A Note on Factorizations of Singular $M$-Matrices

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ABSTRACT

Supposing that $M$ is a singular $M$-matrix, we show that there exists a permutation matrix $P$ such that $PMP^T = LU$, where $L$ is a lower triangular $M$-matrix and $U$ is an upper triangular singular $M$-matrix. An example is given to illustrate that the above result is the best possible one.

I. INTRODUCTION

A real square matrix $A = (a_{i,j})$ is called an $M$-matrix if $a_{i,i} < 0$ whenever $i \neq j$ and all principal minors of $A$ are positive. We will write $B = (b_{i,j}) \geq 0$ if $b_{i,j} \geq 0$ for each pair $(i,j)$. For a real square matrix $A$ with nonpositive off-diagonal elements, it is known (e.g., [1, Theorem 4.3]) that $A$ is an $M$-matrix if and only if $A$ is nonsingular and $A^{-1} \geq 0$. Following Fiedler and Ptak [1], we shall denote by $K$ the class of all $M$-matrices and by $K_0$ the class of all real square matrices $A = (a_{i,j})$ with $a_{i,j} \leq 0$ for $i \neq j$, which have all principal minors nonnegative. A singular matrix in $K_0$ is called a singular $M$-matrix.

It is well known (e.g., [1, Theorem 4.3]) that an $M$-matrix may be written in the form $LU$, where $L \in K$ is lower triangular and $U \in K$ is upper triangular.

In [3], G. Poole and T. Boullion mentioned the possibility of the $LU$-factorizations for singular $M$-matrices. An example is given in Sec. 2 to show that not every matrix in $K_0$ can be factored as $LU$. However, for any matrix $A$ in $K_0$, we show that $PAP^T = LU$ for a suitable permutation matrix $P$, where $L \in K$ is lower triangular and $U \in K_0$ is upper triangular.

The following result will be useful in our work.

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Theorem A [2, Theorem 4, p. 47]. If a rectangular matrix $R$ is represented in partitioned form

$$ R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, $$

where $A$ is a square nonsingular matrix of order $n$, then the rank of $R$ is equal to $n$ if and only if $D = CA^{-1}B$.

II. RESULTS

Theorem 1. Let $M \in K_0$. If $M$ can be partitioned into the form

$$ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, $$

such that $A$ is nonsingular and $\text{rank } M = \text{rank } A$, then $M = LU$, where $L \in K$ is lower triangular and $U \in K_0$ is upper triangular.

Proof. We note first that $D = CA^{-1}B$ by Theorem A. Since $A \in K_0$ and $A$ is nonsingular, we have $A \in K$. Thus, $A = L_1U_1$, where $L_1 \in K$ is lower triangular and $U_1 \in K$ is upper triangular. $L_1$ and $U_1$ are nonsingular; moreover, $L_1^{-1} > 0$ and $U_1^{-1} > 0$. Now let

$$ L = \begin{bmatrix} L_1 & 0 \\ CU_1^{-1} & I \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U_1 & L_1^{-1}B \\ 0 & 0 \end{bmatrix}, $$

where $I$ is the identity matrix of appropriate order. Since $C < 0$ and $B < 0$, we have $CU_1^{-1} < 0$ and $L_1^{-1}B < 0$. Clearly, all principal minors of $L$ are positive and all principal minors of $U$ are nonnegative. Hence, $L \in K$ and $U \in K_0$, and $M = LU$.

Corollary. Let $M \in K_0$ be irreducible. Then $M = LU$, where $L$ and $U$ are the same as in Theorem 1.

Proof. If $M \in K$, then the statement is true. So we assume that $M$ is singular. By Theorem 5.7 of [1], all proper principal minors of $M$ are positive.
Thus, we can partition $M$ into the form

$$M = \begin{bmatrix} M_{n-1} & b \\ c & d_{n,n} \end{bmatrix},$$

where $M_{n-1} \in K$ and $\text{rank} M = \text{rank} M_{n-1}$. Therefore, the corollary follows from Theorem 1.

Next, we prove a lemma.

**Lemma.** Let $M \in K_0$ be partitioned into the form

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

such that $A$ and $D$ are irreducible. Then $M = LU$, where $L$ and $U$ are the same as in Theorem 1.

**Proof.** It is clear that $A \in K_0$ and $D \in K_0$. By the above corollary $A = A_1 A_2$ and $D = D_1 D_2$, where $A_1$ and $D_1$ are lower triangular matrices in $K_0$ and $A_2$ and $D_2$ are upper triangular matrices in $K_0$. Let

$$L = \begin{bmatrix} A_1 & 0 \\ 0 & D_1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} A_2 & A_1^{-1}B \\ 0 & D_2 \end{bmatrix}.$$ 

Then, $L \in K$ is lower triangular and $U \in K_0$ is upper triangular, and $M = LU$.

Our main result is the following.

**Theorem 2.** Let $M \in K_0$. Then there exists a permutation matrix $P$ such that $PMP^T = LU$, where $L \in K$ is lower triangular and $U \in K_0$ is upper triangular.

**Proof.** It is sufficient to consider the case that $M \neq 0$ is singular and reducible. Let $P$ be a permutation matrix such that $PMP^T$ can be partitioned into the form

$$PMP^T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$
where $A$ is irreducible. If $D$ is also irreducible, then $PMP^T = LU$ by the Lemma. If $D$ is reducible, then the proof is completed by using induction.

It is clear that we can obtain another factorization for matrices in $K_0$, i.e., for any $M \neq 0$ in $K_0$, there exists a permutation matrix $P$ such that $PMP^T = LU$, where $L \in K_0$ is lower triangular and $U \in K$ is upper triangular. Also, we can obtain similar results for factorizations of type $UL$.

**Example.** The following example will show that Theorem 2 is the best possible result. Let

$$M = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$  

If

$$M = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix},$$

then we get $a_{11}b_{11} = 0$, $a_{11}b_{13} = -1$, and $a_{21}b_{11} = -1$, which is impossible. Thus, there is no factorization of the type $LU$ for $M$. But if we let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$PMP^T = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = I \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$  

**References**