COHOMOLOGY OF SUB-HOPF-ALGEBRAS OF THE STEENROD ALGEBRA

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1. Introduction

Let $A$ denote the mod 2 Steenrod algebra. The object of this paper is to study certain interesting properties of the cohomology of sub-Hopf-algebras of $A$ which we hope might be helpful for the study of $H^*(A)$, the cohomology of $A$ itself.

Before stating our main theorems we recall the work of Margolis [7], Adams-Margolis [4] and Anderson-Davis [5] on the structure of sub-Hopf-algebras of $A$. For any sequence $(n_1, n_2, \ldots)$ of non-negative integers (possibly equal to infinity) denote by $A(n_1, n_2, \ldots)$ the $\mathbb{Z}_2$-submodule of $A$ generated by all the Milnor basis elements $Sq(r_1, r_2, \ldots)$ with $r_i < 2^n$. (The notation $A(n_1, n_2, \ldots)$ is adopted after [5].) It is a theorem of Margolis [7] that for any sub-Hopf-algebra $B$ of $A$ there is a sequence $(n_1, n_2, \ldots)$ such that $B = A(n_1, n_2, \ldots)$. However not all sequences are realizable as such. It has been shown by Adams and Margolis [4] (also independently by Anderson and Davis [5]) that a sequence $(n_1, n_2, \ldots)$ is realizable if and only if for $i > j \geq 1$, $n_i \geq \min(n_j, n_i - j)$. In particular $A(1, 2, 3, \ldots)$ is a sub-Hopf-algebra of $A$; we denote it by $A_d$.

Our first result is to determine which sub-Hopf-algebra of $A$ has nil-free cohomology. (A commutative algebra is nil-free if it has no non-zero nilpotent elements.)

Theorem 1.1. Let $B$ be a sub-Hopf-algebra of $A$. Then $H^*(B)$ is nil-free if and only if $B$ is either of the following two types:

1. For some integer $t \geq 1$, $B = A(0, \ldots, 0, n, n+1, \ldots)$ where $n_i < t$ all $i$.
2. For some $t \geq 1$, $B = A(0, \ldots, 0, 1, n+1, n+2, \ldots)$ where $n_i < t+1$ all $i$ and $n_i = t+1$ for at least one $i$.

Remark. It can be shown that a sub-Hopf-algebra of $A$ is an exterior algebra if and only if it is of the type (1) in Theorem 1.1.
In order to state our next two theorems we introduce two notions. Let \( f: R \to R' \) be an algebra homomorphism of commutative algebras over a field. We say \( f \) is a monomorphism mod nilpotent elements if for any non-nilpotent element \( x \in R \), \( f(x) \in R' \) is also non-nilpotent and is an epimorphism mod nilpotent elements if for any non-nilpotent element \( x' \in R' \) there exist an integer \( m \) and an element \( x \in R \) such that \( f(x) = (x')^m \). Note that \( f \) is a monomorphism mod nilpotent elements if and only if the induced homomorphism \( \bar{f}: R/N(R) \to R'/N(R') \) is a monomorphism where \( N(R) \) and \( N(R') \) are the ideals consisting of all the nilpotent elements of \( R \) and \( R' \) respectively.

**Theorem 1.2.** Let \( B = A(\kappa_1, \kappa_2, \ldots) \) be any sub-Hopf-algebra of \( A \) such that each \( \kappa_i \) is finite. Let \( i_d: B \cap A_d \to B \) be the inclusion. Then the induced homomorphism \( i_d^*: H^*(B) \to H^*(B \cap A_d) \) is a monomorphism mod nilpotent elements.

**Theorem 1.3.** Under the same assumption as in Theorem 1.2 the induced homomorphism

\[
i_d^*: H^*(B) \to H^*(B \cap A_d)
\]

is an epimorphism mod nilpotent elements.

**Corollary 1.4.** Let \( B \) be as in Theorem 1.2. Then \( B \cap A_d \) is the best possible sub-Hopf-algebra of \( B \) such that the conclusion of Theorem 1.2 holds. This means that if \( B \cap A_d \) is replaced by any proper sub-Hopf-algebra of \( B \cap A_d \) in Theorem 1.2 then the conclusion does not hold any longer.

Corollary 1.4 follows from Theorem 1.3 and the fact that for any proper sub-Hopf-algebra \( D \) of \( B \cap A_d \) there is some indecomposable element \( P_i = \text{Sq}(0, \ldots, 0, 2^i, 0, \ldots) \in B \cap A_d \) with \( P_i \notin D \) and the class \( z_i \in H^{1,*}(B \cap A_d) \) corresponding to \( P_i \) is non-nilpotent but its image in \( H^*(D) \) is zero.

**Conjecture 1.5.** Theorem 1.2 is true for all sub-Hopf-algebras of the Steenrod algebra \( A \), in particular for \( A \) itself.

Theorem 1.3, however, does not hold for the Steenrod algebra \( A \). For \( P_2^* \) is an indecomposable element of \( A_d \) and the class \( z_2^* \in H^{1,*}(A_d) \) corresponding to \( P_2^* \) is non-nilpotent, but there are no non-zero elements of \( H^*(A) \) of bidegree \((s, 3s)\). (See [2].)

We have completed the statements of our main theorems. The rest of the paper is organized as follows. In Section 2 we work out some basic results which relate the cohomology of a Hopf algebra \( \Gamma \) to that of a sub-Hopf-algebra \( \Lambda \) with \( \Gamma \) being obtained from \( \Lambda \) by adding one generator and infer some of their consequences that will be the basic tools to prove our main theorems. In Section 3 we prove a key technical result regarding the nilpotency of some cohomology classes of the
cohomology of sub-Hopf-algebras of the Steenrod algebra. In Section 4 we prove our main theorems using results obtained in Sections 2 and 3.

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2. Some relations between the cohomology of a pair \((\Gamma, \Lambda)\) of Hopf algebras with \(\Gamma\) obtained from \(\Lambda\) by adding one generator and their consequences

Let \(\Gamma\) be a connected locally finite cocommutative Hopf algebra over \(\mathbb{Z}_2\) and let \(\Lambda\) be a normal sub-Hopf-algebra of \(\Gamma\) such that \(\Gamma/\Gamma\Lambda = \{1, \bar{x}\} = E[\bar{x}]\) where \(\bar{\Lambda}\) is the augmentation ideal of \(\Lambda\) and \(\bar{x}\) is the image in \(\Gamma/\Gamma\bar{\Lambda}\) of a homogeneous indecomposable element \(x\) of \(\Gamma\). Let \(\alpha \in H^{1,*}(\Gamma)\) be the class corresponding to \(x\), \(i: \Lambda \to \Gamma\) be the inclusion and \(i^*: H^*(\Gamma) \to H^*(\Lambda)\) the induced map in cohomology.

**Theorem 2.1.** Let \(\Theta \in H^*(\Gamma)\) be an element such that \(i^*(\Theta) \in H^*(\Lambda)\) is nilpotent. Suppose that \(\Theta\alpha\) is nilpotent. Then \(\Theta\) is also nilpotent.

**Theorem 2.2.** Suppose \(\alpha\) is nilpotent. Then the induced homomorphism \(i^*\) is a monomorphism mod nilpotent elements.

**Theorem 2.3.** Suppose \(\alpha\) is nilpotent. Then the induced homomorphism \(i^*\) is an epimorphism mod nilpotent elements.

It is easy to see that Theorem 2.2 follows immediately from Theorem 2.1 and the latter theorem follows immediately from the following.

**Proposition 2.4.** \(\text{Ker } i^* = \text{ideal of } H^*(\Gamma) \text{ generated by } \alpha.\)
Proof. First we note that \( E[\bar{x}] = \Gamma / \Gamma \bar{\Lambda} = \Gamma \otimes \Lambda Z_2 \). We have an exact sequence of \( \Gamma \)-modules
\[
0 \to \Sigma^{|x|} Z_2 \to \Gamma \otimes \Lambda Z_2 \to Z_2 \to 0
\]
where \( \Sigma^{|x|} Z_2 \) means \( Z_2 \) in degree \( |x| \). Applying the functor \( \text{Ext}_\Gamma (\cdot, Z_2) \) to the above exact sequence and applying the change of rings theorem to identify \( \text{Ext}_\Gamma (\Gamma \otimes \Lambda Z_2, Z_2) \) with \( \text{Ext}_\Lambda (Z_2, Z_2) \) we get the long exact sequence
\[
\cdots \to H^s(\Gamma) \to H^{s+1, r+|x|}(\Gamma) \xrightarrow{i^*} H^{s+1, r+|x|}(\Lambda) \to \cdots .
\]
(The differential is due to [1; Theorem 2.6.1].) So \( \ker i^* = H^s(\Gamma) \cdot \alpha \). This completes the proof of Proposition 2.4.

To prove Theorem 2.3 we shall use a spectral sequence considered by Adams in [1]. Define a filtration on the cobar construction \( F(\Gamma^*) \) as follows. \( [\alpha_1 | \alpha_2 | \cdots | \alpha_s] \in F^{(p)}(\Gamma^*) \) if \( \alpha_i \) annihilates \( \bar{\Lambda} \) for at least \( p \) values of \( i \).

Theorem 2.5. This filtration defines a multiplicative spectral sequence \( \{ E_s \} \) converging to \( H^*(\Gamma) \) and we have:
(i) \( E_1^{|a|} = H^{|a|}(E[\bar{x}]) \otimes H^{|a|}(\Lambda) \).
(ii) The isomorphism
\[
E_1^{|a|} = H^{|a|}(E[\bar{x}]) \otimes Z_2 = H^{|a|}(E[\bar{x}])
\]
is induced by the natural map \( F(E[\bar{x}]) \to F(\Gamma^*) \). The isomorphism
\[
E_1^{|a|} = Z_2 \otimes H^{|a|}(\Lambda) = H^{|a|}(\Lambda)
\]
is induced by the natural map \( F(\Gamma^*) \to F(\Lambda^*) \).

For the proof of this theorem see [1]. Now we make a closer study of the resulting spectral sequence. It is well known that \( H^*(E[\bar{x}]) = Z_2[\bar{a}] \). It is clear that \( E_r^{s, r} = \{ \sigma \in H^*(\Lambda) \mid \delta_s(\sigma) = 0 \text{ all } s < r \} \), \( 1 < r < \infty \) and that all \( \bar{a}^s \) are permanent cycles with \( \bar{a}^s \) surviving in the spectral sequence which represents \( \alpha \in H^*(\Gamma) \). From Theorem 2.5 (ii) we see that \( E_\infty^{s, r} = \text{Im } i^* \).

Proposition 2.6. Suppose \( \alpha \) is nilpotent. Then for some integer \( k \) we have \( E_{k+1} = E_{k+2} = \cdots = E_r \). In particular \( E_{k+1}^{s, r} = E_{k+2}^{s, r} = \cdots = E_r^{s, r} = \text{Im } i^* \).

Proof. Since \( E_1^{s, 0} = H^*(E[\bar{x}]) = Z_2[\bar{a}] \) has only one indecomposable element it follows from Theorem 2.5 (i) that each \( E_r \) \( (1 < r < \infty) \) is generated by \( E_0^{s, r} \) and \( \bar{a}^s \) as an algebra. So the differential \( \delta_s : E_r \to E_{r-1} \) is determined by \( \delta_s : E_0^{s, r} \to E_0^{s, r} \). Since \( \alpha \) is nilpotent there is some integer \( k \) such that \( \bar{a}^k \) gets hit in the spectral sequence. It follows that \( \delta_s : E_r^{s, r} \to E_{r-1}^{s, r} \) is a trivial map for each \( r > k + 1 \). Thus \( E_{k+1} = E_{k+2} = \cdots = E_r = \text{Im } i^* \). This completes the proof of the proposition.
Proof of Theorem 2.3. It suffices to prove that if \( \tau \) is any non-nilpotent element of \( H^*(\Lambda) = E_0^\infty \) then some power of \( \tau \) lies in \( \operatorname{Im} i^* \). It is evident that \( \delta_m(\tau^{2^m}) = 0 \) for all integers \( m \). Thus \( \tau^{2^m} \in E_{m+1}^0 \). By assumption \( \alpha \) is nilpotent. By Proposition 2.6 there is some integer \( k \) such that \( \tau^{2^k} \in E_{k+1}^0 = E_\infty^0 = \operatorname{Im} i^* \). This proves Theorem 2.3.

Now we generalize Theorems 2.1, 2.2 and 2.3 a little bit further. We shall no longer insist that \( \Gamma \) is obtained from \( \Lambda \) by adding one generator. We shall assume that \( \Gamma \) is obtained from \( \Lambda \) by adding a finite number of generators. By this we mean that there is a finite sequence \( \Lambda = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma \) of sub-Hopf-algebras of \( \Gamma \) such that each \( \Gamma_{i+1} \) is obtained from \( \Gamma_i \) by adding one generator \( x_{i+1} \in \Gamma_{i+1} \). Let \( \alpha_{i+1} \in H^*(\Gamma_{i+1}) \) be the class corresponding to \( x_{i+1} \). Let \( \phi_i : \Gamma_i \rightarrow \Gamma \) be the inclusion and \( \phi_i^* : H^*(\Gamma) \rightarrow H^*(\Gamma_i) \) the induced map in cohomology.

Theorem 2.7. Suppose \( \Theta \in H^*(\Gamma) \) is an element such that \( \phi_i^*(\Theta) \in H^*(\Gamma_0) = H^*(\Lambda) \) is nilpotent and such that each \( \phi_i^*(\Theta) \alpha_i \in H^*(\Gamma_i) \) is nilpotent for \( 1 \leq i \leq n \). Then \( \Theta \) is also nilpotent.

Theorem 2.8. Suppose each \( \alpha_i \in H^*(\Gamma_i) \) is nilpotent. Then the induced homomorphism \( \phi_i^* : H^*(\Gamma) \rightarrow H^*(\Lambda) \) is a monomorphism mod nilpotent elements.

Theorem 2.9. Suppose each \( \alpha_i \in H^*(\Gamma_i) \) is nilpotent. Then the induced homomorphism \( \phi_i^* \) is an epimorphism mod nilpotent elements.

Theorems 2.7, 2.8 and 2.9 follow respectively from Theorems 2.1, 2.2 and 2.3. We will need these theorems later in the work.

3. Nilpotency of certain cohomology classes

Before applying the results of the previous section to the Steenrod algebra we need the following key technical result.

Theorem 3.1. Let \( B = A(n_1, n_2, \ldots, n_{m-1}, n_m, n_{m+1}, \ldots) \) be a sub-Hopf-algebra of the Steenrod algebra \( A \) such that \( n_i < \infty \) for \( i \leq m - 1 \). Suppose \( [\xi_i^2] \) and \( [\xi_m^2] \) are cocycles in the cobar construction \( F(B^*) \) such that \( l \leq m \) and \( l \leq \mu \). Then the class \( [\xi_i^2] [\xi_m^2] \) is a nilpotent element of \( H^*(B) \).

Corollary 3.2. Under the same assumption as in Theorem 3.1 the class \( [\xi_m^2] \) is a nilpotent element of \( H^*(B) \) if \( m \leq \mu \).

Corollary 3.2 follows from Theorem 3.1 by taking \( l = m \) and \( \lambda = \mu \) and is a generalization of the following result of Anderson and Davis.
Theorem 3.3 [S]. Let $\mathcal{B} = A(k, k, \ldots, k, n_1, n_2, \ldots, n_m)$ be a sub-Hopf-algebra of $A$ such that $1 + m + k \leq n_m$ where $k \geq 0$. Then the class $[\xi_m^{2^n}]$ is a nilpotent element of $H^*(\mathcal{B})$ if $m + k \leq \mu$.

Before carrying out the proof of Theorem 3.1 we note that we may assume that the sub-Hopf-algebra $\mathcal{B}$ satisfies $n_i = 0$ if $i \geq m$. For if this is not the case then we can extend $\mathcal{B}$ to a larger sub-Hopf-algebra $\mathcal{B}'$ of $A$ such that $\mathcal{B} = A(n_1, n_2, \ldots, n_{m-1}, \infty, \infty, \ldots)$ and the nilpotency of the class $[\xi_1^{2^n}] [\xi_m^{2^n}]$ in $H^*(\mathcal{B}')$ will force the nilpotency of the same class in $H^*(\mathcal{B})$.

Our proof of Theorem 3.1 is by induction. If $m = 1$ then the class $[\xi_1^{2^n}] [\xi_m^{2^n}]$ is nilpotent by Theorem 3.3. Suppose $m > 1$. Assume that the result is true for classes $[\xi_i^{2^n}] [\xi_m^{2^n}]$ if $l' \leq m' < m$ and also if $m' < l, m' = m$.

To start working on the inductive step we have to work in a suitable sub-Hopf-algebra. We suppose given an integer $\bar{a}$ which we shall later choose suitably in terms of $l, m, \lambda$ and $\mu$. We shall work in a sub-Hopf-algebra $\mathcal{C} = A(n_l, n_2, \ldots)$ such that

(i) $n_i \leq \bar{a}$ for $i < l$,
(ii) $n_i \leq \bar{a} + l + m - i$ for $i < m - l$ and
(iii) $n_i < \infty$ for $i < m, n_i = \infty$ for $i \geq m$.

Lemma 3.4. Under these assumptions the class $[\xi_i^{2^n}] [\xi_m^{2^n}]$ is a nilpotent element of $H^*(\mathcal{C})$.

Proof. In the cobar construction $F(C^*)$ consider the equation

$$\delta[\xi_i^{2^n}] = \sum_{i+l'=l+m} [\xi_i^{2^{n'}}] [\xi_l^{2^n}]$$

By assumption (i) we have $\xi_j^{2^n} = 0$ for $0 < j < l$, and similarly $\xi_i^{2^{n'}} = 0$ for $0 < i < l$. Since $\bar{a} + j = \bar{a} + l + m - i$, we have $\xi_i^{2^{n'}} = 0$ for $0 < i \leq m - l$ by assumption (ii). Therefore we can run our sum on the right side of the above equation over the range $R$ specified by

$$i + j = l + m$$

$$l \leq i, m - l < i, l \leq j$$

whence

$$j \leq m, j < 2l, i \leq m.$$
are separately cocycles. In fact, we have
\[
\delta [\xi_i^{2a}] = \sum_{p+q=i} [\xi_p^{2a} | \xi_q^{2a}]
\]
and this is zero if \(j < 2l\), for in each term we have either \(p < l\) or \(q < l\); similarly we have
\[
\delta[\xi_i^{2a' + j}] = \sum_{p+q=i} [\xi_p^{2a' + j} | \xi_q^{2a' + j}]
\]
and this is zero if \(i \leq m\), for in each term we have either \(p \leq m - l\) or \(q \leq l\). So in \(H^*(C)\) we have
\[
\sum_{R} [\xi_i^{2a}] [\xi_i^{2a'}] = 0.
\]
Now let us multiply this identity by \([\xi_i^{2a}]\). According to our inductive hypothesis, each product
\[
[\xi_i^{2a}] [\xi_i^{2a'}]
\]
with \(l \leq i < m\) is nilpotent (since \(l \leq a + j\)). Therefore the remaining term
\[
[\xi_m^{2a'}] [\xi_i^{2a}]
\]
is nilpotent. This proves Lemma 3.4.

**Lemma 3.5.** Under the same assumptions as in Lemma 3.4, the classes
\[
[\xi_i^{2a}] [\xi_m^{2a' + b}] \quad \text{and} \quad [\xi_i^{2a' + b}] [\xi_m^{2a'}]
\]
(with \(\bar{b} \geq 0\)) are nilpotent elements of \(H^*(C)\).

To see the use of this lemma, we note that any class \([\xi_i^{2a'}] [\xi_m^{2a'}]\) with \(l \leq \mu\) can be written in one of the two forms given by taking
\[
\bar{a} = \min (\lambda, \mu - l).
\]
\[
\bar{b} + \bar{a} = \max (\lambda, \mu - l).
\]

**Proof of Lemma 3.5.** The case \(\bar{b} = 0\) of Lemma 3.5 is just Lemma 3.4; so we can assume \(\bar{b} > 0\).

We recall [6] that there are Steenrod operations
\[
\mathsf{Sq}^l: H^{s+t}(C) \to H^{s+l,2t}(C)
\]
on the cohomology \(H^*(C)\). These satisfy the following properties:
(1) \( \text{Sq}^0 : H^* (C) \to H^{*+1} (C) \) is just the homomorphism of cohomology induced by the map
\[ \gamma \to \gamma^2 : C^* \to C^*. \]

(2) If \( x \in H^{1,*} (C) \) then
\[ \text{Sq}^1 x = x^2 \quad \text{and} \quad \text{Sq}^i x = 0 \quad \text{for } i > 1. \]

(3) The Cartan formula holds:
\[ \text{Sq}^i (xy) = \sum_{j+k=i} (\text{Sq}^j x)(\text{Sq}^k y). \]

From (3) we see that
\[ \text{Sq}^2 (x^2) = (\text{Sq}^1 x)^2. \]

Hence by induction
\[ \text{Sq}^{2^i} (x^{2^i}) = (\text{Sq}^1 x)^{2^i}. \]

If \( x \) is nilpotent then we can find a power \( 2^e \) of 2 such that \( x^{2^e} = 0 \); it follows that
\( \text{Sq}^i x \) is nilpotent all \( i \).

Now we introduce the operation
\[ Q_6 = \text{Sq}^{6^1} \text{Sq}^{6^2} \cdots \text{Sq}^6. \]

Two things follow from the work above. First if \( x \) is nilpotent then so is \( Q_6 (x) \). Secondly, if \( x \) and \( y \) lie in \( H^{1,*} (C) \) then
\[ Q_6 (xy) = (x^{2^e})((\text{Sq}^0)^6 y) + ((\text{Sq}^0)^6 x)(y^{2^e}) \quad \text{(for } 6 > 0). \]

Let us take \( x = [\xi^*_2] \), \( y = [\xi^*_m] \) and apply the operation \( Q_6 \) on \( xy \). By Lemma 3.4 \( xy \) is nilpotent. Therefore the class
\[ Q_6 (xy) = [\xi^*_i]^{2^6} [\xi^m]^{2^6} + [\xi^*_i]^{2^6} [\xi^m]^{2^6} \]
is nilpotent. If we multiply this element by \( [\xi^*_2] \) then the second term of the resulting sum is nilpotent by Lemma 3.4, whence the class
\[ [\xi^*_i^{2^6}][\xi^m]^{2^6} \]
is nilpotent. On the other hand if we multiply by \( [\xi^*_i]^{2^6} \) then the first term becomes nilpotent by Lemma 3.4 with \( \bar{a} \) replaced by \( \bar{a} + \bar{b} \); so we see that
\[ [\xi^*_i]^{2^6} [\xi^m]^{2^6} \]
is nilpotent, whence
\[ [\xi^*_i]^{2^6} [\xi^m]^{2^6} \]
is nilpotent. This proves Lemma 3.5.

It remains to complete the inductive step. Let $B = A(n_1, n_2, \ldots)$ be the sub-Hopf-algebra considered in Theorem 3.1 with $n_i = \infty$, $i \geq m$, and let

$$[\xi_{i}^{2^i}][\xi_{m}^{2^m}]$$

be the class we wish to prove nilpotent in $H^*(B)$. Let $\bar{a} = \min (\lambda, \mu - l)$. Let $D$ be the sub-Hopf-algebra $A(\delta_1, \delta_2, \ldots)$ of $A$ defined by

$$\delta_i = \begin{cases} \bar{a} & \text{for } i < l \\ \infty & \text{for } i \geq l; \end{cases}$$

let $E$ be the sub-Hopf-algebra $A(\varepsilon_1, \varepsilon_2, \ldots)$ of $A$ defined by

$$\varepsilon_i = \begin{cases} \bar{a} + m + l - i & \text{for } i \leq m - l \\ \infty & \text{for } i > m - l. \end{cases}$$

Then $C = B \cap D \cap E$ is the sub-Hopf-algebra considered in Lemma 3.4. So the class

$$[\xi_{i}^{2^i}][\xi_{m}^{2^m}]$$

is nilpotent in $H^*(C)$ by Lemma 3.5. Now $B$ is obtained from $C$ by adding a finite number of generators. By Theorem 2.7 we see the class

$$[\xi_{i}^{2^i}][\xi_{m}^{2^m}]$$

will remain nilpotent in $H^*(B)$ provided we prove that certain further classes

$$[\xi_{i}^{2^i}][\xi_{i'}^{2^{i'}}][\xi_{m}^{2^m}]$$

are nilpotent in the cohomology of suitable sub-Hopf-algebras between $C$ and $B$. But then we certainly have

- either (i) $l' < l$ and $\lambda' \geq \bar{a}$
- or (ii) $l' \leq m - l$ and $\lambda' \geq \bar{a} + m + l - l'$.

If $l' < l$ then the class

$$[\xi_{i}^{2^i}][\xi_{m}^{2^m}]$$

is nilpotent by our inductive hypothesis. So it suffices to consider the case in which $l \leq l' \leq m - l$ and $\lambda' \geq \bar{a} + m + l - l'$; but then the class

$$[\xi_{i}^{2^i}][\xi_{i'}^{2^{i'}}]$$

is nilpotent by our inductive hypothesis, because
\[ \lambda' \geq a + m + l - l' \]
\[ \geq a + m + l - (m - l) \]
\[ = a + 2l \]
\[ > l. \]

This completes the proof of Theorem 3.1.

4. The proofs of the main theorems

We will need the following lemma to complete the proof of Theorem 1.1.

Lemma 4.1. Let \( B \) be a sub-Hopf-algebra of the Steenrod algebra \( A \) such that \( [\xi^2_i] \) and \( [\xi^2_m] \) are cocycles in the cobar construction \( \Gamma(B^*) \). Suppose either \( l < m, l < \mu, \lambda = 0 \) or \( l < m, l = \mu, \lambda = 1 \). Then the class

\[ [\xi^2_i][\xi^2_m] \]

is a non-zero element of \( H^*(B) \).

Proof. We prove the lemma for the case in which \( l < m, l < \mu, \lambda = 0 \) and leave the proof of the other case to the reader.

The Milnor basis elements dual to \( \xi_i \) and \( \xi^2_m \) are \( P_i^0, P_m^0 \) respectively. We have

\[ P_m^0 P_i^0 = \text{Sq}(0, \ldots, 0, 1, 0, \ldots, 2^n, 0, \ldots) \]
\[ + \text{Sq}(0, \ldots, 0, 2^n - 2^l, 0, \ldots, 0, 1, 0, \ldots) \]

and

\[ P_m^0 \text{Sq}(0, \ldots, 0, 2^n - 2^l, 0, \ldots) = \text{Sq}(0, \ldots, 0, 2^n - 2^l, 0, \ldots, 0, 1, 0, \ldots). \]

So \( \xi^2_m \) and \( \xi^2_m - 2^l \xi_{m+l} \) are the only two monomials of \( B^* \) such that \( \xi^2_m \otimes \xi_i \in \Delta(\xi^2_m), \Delta(\xi^2_m \xi_{m+l}) \) and \( \xi^2_m - 2^l \xi_{m+l} \) is the only monomial of \( B^* \) such that \( \xi_{m+l} \otimes \xi^2_m - 2^l \in \Delta(\xi_{m+l} \xi^2_m - 2^l) \) where \( \Delta: B^* \rightarrow B^* \otimes B^* \) is the coproduct. It follows that \( [\xi^2_m \xi_i] = [\xi^2_m][\xi_i] \) is not a coboundary in \( F(B^*) \) and therefore represents a non-zero class in \( H^*(B) \). This proves Lemma 4.1.

Proof of Theorem 1.1. First we prove that if the sub-Hopf-algebra \( B = A(n_1, n_2, \ldots) \) is neither of the type (1) nor of the type (2) then \( H^*(B) \) has at least a non-zero nilpotent element.

Recall that the sub-Hopf-algebras of the type (1) are of the form \( A(0, \ldots, 0, n_1, n_2, \ldots) \) where \( n_i < t \) all \( i \) and are exterior algebras and those of the type (2) are of the form \( A(0, \ldots, 0, 1, n_1, n_2, \ldots) \), \( n_i < t + 1 \) all \( i \) and \( n_j = t + 1 \) for at least one \( j \).
In case $B \not\subset A_d$, there exists an integer $m$ such that $n_m \geq m + 1$ and $n_i \leq i$, $i \leq m - 1$. By Corollary 3.2 $[\xi_m^*]$ is a non-zero nilpotent element of $H^*(B)$.

In case $B \subset A_d$, let $l = \min\{j \mid n_j > 0\}$. The sequence $(n_1, n_2, \ldots)$ is either unbounded or bounded. In the first case we can find a pair of integers $(m, \mu)$ such that $l < m$, $l < \mu$, and such that $[\xi_m^*]$ is a cocycle in the cobar construction $F(B^*)$.

By Theorem 3.1 and Lemma 4.1 the class $[\xi_i][\xi_m^{2i-1*}]$ is a non-zero nilpotent element of $H^*(B)$. If the sequence $(n_1, n_2, \ldots)$ is bounded, let $k = \max\{n_i\}$, $m = \min\{j \mid n_j = k\}$. Then $[\xi_m^{2i-1*}]$ is a cocycle in $F(B^*)$. Clearly $k \geq l + 1$ (otherwise $B$ would be of the type (1)) and if $k = l + 1$ then $n_i > 1$ (otherwise $B$ would be of the type (2)). Again by Theorem 3.1 and Lemma 4.1 the class $[\xi_i][\xi_m^{2i-1*}]$ is a non-zero nilpotent element of $H^*(B)$ if $k > l + 1$ and $[\xi_i][\xi_m^{2i-1*}]$ is a non-zero nilpotent element of $H^*(B)$ if $k = l + 1$.

It remains to prove that if $B$ is either of the type (1) or the type (2) then $H^*(B)$ is nil-free. If $B$ is of the type (1) then it is an exterior algebra; so $H^*(B)$ is a polynomial algebra which is nil-free.

If $B$ is of the type (2) then for some integer $m$ we have $B = A(0, \ldots, 0, 1, n_{m+1}, n_{m+2}, \ldots)$ where $n_i \leq m + 1$ all $i$ and $n_i = m + 1$ for at least one $i$. Note that $B$ is generated by all $P^i \subset B$ as an algebra. Redefine a coalgebra structure on $B$ by requiring that each $P_i \in B$ be a primitive element. We prove that $B$ becomes a primitively generated Hopf algebra over $\mathbb{Z}_2$. Let $M = \{j \geq m + 1 \mid n_j = m + 1\}$. Let $a = P_{m+1}^0$, $b_i = P_{m+1}^i$, $c_i = P_{m+2}^i$ for $i \in M$ and let $\{d_k\}$ be the set of the rest of the generators $P_i \in B$. Then $B$, as an algebra, is generated by $a$, $b_i$, $c_i$, $d_k$ subject to the following relations:

\[
a^2 = 0, \quad b_i^2 = 0, \quad c_i^2 = 0, \quad d_i^2 = 0, \quad [a, b_i] = c_i, \quad [a, c_i] = 0, \quad [a, d_k] = 0, \\
b_i c_j = 0, \quad [b_i, d_k] = 0, \quad [c_i, d_k] = 0, \quad [b_i, b_j] = 0, \quad i \neq j, \\
c_i c_j = 0, \quad i \neq j, \quad [d_k, d_l] = 0, \quad k \neq l
\]

where by $[x, y]$ we mean $xy + yx$. It is then easy to check that $B$ is indeed a Hopf algebra over $\mathbb{Z}_2$ with the new coalgebra structure defined above and that $\{a, b_i, c_i, d_k\}$ is a $\mathbb{Z}_2$-base for the primitive elements of $B$. So $B$ is a primitively generated Hopf algebra over $\mathbb{Z}_2$. By Milnor-Moore Theorem [9] $B$ is isomorphic to the universal enveloping algebra of the restricted Lie algebra of its primitive elements. By May's Theorem [8] $H^*(B)$ is the homology of the differential algebra $\mathbb{Z}_2[\alpha, \beta_n, \gamma_n, \theta_k]$ where $\alpha$, $\beta_n$, $\gamma_n$, $\theta_k$ correspond to $a$, $b_i$, $c_i$, $d_k$ respectively; the differential is determined by

\[
\delta(\alpha) = 0, \quad \delta(\beta_i) = 0, \quad \delta(\gamma_i) = \beta_i \alpha, \quad \delta(\theta_k) = 0.
\]

Let $D_1$ be the differential subalgebra of $D = \mathbb{Z}_2[\alpha, \beta_n, \gamma_n, \theta_k]$ generated by $\alpha$, $\beta_n$, $\gamma_n$ and let $D_2$ be the subalgebra of $D$ generated by $\theta_k$. Then $D$ is isomorphic to $D_1 \otimes D_2$ as a differential algebra where $D_2$ has trivial differential. So we have

\[
H^*(B) = H(D, \delta) = H(D_1 \otimes D_2).
\]
To prove that $H^*(B)$ is nil-free it suffices to prove that $H(D_1, \delta)$ is nil-free. Let $\Sigma$ be the collection of all the finite subsets of $M$ containing more than one element (if $M$ has only one element let $\Sigma = \emptyset$). To each $S = \{i_1, i_2, \ldots, i_n\} \in \Sigma$ assign an element $h(S) \in D_1$ by

$$h(S) = \sum_{m=1}^{n} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{m-1}} \beta_{i_m} \gamma_{i_{m+1}} \cdots \gamma_{i_n}.$$ 

Let $\bar{\gamma}_i = \gamma_i^2$. It is not hard to prove that $H(D_1, \delta)$, as an algebra over $\mathbb{Z}_2$, is generated by $\alpha, \beta_n \bar{\gamma}_i$ and $h(S), S \in \Sigma$ subject to the following relations

$$\beta_n \alpha = 0,$$

and

$$h(S) \alpha = 0, S \in \Sigma$$

and

$$h(S) \bar{\gamma}_i = \sum_{m=1}^{n} \bar{\gamma}_{i_1} \bar{\gamma}_{i_2} \cdots \bar{\gamma}_{i_{m-1}} \beta_{i_m} \bar{\gamma}_{i_{m+1}} \cdots \bar{\gamma}_{i_n} \quad \text{if } S = \{i_1, i_2, \ldots, i_n\}.$$ 

It is easy to see that $H^*(D_1, \delta)$ is nil-free. This proves that $H^*(B)$ is nil-free. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** We have $B = A(n_1, n_2, \ldots)$ where each $n_i$ is finite. Assume $n_i > i$ all $i$. We prove the theorem for such cases. Our proof equally applies to the general cases. Let $B_{(k)} = A(1, 2, \ldots, k-1, n_2, \ldots)$. Then

$$B = B_{(1)} \supset B_{(2)} \supset \cdots \supset A_d = B \cap A_d.$$ 

It is easy to see that for each integer $N > 0$ there is a $k$ (depending on $N$) such that $H^i(A_d) = H^i(B_{(k)})$ if $l \leq N$.

Now fix any $k$. Construct a sequence $B_{(k)} = B_0 \subset B_1 \subset \cdots \subset B_{\rho_k} = B$ of sub-Hopf-algebras as follows:

$$B_0 = B_{(k)},$$

$$B_1 = A(1, 2, \ldots, k-2, k, n_k, n_{k+1}, \ldots)$$

$$\vdots$$

$$B_{n_k-1-k+1} = A(1, 2, \ldots, k-2, n_{k-1}, n_k, n_{k+1}, \ldots)$$

$$B_{n_k-1-k+2} = A(1, 2, \ldots, k-3, k-1, n_{k-1}, n_k, n_{k+1}, \ldots)$$

$$\vdots$$

$$B_{n_k-1-k+1+n_k-2-k+2} = A(1, 2, \ldots, k-3, n_{k-2}, n_{k-1}, n_k, n_{k+1}, \ldots)$$

$$\vdots$$

$$B_{\rho_k} = A(n_1, n_2, \ldots, n_k, n_{k+1}, \ldots)$$

where $\rho_k = n_{k-1} + n_{k-2} + \cdots + n_1 - ((k-1) + (k-2) + \cdots + 1)$. Each $B_{i+1}$ is obtained from $B_i$ by adding an indecomposable element $x_{i+1} \in B_{i+1}$. Thus $B$ is obtained from $B_0 = B_{(k)}$ by adding a finite number of generators. The class $\alpha_{i+1} \in H^{i+1}(B_{i+1})$
corresponding to $x_{i+1}$ is nilpotent for each $i$ by Corollary 3.2. By Theorem 2.8 the induced homomorphism

$$i_k^*: H^*(B) \to H^*(B_{\langle k \rangle})$$

is a monomorphism mod nilpotent elements. This is true for all $k$.

Now let $\Theta \in H^*(B)$ be any non-nilpotent element. To prove the theorem we have to prove that $i_k^*\Theta \in H^*(B \cap A_d) = H^*(A_d)$ is also non-nilpotent. Suppose not, say $(i_k^*\Theta)^q = 0$ for some $q \geq 1$. By (i) there is a $k$ such that $(i_k^*(\Theta))^r = 0$ holds in $H^*(B_{\langle k \rangle})$. On the other hand $\Theta^r$ is non-nilpotent. So $(i_k^*(\Theta))^r = i_k^*(\Theta^r)$ is also non-nilpotent by (ii); in particular it is not zero. Thus we have got a contradiction. This shows that $i_k^*(\Theta)$ is non-nilpotent. This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is similar to that of Theorem 1.2 given above; instead of using Theorem 2.8 we use Theorem 2.9. We leave the details of the proof to the reader.

References