The Moore-Penrose Inverses of Singular M-Matrices

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ABSTRACT

Suppose M is a real square matrix such that off-diagonal elements of M are nonpositive and all principal minors of M are nonnegative. Necessary and sufficient conditions are given in order that M have a nonnegative Moore-Penrose inverse M⁺.

INTRODUCTION

A real square matrix $A = (a_{i,j})$ is called an M-matrix if $a_{i,j} < 0$ whenever $i \neq j$ and all principal minors of $A$ are positive. Such matrices were introduced in 1937 by Ostrowski [5] and arise in investigations concerning the convergence of iteration processes in linear algebra and spectral properties of matrices. In 1953, Schneider [7, 8] extended the M-matrix to the concept of singular M-matrix by establishing some analogues to some results of Ostrowski.

For a real square matrix $A$ with nonpositive off-diagonal elements, it is known that $A$ is an M-matrix if and only if $A$ is nonsingular and $A^{-1} > 0$. The purpose of this paper is to investigate the Moore-Penrose inverses of singular M-matrices. Section 1 contains the notation and preliminaries. In Sec. 2, a necessary condition for the nonnegativity of the Moore-Penrose inverse of a singular M-matrix is given. In Sec. 3, we characterize all singular M-matrices whose Moore-Penrose inverses are nonnegative.

1. NOTATION AND PRELIMINARIES

Throughout this paper, all matrices considered are real. A square matrix is a real function on $N \times N$, where $N$ is the set of indices 1, 2, ..., $n$, and $n$ is a
positive integer. If A is a matrix, we shall denote by \( a_{i,j} \) the value of A at \((i, j)\). The transpose of A will be denoted by \( A^T \), the range of A by \( R(A) \), and the null space of A by \( N(A) \). The spectral radius of A is the maximum of the moduli of the eigenvalues of A and will be denoted by \( \rho(A) \). The determinant of A will be denoted by \( \det A \). If \( M \subseteq N \) and if A is a matrix on \( N \times N \), we define \( A(M) \) to be the restriction of A to \( M \times M \). \( A(M) \) is called the principal submatrix of A, and \( \det A(M) \) is called the principal minor of A corresponding to \( M \). A matrix A = \((a_{i,j})\) is said to be nonnegative, or \( A \geq 0 \), if \( a_{i,j} \geq 0 \) for each \((i, j)\). If \( a_{i,j} > 0 \) for each \((i, j)\), we say A is positive, or \( A > 0 \). A vector \( X = (x_i) \) in \( \mathbb{R}^N \) is said to be nonnegative, or \( X \geq 0 \), if \( x_i \geq 0 \) for each \( i \in N \). We write \( X > 0 \) if \( x_i > 0 \) for each \( i \in N \). If \( A \) and \( B \) are two matrices, we write \( B \geq A \) if \( B - A > 0 \). We shall denote by \( Z \) the class of all real square matrices whose off-diagonal elements are all nonpositive.

The following theorem contains most of the important characterizations of \( M \)-matrices.

**Theorem 1.1** [3, Theorem 4.3; 4, Theorem 2.1]. Suppose \( A \in Z \). Then the following statements are equivalent:

1a) \( A = \lambda I - B \), where I is the identity matrix, \( B \geq 0 \), and \( \lambda > \rho(B) \), \( \rho(B) \) being a maximal eigenvalue of \( B \).
1b) The real part of each eigenvalue of \( A \) is positive.
1c) All principal minors of \( A \) are positive.
1d) \( \lambda^{-1} \) exists and \( \lambda^{-1} > 0 \).
1e) There exists a vector \( X > 0 \) such that \( AX > 0 \).

Following Fiedler and Ptak [3], we shall denote by \( K \) the class of all matrices \( A \in Z \) fulfilling one of the conditions in Theorem 1.1. Also, we denote by \( K_0 \) the class of all matrices \( A \in Z \) which have all principal minors nonnegative. A singular matrix in \( K_0 \) is called a singular \( M \)-matrix.

The following theorem characterizes a matrix \( A \in Z \) which has nonnegative principal minors.

**Theorem 1.2** [3, Theorem 5.1; 4, Theorem 2.1] Suppose \( A \in Z \). Then the following statements are equivalent:

2a) \( A = \lambda I - B \), where I is the identity matrix, \( B \geq 0 \), and \( \lambda > \rho(B) \), \( \rho(B) \) being a maximal eigenvalue of \( B \).
2b) The real part of each eigenvalue of \( A \) is nonnegative.
2c) \( A \in K_0 \).
2. REDUCIBILITY AND NONNEGATIVITY OF THE MOORE-PENROSE INVERSE

Let $A$ be an arbitrary $m \times n$ matrix. The Moore-Penrose inverse $[1]$ of $A$ is the unique $n \times m$ matrix $A^+$ satisfying $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^T = AA^+$, and $(A^+A)^T = A^+A$. The following results $[1]$ are basic properties of $A^+$ for an $m \times n$ matrix $A$.

(2.1) $A^+ = A^{-1}$ if $A$ is nonsingular.
(2.2) $(A^T)^+ = (A^+)^T$.
(2.3) If $U$ and $V$ are orthogonal matrices, then $(UAV)^+ = V^TA^+U^T$.
(2.4) $A^+A$ is the projection on $R(A^T)$ along $N(A^T)$.
(2.5) $R(A^+) = R(A^T)$ and $N(A^+) = N(A^T)$.

A matrix $A$ of order $n$, $n \geq 2$, is said to be reducible if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11}$ and $A_{22}$ are square submatrices of $A$. A matrix is irreducible if it is not reducible. In this paper, a one-by-one matrix is also said to be irreducible; a one-by-one matrix is nonsingular (singular) if it is nonzero (zero). We shall use the following well-known results about an irreducible matrix in $K_0$ throughout this paper.

**Theorem 2.1** [3, Theorems 5.6, 5.7]. Let $M \in K_0$ be an irreducible matrix of order $n$.

(a) If $M$ is singular, then $\text{rank } M = n - 1$, and there exists a vector $X > 0$ such that $MX = 0$.

(b) All proper principal minors of $M$ are positive.

**Lemma 2.2.** Let $M$ be a square matrix with $M^+ > 0$. If $Q$ is a vector in $R(M^T)$ and $Y$ is a vector in $N(M^T)$ such that $MQ - dY > 0$ for some real number $d$, then $Q > 0$.

**Proof.** Let $MQ = dY + b$ for some vector $b > 0$. Since $M^+M$ is the projection on $R(M^T)$ along $N(M^T)$, we have $Q = M^+MQ = M^+b > 0$. $lacksquare$
Theorem 2.3. If $M$ is an $n \times n$ ($n \geq 2$) singular irreducible matrix in $K_0$, then $M^+ \neq 0$.

Proof. Partition $M$ as follows:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11} \in K$ and rank $M = \text{rank } M_{11} = n - 1$ by Theorem 2.1. There exists a vector $X > 0$ such that $M_{11}^T X > 0$ by (1e) of Theorem 1.1. Let $Q = M^T (X^T | 0)^T$; then $Q \in R(M^T)$, $Q \neq 0$, and $Q \neq 0$. Since $M$ is singular and irreducible, $M^T Y = 0$ for some vector $Y > 0$ by Theorem 2.1. Thus, we can find a real number $d$ such that $MQ - dY > 0$. If $M^+ > 0$, then $Q > 0$. But this contradicts the fact that $Q$ is nonzero and nonpositive. Hence, $M^+ \neq 0$.

3. CHARACTERIZATIONS OF ELEMENTS OF $K_0$ WITH NONNEGATIVE MOORE-PENROSE INVERSES

It is obvious, from Theorem 2.3, that a necessary condition for a singular matrix $M \in K_0$ to have $M^+ > 0$ is that $M$ must be reducible.

Theorem 3.1. Let $M \in K_0$ be partitioned as follows:

$$M = \begin{bmatrix} M_1 \\ B_{11} & \cdots & B_{1s} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{ss} \end{bmatrix},$$

where $B_{ii}$ is singular and irreducible for $i = 1, \ldots, s$. If $M^+ > 0$, then $B_{ij} = 0$ for $i \neq j$.

Proof. There exists a vector $X_i > 0$, by Theorem 2.1, such that $B_{ii}^T X_i = 0$ for $i = 1, \ldots, s$. We claim $B_{s-1,s} = 0$.

Let $Q = M^T(0|0, \ldots, 0, X_{s-1}^T, 0)^T$. Then $Q < 0$, and all blocks in $MQ$ are nonnegative except possibly the last block $B_{x,s} B_{s-1,s}^T X_{s-1}$. Let $X =$
(0|0|...,0,X_\mathcal{S}^T)^T$, then $X \in N(M^T)$. Clearly, there exists a real number $d$ such that $MQ - dX > 0$ since $X \succ 0$. Hence, $Q \succ 0$ by Lemma 2.2. If $B_{s-1,s} \neq 0$, then $Q$ is nonzero and nonpositive, and we get a contradiction. Therefore, $B_{s-1,s} = 0$.

We now assume that $B_{ij} = 0$ for $j = i + 1, ..., s$, and $i = k + 1, ..., s - 1$, and at least one of $B_{kl}$, $l = k + 1, ..., s$, is not zero. Let $Q_1 = M^T$. $(0|0|...,0,X_\mathcal{K}^T,0|0,0)^T$; then $Q_1$ is a nonzero and nonpositive vector in $R(M^T)$. And $MQ_1 = (V|Y_1^T, ..., Y_s^T)^T$, where $V$ is a nonnegative row vector, $Y_i = \sum_{l=k+1}^s B_{il} B_{kl} X_k$ for $j = 1, ..., k$, and $Y_i = \sum_{l=k}^s B_{il} B_{kl} X_k$ for $i = k + 1, ..., s$. Since $Y_i > 0$ for $h = 1, ..., k$, all blocks in $MQ_1$ are nonnegative except possibly the blocks $Y_{k+1}, ..., Y_s$. Let $X_1 = (0|0|...,0,X_{k+1}^T, ..., Y_s^T)^T$; then $X_1 \in N(M^T)$. Thus, we can find a real number $d_1$ so that $MQ_1 - d_1 X_1 > 0$. Hence, $Q_1 \succ 0$. But this is a contradiction to the fact that $Q_1$ is nonzero and nonpositive. Therefore, $B_{kl} = 0$ for $l = k + 1, ..., s$.

Repeating the same process, we finally obtain $B_{ij} = 0$ for $i \neq j$. 

**Corollary 3.2.** If $M$ is a matrix in $K_0$ such that $M$ is partitioned into the form

$$M = \begin{bmatrix} B_{11} & \cdots & B_{1s} \\ & \ddots & \\ 0 & & B_{ss} \end{bmatrix},$$

where $B_i$ is singular and irreducible for $i = 1, ..., s$, then $M^+ \succ 0$ if and only if $M = 0$.

**Proof.** The corollary follows from Theorem 3.1. and Theorem 2.2.

Before we proceed, we need the following results about an irreducible $M$-matrix.

**Theorem 3.3** [9, Theorem 3.9]. If $B \succ 0$ is an $n \times n$ matrix, then the following are equivalent:

1. $\alpha \succ \rho(B)$, and $B$ is irreducible;
2. $\alpha I - B$ is nonsingular, and $(\alpha I - B)^{-1} > 0$.

We now prove a key theorem in the characterizations of matrices $M \in K_0$ with the property $M^+ \succ 0$. 


Theorem 3.4. Let $M \in K_0$ be partitioned as follows:

\[
M = \begin{bmatrix}
M_1 & M_2 & M_3 & M_4 \\
0 & D & E & F \\
0 & 0 & A & C \\
0 & 0 & 0 & B
\end{bmatrix},
\]

where

\[
D = \begin{bmatrix}
D_{11} & \cdots & D_{1s} \\
& \ddots & \vdots \\
0 & \cdots & D_{ss}
\end{bmatrix}, \quad E = \begin{bmatrix}
E_{1,s+1} & \cdots & E_{1,t} \\
& \ddots & \vdots \\
E_{s,s+1} & \cdots & E_{s,t}
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
F_{1,t+1} & \cdots & F_{1,n} \\
& \ddots & \vdots \\
F_{s,t+1} & \cdots & F_{s,n}
\end{bmatrix}, \quad A = \begin{bmatrix}
A_{s+1,s+1} & \cdots & A_{s+1,t} \\
& \ddots & \vdots \\
& & A_{t,t}
\end{bmatrix},
\]

and

\[
B = \begin{bmatrix}
B_{t+1,t+1} & \cdots & B_{t+1,n} \\
& \ddots & \vdots \\
& & B_{n,n}
\end{bmatrix}
\]

are such that $D_{ii}$ and $B_{ij}$ are singular and irreducible for $i = 1, \ldots, s$ and $j = t+1, \ldots, n$, and $A_{hh}$ is nonsingular and irreducible for $h = s+1, \ldots, t$. If $M^+ > 0$, then

(1) $B_{ij} = 0$ and $D_{ij} = 0$ for $i \neq j$;
(2) $E = 0$ and $F = 0$.

Proof. We first note that $B_{ij} = 0$ for $i \neq j$, by Theorem 3.1. There exist vectors $X_i > 0$ and $X_f > 0$, by Theorem 2.1., such that $D_{ii}^T X_i = 0$ for $i = 1, \ldots, s$, and $B_{ji}^T X_j = 0$ for $j = t+1, \ldots, n$. We define

\[
Z_l = E_{s,t} X_s \quad \text{for} \quad l = s+1, \ldots, t,
\]
and

\[ Y_{s+1} = -\left(A_{s+1,s+1}^T\right)^{-1}Z_{s+1}, \]

\[ Y_l = -\left(A_l^T\right)^{-1}\left(Z_l + \sum_{j=s+1}^{l-1} A_l^T Y_j\right) \quad \text{for} \quad l = s+2, \ldots, t. \]

It is clear that \( Z_l < 0 \), since \( X_s > 0 \) and \( E_{s,l} < 0 \) for \( l = s+1, \ldots, t \). Now \( A_l^T \) is an irreducible \( M \)-matrix, so \( (A_l^T)^{-1} > 0 \) by Theorem 3.3. This implies \( Y_l \geq 0 \) for \( l = s+1, \ldots, t \).

We claim \( F_{s,j} = 0 \) for \( j = t+1, \ldots, n \). Let

\[ Q = M^T \cdot (0|0, \ldots, 0, X_s^T|Y_{s+1}^T, \ldots, Y_t^T|0)^T. \]

Since

\[ E^T \cdot (0, \ldots, 0, X_s^T)^T + A^T \cdot (Y_{s+1}^T, \ldots, Y_t^T)^T = (Z_{s+1}^T, \ldots, Z_t^T)^T + (-Z_{s+1}^T, \ldots, -Z_t^T)^T = 0, \]

we have \( Q = (0|0|0|W^T)^T \), where

\[ W = F^T \cdot (0, \ldots, 0, X_s^T)^T + C^T \cdot (Y_{s+1}^T, \ldots, Y_t^T)^T \]

\[ MQ = (W^T M_s^T|W^T F^T|W^T C^T|W^T B^T)^T \]

and all blocks in \( MQ \) are nonnegative except possibly the block \( BW \). Let \( X = (0|0|0|X_{s+1}^T, \ldots, X_n^T)^T \); then \( X \in N(M^T) \), and thus there exists a real number \( d \) such that \( MQ - dX \geq 0 \). By Lemma 2.2, \( Q \geq 0 \). If one of \( F_{s,j} \), \( j = t+1, \ldots, n \), is not zero, \( Q \) will be nonzero and nonpositive, which is a contradiction. Therefore, \( F_{s,j} = 0 \) for \( j = t+1, \ldots, n \).

Next, we claim \( E_{s,j} = 0 \) for \( j = s+1, \ldots, t \). Suppose that one of \( E_{s,j} \), \( j = s+1, \ldots, t \), is not zero. Let \( U_k = E_{s,k}^T X_s \) for \( k = s+1, \ldots, t \). Then \( E_k < 0 \) for \( k = s+1, \ldots, t \), and at least one of these \( E_k \) is nonzero. We define \( V_{s+1} \) and \( V_k \) for \( k = s+2, \ldots, t \), as follows:

\[ V_{s+1} = -\left(A_{s+1,s+1}^T\right)^{-1}U_{s+1}, \quad \text{and} \quad V_k = -\left(A_{k,k}^T\right)^{-1}\left(U_k + \sum_{j=s+1}^{k-1} A_{j,k}^T V_j\right). \]
Now let \( V = (V_{s+1}^T, \ldots , V_t^T)^T \); then \( V \) is nonzero and nonnegative.

**Case 1:** \( C^TV \neq 0 \). Let

\[
Q_1 = M^T \cdot (0|0, \ldots , 0, X_s^T|V^T|0)^T;
\]

then \( Q_1 < 0 \) and \( Q_1 \neq 0 \), since \( E^T \cdot (0, \ldots , 0, X_s^T) + A^T \cdot (V_{s+1}^T, \ldots , V_t^T)^T = 0 \). Now

\[
MQ_1 = (V^T CM_3^T|V^T CF^T|V^T CC^T|V^T CB^T)^T,
\]

and all blocks in \( MQ_1 \) are nonnegative except possibly the block \( BC^TV \). Let

\[
X_{Q_1} = (0|0|0|X_{s+1}^T, \ldots , X_n^T)^T;
\]

then \( X_{Q_1} \in N(M^T) \). Thus, \( MQ_1 - d_1X_{Q_1} \geq 0 \) for some real number \( d_1 \). By Lemma 2.2, \( Q_1 > 0 \), which is a contradiction.

**Case 2:** \( C^TV = 0 \). Let

\[
Q_2 = M^T \cdot (0|0, \ldots , 0, X_s^T|0|0)^T;
\]

then \( Q_2 < 0 \) and \( Q_2 \neq 0 \). Now

\[
MQ_2 = (U^TM_3^T|U^TE^T|P_{s+1}^T, \ldots , P_t^T|0)^T,
\]

where \( U = (U_{s+1}^T, \ldots , U_t^T)^T \), and \( P_k = \Sigma_j A_k U_j \) for \( k = s+1, \ldots , t \). All blocks in \( MQ_2 \) are nonnegative except possibly the blocks \( P_{s+1}, \ldots , P_t \). Now, let

\[
X_{Q_2} = (0|0, \ldots , 0, X_s^T|V^T|0)^T;
\]

then \( X_{Q_2} \in N(M^T) \), since \( C^TV = 0 \). If \( U_k = 0 \), then \( P_k = \Sigma_j A_k U_j \geq 0 \). If \( U_k \neq 0 \), then \( V_k > 0 \). It is easy to see that \( MQ_2 - d_2X_{Q_2} \geq 0 \) for some real number \( d_2 \). By Lemma 2.2, \( Q_2 > 0 \). We again get a contradiction. Therefore, \( E_{s,t} = 0 \) for \( j = s+1, \ldots , t \).

We now prove \( D_{s-1,s} = 0 \). We define \( Y'_l \ (>0) \) similarly to the way we define \( Y_l \ (>0), \ l = s+1, \ldots , t \), so that

\[
E^T \cdot (0, \ldots , 0, X_{s-1}^T, 0)^T + A^T \cdot (Y_{s+1}^T, \ldots , Y_t^T)^T = 0.
\]
Let
\[ Q_3 = M^T \begin{pmatrix} 0 | 0, \ldots, 0, X_{s-1}^T, 0 | Y_{s+1}^T, \ldots, Y_t^T | 0 \end{pmatrix}^T, \]
and let
\[ X_Q = \begin{pmatrix} 0 | 0, \ldots, 0, X_s^T, 0 | X_{t+1}^T, \ldots, X_n^T \end{pmatrix}^T. \]

Then \( MQ_3 - d_3X_Q \geq 0 \) for some real number \( d_3 \), and thus \( Q_3 > 0 \) by Lemma 2.2. If \( D_{s-1,s} \neq 0 \), then \( Q_3 \) will be nonzero and nonpositive, which is a contradiction. Hence, \( D_{s-1,i} = 0 \).

By using the same argument as we did to show \( F_{s,i} = 0 \) and \( E_{s,i} = 0 \), we can show that \( F_{s-1,i} = 0 \) and \( E_{s-1,i} = 0 \) for \( i = s+1, \ldots, t \) and \( j = t+1, \ldots, n \). Repeating the same process, we finally obtain \( B_{s,s} = 0 \), \( D_{s,s} = 0 \) for \( i \neq j \), \( E = 0 \), and \( F = 0 \).

**Corollary 3.5.** Let \( M \in K_0 \) be partitioned as follows:
\[
M = \begin{bmatrix}
M_1 & M_2 & M_3 \\
0 & D & E \\
0 & 0 & A
\end{bmatrix},
\]
where \( A, D, \) and \( E \) are the same as in Theorem 3.4. If \( M^+ > 0 \), then \( D_{s,s} = 0 \) for \( i \neq j \), and \( E = 0 \).

**Corollary 3.6.** Let \( M \in K_0 \) be partitioned into the form
\[
M = \begin{bmatrix}
D & E \\
0 & A
\end{bmatrix},
\]
where \( A, D, \) and \( E \) are the same as those in Theorem 3.4. Then \( M^+ > 0 \) if and only if \( D = 0 \) and \( E = 0 \).

**Proof.** The corollary follows from Corollary 3.5 and Theorem 2.3.

**Theorem 3.7.** Let \( M \in K_0 \) be partitioned as follows:
\[
M = \begin{bmatrix}
A & C \\
0 & B
\end{bmatrix},
\]
where $A$, $B$, and $C$ are the same as in Theorem 3.4. Then $M^+ > 0$ if and only if $B = 0$ and $C = 0$.

**Proof.** The proof is similar to that of Theorem 3.4.

If $M$ is a reducible matrix, then by definition there exists a permutation matrix $P$ such that

$$PMP^T = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{nn} \end{bmatrix}$$

where $M_{ii}$ is square and irreducible for $i = 1, \ldots, n$. We will regroup the blocks on the diagonal in the following way:

Suppose that $M_{ii}, i = k, k+1, \ldots, l$, is singular, and suppose that $M_{k-1,k-1}$ and $M_{l+1,l+1}$ are nonsingular. Then we group $M_{kk}, M_{k+1,k+1}, \ldots, M_{l,l}$ together to form a new block on the diagonal and call it $D_{kk}$. That is,

$$D_{k,k} = \begin{bmatrix} M_{k,k} & \cdots & M_{k,l} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{l,l} \end{bmatrix}$$

where all blocks on the diagonal are singular, and $M_{k-1,k-1}$ and $M_{l+1,l+1}$ in $PMP^T$ are nonsingular. We perform the same regrouping for nonsingular blocks on the diagonal of $PMP^T$. Thus, we can rewrite $PMP^T$ in the following form:

$$PMP^T = \begin{bmatrix} D_{1,1} & \cdots & D_{1,t} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{t,t} \end{bmatrix} \quad (3.0)$$

where $D_{i,i}$ is a submatrix (of $PMP^T$) of the form

$$D_{i,i} = \begin{bmatrix} M_{i,i} & \cdots & M_{i,i} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{i,i} \end{bmatrix}$$
such that either (1) every block on the diagonal of $D_{i,i}$ is singular and every block on the diagonals of $D_{i-1,i-1}$ and $D_{i+1,i+1}$ is nonsingular, or (2) every block on the diagonal of $D_{i,i}$ is nonsingular and every block on the diagonals of $D_{i-1,i-1}$ and $D_{i+1,i+1}$ is singular.

We now characterize all matrices $M \in K_0$ whose $M^+ \succ 0$.

**Theorem 3.8.** Let $M$ be a nonzero matrix in $K_0$. A necessary and sufficient condition for $M^+ \succ 0$ is that there exists a permutation matrix $P$ such that

$$PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

with $A \in K$.

**Remark.** The zero blocks in $PMP^T$ may not be present.

**Proof.**

**Necessity:** We assume $M^+ \succ 0$. If $M \in K$, then the statement is true. If $M$ is singular, then $M$ must be reducible by Theorem 2.3. Let $P$ be a permutation matrix such that

$$PMP^T = \begin{bmatrix} D_{1,1} & \ldots & D_{1,t} \\ & \ddots & \\ 0 & & D_{t,t} \end{bmatrix}$$

with $A \in K$.

**Case 1.** Every block on the diagonal of $D_{k,k}$ is nonsingular. We rewrite $P_1MP_1^T$ in the following form:

$$P_1MP_1^T = \begin{bmatrix} M_1 & M_2 & M_3 \\ 0 & D_{k-1,k-1} & D_{k-1,k} \\ 0 & 0 & D_{k,k} \end{bmatrix}$$
Since \((P_1 M P_1^T)^+ = P_1 M^+ P_1^T \geq 0\), we obtain \(D_{k-1,k} = 0\) by Corollary 3.5. Let \(P_2\) be a permutation matrix such that
\[
P_2 M P_2^T = \begin{bmatrix}
M_1 & M_3 & M_2 \\
0 & D_{k,k} & 0 \\
0 & 0 & D_{k-1,k-1}
\end{bmatrix}.
\]

Thus \(D_{k-2,k-2}\) and \(D_{k,k}\) are merged into one block. By the induction hypothesis, there exists a permutation matrix \(P\) such that
\[
P M P^T = \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\] with \(A \in K\).

**Case 2.** Every block on the diagonal of \(D_{k,k}\) is singular. We rewrite \(P_1 M P_1^T\) in the following form:
\[
P_1 M P_1^T = \begin{bmatrix}
M_1 & M_2 & M_3 & M_4 \\
0 & D_{k-2,k-2} & D_{k-2,k-1} & D_{k-2,k} \\
0 & 0 & D_{k-1,k-1} & D_{k-1,k} \\
0 & 0 & 0 & D_{k,k}
\end{bmatrix},
\]
such that every block on the diagonal of \(D_{k-2,k-2}\) is singular and every block on the diagonal of \(D_{k-1,k-1}\) is nonsingular.

Since \((P_1 M P_1^T)^+ = P_1 M^+ P_1^T \geq 0\), we have \(D_{k-2,k-1} = 0\) by Theorem 3.4. Hence, there exists a permutation matrix \(P_3\) such that
\[
P_3 M P_3^T = \begin{bmatrix}
M_1 & M_3 & M_2 & M_4 \\
0 & D_{k-1,k-1} & 0 & D_{k-1,k} \\
0 & 0 & D_{k-2,k-2} & D_{k-2,k} \\
0 & 0 & 0 & D_{k,k}
\end{bmatrix}.
\]

Thus \(D_{k-2,k-2}\) and \(D_{k,k}\) are merged into one block. By the induction hypothesis, there exists a permutation matrix \(P\) such that
\[
P M P^T = \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\] with \(A \in K\).
Sufficiency: Clearly,

\[(PMP^T)^+ = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \implies M = P^T \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \geq 0.\]

For any square matrix of order \(n\), an \(n \times n\) matrix \(X\) which satisfies \(AXA = A\), \(XAX = X\), and \(AX = XA\) is called the group inverse of \(A\). It is known that the group inverse of a matrix \(A\) does not always exist, but when it exists it is unique and is denoted by \(A^\#\). The existence of \(A^\#\) is equivalent to the condition that \(\text{rank} A = \text{rank} A^2\), which in turn is equivalent to the requirement that \(R(A) \cap N(A) = \{0\}\) [1, pp. 162, 165]. There exists a class of matrices such that the group inverse and the Moore-Penrose inverse are the same. We shall call a square matrix \(A\) range-Hermitian if \(R(A) = R(A^T)\). It is well known that \(A^\# = A^+\) if and only if \(A\) is range-Hermitian [1, p. 164].

For any square matrix \(A\), \(A\) and \(A^\#\) have the same range and the same null space, by the defining equations of \(A^\#\), if \(A^\#\) exists. Since \(A^\# A\) is an idempotent matrix, \(A^\# A\) is the projection on \(R(A^\# A)\) along \(N(A^\# A)\). But \(R(A^\# A) = R(A^\#) = R(A)\); hence \(A^\# Ax = x\) for \(x\) in \(R(A)\). Also, we have \(A^\# = A^{-1}\) if \(A\) is nonsingular, \((A^T)^\# = (A^\#)^T\), and \((PAP^T)^\# = PA^\# P^T\) for any permutation matrix \(P\).

By using the same argument as we did before, we can obtain the same results about \(M^\#\) as those in Theorem 2.3, Theorem 3.1, Corollary 3.2, Theorem 3.4, Corollary 3.5, Corollary 3.6, Theorem 3.7, and Theorem 3.8. Therefore, we obtain the following equivalent statements.

**Theorem 3.9.** Let \(M\) be a nonzero matrix in \(K_0\). The following statements are equivalent:

1. \(M^+ \geq 0\).
2. There exists a permutation matrix \(P\) such that

\[
PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\quad \text{with} \quad A \in K.
\]

3. \(\text{rank} M = \text{rank} M^2\) and \(M^\# \geq 0\).

Furthermore, if one of (1), (2), and (3) holds, then \(M^+ = M^\#\).

I gratefully acknowledge the help and guidance of Dr. T. L. Markham in the preparation of this paper, which forms part of my dissertation at the University of South Carolina.
REFERENCES


Received 6 February 1976