decreasing with the current price of the underlying asset and the risk-free interest rate and increasing with the volatility of the underlying asset. However, from Table 3, it is worth to note that the influence of the barrier level \( F \) or the number of monitoring dates on the value of Taiwanese floored options is uncertain. As a result, the values of Taiwanese floored options are not always larger than the values of plain vanilla put options. (For example, when \( \sigma = 50\% \), \( S = 115 \) and \( F = 80 \), the values of plain vanilla put options are larger than the values of Taiwanese floored options.)

**Taiwanese capped or floored options with exponential barrier**

We now consider the cases where the barrier of Taiwanese capped or floored options is exponentially increasing or decaying with the expression used by Ritchken (1995) as \( \Delta H(t) = aH(t)dt \), or equivalently, given \( H(t) = H_0 \), \( H(T) = H_0 e^{a(T-t)} \), where \( H = B \) or \( F \). We can then easily obtain the closed-form solution by assuming the volatility of \( S_2 \) (for Taiwanese capped options with exponential barrier) or \( S_1 \) (for Taiwanese floored options with exponential barrier) is equal to zero and \( u_i(t) - \delta_i(t) = \alpha_i \), \( i = 1 \) (for Taiwanese floored options) or 2 (for Taiwanese capped options). The closed-form solutions are the same as (10) or (37) with the following (slightly changed) parameters:

\[
g_h(x) = \frac{\ln(S(t)/x(t)) + (r - \delta - \alpha + (1/2)\sigma^2)(T_h - t)}{\sigma \sqrt{T_h - t}}
\]

\( h = 1, \ldots, n, x = B \text{ or } F \) \hspace{1cm} (40)

\[
g_h(x) = \frac{\ln(S(t)/x(t)) + (r - \delta - \alpha - (1/2)\sigma^2)(T_h - t)}{\sigma \sqrt{T_h - t}}
\]

\( h = 1, \ldots, n, x = B \text{ or } F \) \hspace{1cm} (41)

**Other extensions**

The volatility and continuous dividend payout rate in (10) and (37) are both independent of time. We can straightforwardly extend to the case of a deterministic time-varying model and the closed-form solution will be similar to (26).

If \( B(t, T) = \exp(-r(T - t)) \) and \( S_2 = B \), \( S_1 = K \), the closed form solution in (26) is the capped option with the trigger being another underlying asset under a constant interest rate. Similarly, if \( S_1 = B \), \( S_3 = K \), the closed form solution in (26) is the floored option with the trigger being another underlying asset.

If \( B(t, T) = \exp(-r(T - t)) \) and \( S_2 = B \), \( S_1 = F \), then (26) is the closed-form solution of the capped (floored) exchange option with another related asset as the trigger. This kind of capped (floored) exchange option with another asset as trigger can be used to obtain the nonnegative return of the difference of two stock indices if the related indices (such as exchange rate, CPI index and so on) are greater (less) than some pre-specified level. It can make the exchange option be early exercised to avoid a bad outcome such as erasing the payoff completely.

**VI. Conclusion**

In this article we first derive the closed-form solution of Taiwanese capped options. We also provide the properties of Taiwanese capped (floored) options and the phenomenon of delta jump. Then, we derive the closed-form solution of generalized discrete monitoring capped exchange options with a stochastic barrier and related asset as trigger under the Hull and White model. We allow the volatility or continuous dividend-payout rate of stock to be time varying and the spot interest rate to fluctuate in the short run. The formulas can also be applied for options with long-dated maturities.

Special cases of generalized time-varying discrete monitoring capped exchange options with a stochastic barrier are abundant: for example, capped (floored) options, capped (floored) options with exponentially decayed or increasing barriers, capped (floored) options with related assets or indices as triggers, capped (floored) options with related assets or indices as triggers and the other related assets as barriers and so on. Those extensions of generalized capped exchange options would be potential financial derivatives for practical application.

**References**


Appendix A

Proof of Theorem 1: The key elements of the proof of (10) are
\[
\exp[-r(T_j - t)]E_P([S(T_j) - K] \\
\times I(S(T_j) \geq B, S(T_h) < B, h < j)|F_t] \\
= \exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
- K \exp[-r(T_j - t)]E_P(I_A|F_t) \\
= A_1 - A_2 \tag{A1}
\]
and
\[
\exp[-r(T - t)]E_P([S(T) - K] \\
\times I(S(T) < B, h = 1, \ldots, n)|F_t) \\
= \exp[-r(T - t)]E_P(S(T)I_L|F_t) \\
\times -K \exp[-r(T - t)]E_P(I_L|F_t) \\
= L_1 - L_2 \tag{A2}
\]
where \( A \) and \( L \) stand for the sets \{\( S(T_j) \geq B, S(T_h) < B, h < j \)\} and \{\( S(T_h) < B, h = 1, \ldots, n \)\}, respectively. For \( A_1 \) we have
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
Let the probability measure \( P_G \) be defined by the Radon–Nikodym derivative \( dP_G/dP = \exp(\nu W_T^G - (1/2)\nu^2 T) \). Then, \( W_T^G \), defined by \( dW_T^G = dW_T^g - \nu dt \) is a standard Brownian motion under probability measure \( P_G \). Therefore,
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
Under \( P_G \), \( S(t) \) satisfies
\[
S(T_j) = S(t)\exp\left[ \left( r - \delta + \frac{1}{2}\sigma_S^2 \right) (T_j - t) \right] \\
\times (T_j - t) + \sigma_S \cdot \left( W_{T_j}^G - W_T^g \right) \tag{A5}
\]
Using (A5), we have
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
\[
\exp[-r(T_j - t)]E_P(S(T_j)I_A|F_t) \\
= S(t)\exp[-\delta(T_j - t)]E_P\left\{ \exp\left[ \sigma_S \cdot (W_{T_j}^G - W_T^g) - \frac{1}{2}\sigma_S^2(T_j - t) \right]I_A|F_t \right\} \tag{A3}
\]
Pricing generalized capped exchange options

Similarly, following the same procedure, we can find that

\[ A_2 = K \exp[-r(T_i - t)] \times N_j[-q_l(B), \ldots, -q_{l-1}(B), q_l(B), \Psi_j] \]  

where \( D = \exp[-r(T - t)] \times N_{n+1}[-g_1(B), \ldots, g_n(B), q_{n+1}(K), \Psi_{n+1}] \) are as in (10). This completes the proof of the theorem.

Appendix B

Proof of the Theorem 2: The key elements of the proof of (26) are

\[ B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \right] \times I(S_1(T_j) \geq S_2(T_j), S_1(T_h) < S_2(T_h), h < j) \right] \]  

and

\[ B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \right] \times I(S_1(T_h) < S_2(T_h), h = 1, \ldots, n) \right] \]  

for \( I_1 \) we have

\[ I_1 = B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \right] \times I(S_1(T_h) < S_2(T_h), h = 1, \ldots, n) \]  

and

\[ B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \right] \times I(S_1(T_h) < S_2(T_h), h = 1, \ldots, n) \]

where \( D \) and \( Z \) stand for the sets \( \{ S_3(T_j) \geq S_4(T_j), S_1(T_j) \geq S_2(T_j), S_1(T_h) < S_2(T_h), h < j) \} \) and \( \{ S_3(T) \geq S_4(T), S_1(T_j) < S_2(T_h), h = 1, \ldots, n) \) respectively. For \( I_1 \) we have

\[ I_1 = B(t, T_j)E_j \left[ [S_3(T_j) - S_4(T_j)]^+ \right] \times I(S_1(T_h) < S_2(T_h), h = 1, \ldots, n) \]  

Let probability measure \( P_{T_j} \) on \( (\Omega, F) \) be equivalent to \( P_T \) with the Radon–Nikodym derivative given by

\[ \frac{dR_{T_j}}{dP_T} = \exp \left[ \int_0^T \gamma_3(v, T) \cdot dW^T_v - \frac{1}{2} \int_0^T |\gamma_3(v, T)|^2 dv \right] \]

By Girsanov’s theorem,

\[ W^R_{T_i} = W^T_{T_i} - \int_0^T \gamma_3(v, T) \cdot dW^T_v \quad \forall t \in [0, T] \]  

is a \( R_{T_j} \)-Brownian motion. Therefore

\[ I_3 = P_{R_{T_j}}(S_3(T_j) \geq S_4(T_j), S_1(T_j) \geq S_2(T_j), S_1(T_h) < S_2(T_h), h < j) \]  

\[ = P_{R_{T_j}} \left[ \begin{array}{c} \ln F_3^3(T_j, T_j) \geq \ln F_3^3(T_j, T_j), \\ \ln F_3^3(T_j, T_j) \geq \ln F_3^3(T_j, T_j), \\ \ln F_3^3(T_j, T_j) < \ln F_3^3(T_j, T_j), h < j \end{array} \right] \]

where \( Z_{j+1} \) are all standard normal random variables. As a result, we have

\[ I_3 = S_3(t)e^{-\int_{T_j}^T \delta_1(u)du} N_{j+1}[-d_3(T_j, T_j), \ldots, \\ -d_3(T_{j-1}, T_j), d_3(T_j, T_j), e_1(T_j), e_2(T_j), \sum_j] \]

Similarly, following the same procedure, we can find that

\[ I_2 = S_3(t)e^{-\int_{T_j}^T \delta_1(u)du} N_{j+1}[-d_4(T_j, T_j), \ldots, \\ -d_4(T_{j-1}, T_j), d_4(T_j, T_j), e_2(T_j), \sum_j] \]
and

\[ A_1 = S_3(t)e^{-\int^{T_n}_{T_n} \delta_3(u) \, du} N_{n+1}\left[-d_3(T_1, T_n), \ldots, -d_3(T_n, T_n), e_1(T), \sum_n \right] \]  \hspace{1cm} (B9)

\[ A_2 = S_4(t)e^{-\int^{T_n}_{T_n} \delta_4(u) \, du} N_{n+1}\left[-d_4(T_1, T_n), \ldots, -d_4(T_n, T_n), e_2(T), \sum_n \right] \]  \hspace{1cm} (B10)

Consequently, the valuation of \( C_{Capped} \) is as in (26) and we complete the proof.