A NOTE ON THE SIMPLIFYING ALGORITHMIC STEPS OF THE SIMPLEX METHOD FOR QUADRATIC PROGRAMMING

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INTRODUCTION

Before considering the simplex method for quadratic programming problem, it is necessary to learn how to find an optimal solution to non-linear programming problem. There are the Kuhn-Tucker conditions [1] [2, P.723] which describe such optimal solution as follows:

Let the maximizing objective function, $f(x_1,x_2,x_3,\cdots\cdots x_n)$, and the constraint functions $g_i(x_1,x_2,x_3,\cdots\cdots x_n)\leqq b_i$, for $i=1,2,3,\cdots\cdots m$, be differentiable; but not necessary linear. Then $(x_1',x_2',x_3',\cdots\cdots x_n')$ can be an optimal solution to non-linear programming problem if and only if there exist m values, $v_1,v_2,v_3,\cdots\cdots v_m$, named Lagrange Multipliers or dual variables, such that the following conditions are satisfied:

$$\frac{\partial f}{\partial x_j}-\sum_{i=1}^{m} v_i \frac{\partial g_i}{\partial x_j} \leqq 0 \quad (1)$$

$$x_j'\left[\frac{\partial f}{\partial x_j}-\sum_{i=1}^{m} v_i \frac{\partial g_i}{\partial x_j}\right]= 0 \quad (2)$$

at $x_j=x_j'$, for $j=1,2,3,\cdots\cdots,n$.

$$g_i(x_1',x_2',x_3',\cdots\cdots x_n')-b_i \leqq 0 \quad (3)$$

$$v_i\,[g_i(x_1',x_2',x_3',\cdots\cdots x_n')-b_i] =0 \quad (4)$$

, for $i=1,2,3,\cdots\cdots m$.

$$x_j\geqq 0, \text{ for all } j. \quad (5)$$

$$v_i\geqq 0, \text{ for all } i. \quad (6)$$

Of there existing algorithm for quadratic programming problem, let the maximizing objective function be quadratic and the constraint functions be linear, then the Kuhn-Tucker conditions can be considered as generalizations or extensions of the simplex method for linear programming problem derived by P. Wolfe [3], C. van de Panne and A Whinston [4] and others. The objective of this note is to simplify algorithm steps of the simplex method for computing the maximizing quadratic programming problem.

NOTATIONS

The following useful notations are through-out this note:

N_1 A is an m-by-n matrix.

N_2 C is an n-square symmetric matrix assumed to be positive definite.

N_3 I_r is an identity matrix where r=m, n or m+n.

N_4 p is a column vector with n components $p_1, p_2, p_3, \cdots\cdots p_n$.

N_5 b is a column vector with m components $b_1, b_2, b_3, \cdots\cdots b_m$.

N_6 x is a column vector with n components $x_1, x_2, x_3, \cdots\cdots x_n$.

N_7 v is a column vector with m components $v_1, v_2, v_3, \cdots\cdots v_m$.

N_8 u is a column vector with n components $u_1, u_2, u_3, \cdots\cdots u_n$.

N_9 y is a column vector with m components $y_1, y_2, y_3, \cdots\cdots y_m$.

N_{10} z is a column vector with n components $z_1, z_2, z_3, \cdots\cdots z_n$.

N_{11} The prime (′) indicates the transpose of a matrix or a vector

N_{12} $x'p=p'x$, $u'x=x'u$, $v'b=b'v$, and $x'A'v=v'Ax$.

N_{13} B is an (m+n)−square matrix $\begin{pmatrix} -C & -A' \\ A & 0 \end{pmatrix}$.

N_{14} $\left(\begin{array}{cccc|c} -C & -A' & I_n & 0 & -p \\ A & 0 & 0 & I_m & b \\ \hline p' & b' & 0 & 0 & 2f \end{array}\right)$ is called a simplex matrix[5, P. 375].

N_{15} ↓ indicates the pivot column.

N_{16} ← indicates the pivot row.

N_{17} 0 indicates the pivot element.

N_{18} b.v. represents the basic variable.

PRELIMINARIES

The general convex quadratic programming problem considered here is the following form:

$$\text{Maximize} \qquad f=p'x-\tfrac{1}{2}x'Cx, \tag{7}$$

$$\text{subject to} \qquad Ax \leqq b, \tag{8}$$

$$\text{and} \qquad x \geqq 0. \tag{9}$$

The Kuhn-Tucker conditions,(1)(2)(3)(4)(5)(6), for an optimal solution to (7)(8)(9) can be written in the forms

$$p-Cx-A'v+u=0, \tag{10}$$

$$Ax+y=b, \tag{11}$$

$$u'x+v'y=0, \tag{12}$$

$$\text{and} \qquad x\geqq 0,\ u\geqq 0,\ v\geqq 0,\ y\geqq 0. \tag{13}$$

A routine calculation shows that x, u, v, y satisfy (10)(11)(12) and N_{12}, and then the result of the objective function (7) is

$$f=\frac{1}{2}(p'x+b'v), \tag{14}$$

$$2f=p'x+b'v. \tag{15}$$

Thus, it is obvious that the quadratic programming problem (7)(8)(9) can be changed into the following linear programming problem.

$$\text{Maximize} \quad 2f=p'x+b'v, \tag{16}$$

$$\text{subject to} \quad -Cx-A'v+u=-p, \tag{17}$$

$$Ax+y=b, \tag{18}$$

$$u'x+v'y=0, \tag{19}$$

$$\text{and} \quad x\geqq 0,\ u\geqq 0,\ v\geqq 0,\ y\geqq 0. \tag{20}$$

By use of N_{14}, we have

$$\left|\begin{array}{cccc|c} x & v & u & y & \\ -C & -A' & I_n & 0 & -p \\ A & 0 & 0 & I_m & b \\ \hline p' & b' & 0 & 0 & 2f \end{array}\right| ; \tag{21}$$

where x, y are called the primal variables and u,v called the dual variables, whose coefficients are all elements of the lst, 2nd, 3rd, 4th colmuns in (21) respectively. From (19), there is a property that if each primal variable is basic, then a corresponding dual variable is non-basic and vice versa.

ALGORITHMIC STEPS

Before defining the algorithmic steps for the simplex method, (21) can be represented by the following form:

$$\begin{array}{c} \text{b.v.} \\ u \\ y \\ 0 \end{array} \left|\begin{array}{cccc|c} x & v & u & y & \\ -C & -A' & I_n & 0 & -p \\ A & 0 & 0 & I_m & b \\ \hline p' & b' & 0 & 0 & 2f \end{array}\right| , \tag{22}$$

where u, y along the left hand side of (22) are the basic variables, whose values are equal to the corresponding elements in last column. From (16), 2f is equal to 0 under the basic solution, $u=-p$, $y=b$, $x=0$, $v=0$; where $b\geqq 0$ because the basic solution can be used as an initial basic solution to the quadratic programming problem for the simplex method.

In the simplex method for computing the optimal solution, the iterative steps consist of selecting a non-basic variable, that enters the basic variable; and a basic variable, that leaves the basic variable. According to (19) (20), the following iterative steps can be defined:

Step l. Choose a pivot column to determine the entering basic variable.

(a) If the non-basic primal variable column has the greatest positive value in the bottom row of (22), then this column becomes the pivot column.

(b) Until all bottom elements in the x-columns are non-positive. If the non-basic dual variable has the greatest positive value, then ths column becomes the pivot column.

Step 2. Choose a pivot row to determine the leaving basic variable.

Find the minimun ratio, such that

$$h_r/k_r = \min._r \{h_r/k_r \geqq 0 \mid k_r \neq 0,\ r = n \text{ or } m\},$$

where h_r is the rth element of the last column of (22) and k_r is the rth element of the pivot column except the bottom row. Hence, the rth row becomes the pivot row.

Sten 3. Choose a pivot element.

The pivot element is at the intersection of the pivot column and the pivot row. By use of the elementary matrix row operations [5, P.24], let this pivot element be eoual to 1 and the other elements in this pivot column including the bottom element be equal to 0.

After this elementary matrix row operations, if all primal and dual variables satisfy (19) (20), then the basic solution is optimizing the objective function (16); otherwise return to the iterative steps.

NUMERICAL EXAMPLE

To illustrate the iterative steps, a numerical example can be calculated as follows:

Example 1a. Consider the quadratic programming problem

$$\text{Maximize} \quad f = 10x_1 + 4x_2 - x_1^2 + 4x_1x_2 - 4x_2^2, \tag{23}$$

$$\text{subject to} \quad x_1 + x_2 \leqq 6, \quad 4x_1 + x_2 \leqq 18, \tag{24}$$

$$\text{and} \quad x_1 \geqq 0, \ x_2 \geqq 0; \tag{25}$$

$$\text{where } p' = (10 \quad 4),\ b = \begin{pmatrix} 6 \\ 18 \end{pmatrix},\ A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix},\ \text{and } C = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}. \tag{26}$$

This example can be changed into the following form by (16)(17)(18)(19)(20).

$$\text{Maximize} \quad 2f = 10x_1 + 4x_2 + 6v_1 + 18v_2, \tag{27}$$

$$\text{subject to} \quad -2x_1 + 4x_2 - v_1 - 4v_2 + u_1 = -10, \tag{28}$$

$$4x_1 - 8x_2 - v_1 - v_2 + u_2 = -4, \tag{29}$$

$$x_1 + x_2 + y_1 = 6, \tag{30}$$

$$4x_1 + x_2 + y_2 = 18, \tag{31}$$

$$u_1x_1 + u_2x_2 + v_1x_1 + v_2y_2 = 0, \tag{32}$$

$$\text{and} \quad x_1 \geqq 0,\ x_2 \geqq 0,\ u_1 \geqq 0,\ u_2 \geqq 0,\ v_1 \geqq 0,\ v_2 \geqq 0,\ y_1 \geqq 0,\ y_2 \geqq 0. \tag{33}$$

From (22), (26), we have

b.v.	↓ x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2		
u_1	−2	4	−1	−4	1	0	0	0	−10	
u_2	4	−8	−1	−1	0	1	0	0	−4	
y_1	1	1	0	0	0	0	1	0	6	
y_2	④	1	0	0	0	0	0	1	18	←
0	10	4	6	18	0	0	0	0	2f	

(34)

Since x_1 has the greatest positive values 10 at the bottom, it becomes new basic. Comparing the ratios 6/1, 18/4 (min.). From (34), y_2 must become non-basic, and thus, the pivot element is 4. By use of the elementary matrix row operations, the following simplex matrix is obtained

b.v.	x_1	↓ x_2	v_1	v_2	u_1	u_2	y_1	y_2		
u_1	0	9/2	−1	−4	1	0	0	1/2	−1	
u_2	0	9	−1	−1	0	1	0	−1	−22	
y_1	0	(3/4)	0	0	0	0	1	−1/4	3/2	←
x_1	1	1/4	0	0	0	0	0	1/4	9/2	
0	0	3/2	6	18	0	0	0	−5/2	2f−45	

(35)

Since x_2 has the greatest positive value 3/2 at the bottom, it becomes new basic. Comparing the ratios $\frac{3}{2}/\frac{3}{4}$(min.), $\frac{9}{2}/\frac{1}{4}$. From (35), y_1 must become non-basic, and thus, the pivot element is 3/4. Similarly, we have

b.v.	x_1	x_2	v_1	↓ v_2	u_1	u_2	y_1	y_2		
u_1	0	0	−1	(−4)	1	0	−6	2	−10	←
u_2	0	0	−1	−1	0	1	12	−4	−4	
x_2	0	1	0	0	0	0	4/3	−1/3	2	
x_1	1	0	0	0	0	0	−1/3	1/3	2	
0	0	0	6	18	0	0	−2	−2	2f−48	

(36)

In (36), all bottom elements in the x-variable columns are 0. Since the dual variable v_2 has the greatest positve value 18 at the bottom, it becomes new basic. Comparing the ratios −10/−4(min.), −4/−1. From (36), u_1 must become non-basic, and thus, the pivot element is −4. Similarly, we have

b.v.	x_1	x_2	↓ v_1	v_2	u_1	u_2	y_1	y_2		
v_2	0	0	1/4	1	−1/4	0	3/2	−1/2	5/2	
u_2	0	0	−(3/4)	0	−1/4	1	27/2	−9/2	−3/2	←
x_2	0	1	0	0	0	0	4/3	−1/3	2	
x_1	1	0	0	0	0	1	−1/3	1/3	4	
0	0	0	3/2	0	9/2	0	−29	7	2f−93	

(37)

It is clear that v_1 must become new basic and u_2 must become non-basic, because the minimum ratio is $-\frac{3}{2}/-\frac{3}{4}$. The pivot element is −3/4. Similarly, we have

b.v.	x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2	
v_2	0	0	0	1	$-1/3$	$1/3$	6	-2	2
v_1	0	0	1	0	$1/3$	$-4/3$	-18	6	2
x_2	0	1	0	0	0	0	$4/3$	$-1/3$	2
x_1	1	0	0	0	0	0	$-1/3$	$1/3$	4
0	0	0	0	0	4	2	-2	-2	$2f-96$

. (38)

At last, the optimal solution to this example is $2f-96=0$ or $f=48$, because the basic solution, $x_1=4$, $x_2=2$, $v_1=2$, $v_2=2$, $u_1=0$, $u_2=0$, $y_1=0, y_2=0$, satisfies (19)(20) or (32)(33).

REMARKS

(1) In accordance with the general linear programming technique, the quadratic programming (16)(17)(18)(19)(20) is not an obvious initial basic feasible solution because $u=-p$ from (22). Hence, the artificial variables, $z\geqq 0$, must be introduced and (17) can be written in the following form [2, P. 727],

$$Cx+A'v-u+p'z=p. \tag{39}$$

This technique provides an artificial basic feasible solution to (18) (39), $z=1$, $y=b$, $x=0$, $v=0$, $u=0$. The quadratic programming problem has an optimal solution if and only if $z=0$. Therefore z must be reduced to 0 to obtain the other basic feasible solution except u. We can start with the artificial basic feasible solution as an initial basic solution indicated above and then apply two-phase method [6, P. 113] to find the optimal solution. The optimal solution to the following problem will be obtained by the phase-I method.

$$\text{Minimize } z=\sum_{j=1}^{n} z_j, \tag{40}$$

$$\text{subject to } Cx+A'v-u+p'z=p, \tag{41}$$

$$Ax+y=b, \tag{42}$$

$$\text{and } x\geqq 0,\ v\geqq 0,\ u\geqq 0,\ y\geqq 0,\ z\geqq 0. \tag{43}$$

And then the optimal solution to the quadratic programming problem will be obtained by the phase-II method, if (40) has an optimal solution, $z=0$.

Example lb. The equations, (40) (41) (42) (43), are now applied to the example la, (23) (24) (25) (26), and then we have

$$\text{Minimize } z=z_1+z_2, \tag{44}$$

$$\text{subject to } 2x_1-4x_2+v_1+4v_2-u_1+10z_1=10, \tag{45}$$

$$-4x_1+8x_2+v_1+v_2-u_2+4z_2=4, \tag{46}$$

$$x_1+x_2+y_1=6, \tag{47}$$

$$4x_1+x_2+y_2=18, \tag{48}$$

and $x_1\geqq 0$, $x_2\geqq 0$, $v_1\geqq 0$, $v_2\geqq 0$, $u_1\geqq 0$, $u_2\geqq 0$, $y_1\geqq 0$, $y_2\geqq 0$,

$z_1 \geqq 0, \; z_2 \geqq 0.$ (49)

Phase-I Method.

By use of the phase-I method, the following simplex matrix is obtained

x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2	z_1	z_2	
2	−4	1	4	−1	0	0	0	10	0	10
−4	8	1	1	0	−1	0	0	0	4	4
1	1	0	0	0	0	1	0	0	0	6
4	1	0	0	0	0	0	1	0	0	18
0	0	0	0	0	0	0	0	1	1	z

. (50)

From (50), z_1 and z_2 are really basic and then the elements 10 and 4 must be equal to 1 and the bottom elements 1 and 1 must be equal to 0 respectively. By use of the elementary matrix row operations, the simplex matrix (50) will be written in the following form

b.v.	x_1	x_2	v_1	↓ v_2	u_1	u_2	y_1	y_2	z_1	z_2		
z_1	1/5	−2/5	1/10	(2/5)	−1/10	0	0	0	1	0	1	←
z_2	−1	2	1/4	1/4	0	−1/4	0	0	0	1	1	
y_1	1	1	0	0	0	0	1	0	0	0	6	
y_2	4	1	0	0	0	0	0	1	0	0	18	
0	4/5	−8/5	−7/20	−13/20	1/10	1/4	0	0	0	0	x−2	

. (51)

In order to reduce $z=0$, z_1 and z_2 must become non-basic and then v_1 and v_2 are selected to be basic. Since v_2 has the smallest negative value −13/20 at the bottom, it becomes new basic. Comparing the ratios $1/\frac{2}{5}$ (min.), $1/\frac{1}{4}$. From (51), z_1 must become non-basic and thus, the pivot element is 2/5. Similarly, we have

b.v.	x_1	x_2	v_1	↓ v_2	u_1	u_2	y_1	y_2	z_1	z_2		
v_2	1/2	−1	1/4	1	−1/4	0	0	0	5/2	0	5/2	
z_2	−9/8	9/4	(3/16)	0	1/16	−1/4	0	0	−5/8	1	3/8	←
y_1	1	1	0	0	0	0	1	0	0	0	6	
y_2	4	1	0	0	0	0	0	1	0	0	18	
0	9/8	−9/4	−3/16	0	−1/16	1/4	0	0	13/8	0	z−3/8	

. (52)

It is clear that v_1 must become the new bsaic variable instead of the basic variable z_2 because the minimum ratio is $\frac{3}{8}\Big/\frac{3}{16}$. The pivot element is 3/16. Similarly, we have

b.v.	x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2	z_1	z_2	
v_2	2	−4	0	1	−1/3	1/3	0	0	10/3	4/3	2
v_1	−6	12	1	0	1/3	−4/3	0	0	10/3	16/3	2
y_1	1	1	0	0	0	0	1	0	0	0	6
y_2	4	1	0	0	0	0	0	1	0	0	18
0	0	0	0	0	0	0	0	0	1	1	z−0

, (53)

It will be observed that the optimal solution, $z=0$, is obtained and the useful basic solution, $v_1=2$, $v_2=2$, $y_1=6$, $y_2=18$, $x_1=0$, $x_2=0$, $u_1=0$, $u_2=0$, $z_1=0$, $z_2=0$, can be as an initial basic solution in phase-II method.

Phase-II Method

The phase-II method can be used because the optimal solution is obtained by the phase-I method. Hence, from (53), the following simplex matrix can be formulated by deleted the z_1 and z_2 columns, and the bottom row substituted by (27).

x_1	x_2	v_2	v_2	u_1	u_2	y_1	y_2	
2	−4	0	1	−1/3	1/3	0	0	2
−6	12	1	0	1/3	−4/3	0	0	2
1	1	0	0	0	0	1	0	6
4	1	0	0	0	0	0	1	18
10	4	6	18	0	0	0	0	2f

. (54)

From (54), v_1 and v_2 are really basic and then their bottom elements, 6 and 18, must be equal to 0, By use of the elementary matrix row operations, we have

b.v.	↓x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2		
v_2	2	−4	0	1	−1/3	1/3	0	0	2	
v_1	−6	12	1	0	1/3	−4/3	0	0	2	
y_1	1	1	0	0	0	0	1	0	6	
y_2	④	1	0	0	0	0	0	1	18	←
0	10	4	0	0	4	2	0	0	2f−48	

. (55)

Since x_1 has the greatest positive value 10 at the bottom, it must become new basic. Comparing the ratios, 6/1, 18/4 (min.). From (55), y_2 must become non-basic and thus, the pivot element is 4. Similarly, we have

b.v.	x_1	↓x_2	v_1	v_2	u_1	u_2	y_1	y_2		
v_2	0	−9/2	0	1	−1/3	1/3	0	−1/2	−7	
v_1	0	27/2	1	0	1/3	−4/3	0	3/2	29	
y_1	0	(3/4)	0	0	0	0	1	−1/4	3/2	←
x_1	1	1/4	0	0	0	0	0	1/4	9/2	
0	0	3/2	0	0	4	2	0	−5/2	2f−93	

. (56)

From (56), x_2 must become new basic instead of the basic variable y_1 because the minimum ratio is $\frac{3}{2}\Big/\frac{3}{4}$. The pivot element is 3/4. Similarly, we have

b.v.	x_1	x_2	v_1	v_2	u_1	u_2	y_1	y_2	
v_2	0	0	0	1	−1/3	1/3	6	−2	2
v_1	0	0	1	0	1/3	−4/3	−18	6	2
x_2	0	1	0	0	0	0	4/3	−1/3	2
x_1	1	0	0	0	0	0	−1/3	1/3	4
0	0	0	0	0	4	2	−2	−2	2f−96

. (57)

It is evident that the optimal solution to this example Ia is obtained by the two-phase method because the simplex matrix (57) is indeed quite the same simplex matrix (38).

(2) From (17) (18), We have the following matrix equation:

$$\begin{pmatrix} -C & -A' \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} -p \\ b \end{pmatrix}, \qquad (58)$$

or $$B\begin{pmatrix} x \\ v \end{pmatrix} + I_{n+m}\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} -p \\ b \end{pmatrix}. \qquad (59)$$

The matrix equation (58) or (59) is an obvious consistent and has a basic solution, $u=-p$, $y=b$, $x=0$, $v=0$; but not feasible. From the Kuhn-Tucker Conditions, if the optimal solution to the quadratic programming problem is obtained, the x and v must be non-negative (5) (6). Hence, x and v are the possible basic variables to optimize the quadratic programming problem and the matrix equation (59) can be solved for x and v by use of the inverse of B; if B is non-singular [7, P. 54]. If B is singular, then the optimal solution to the quadratic programming problem can be obtained by use of the revised simplex method proposed by M.H. Rusin [8].

Example lc. From example la, (28)(29)(30)(31), will be changed into the following matrix equation form (59)

$$\begin{vmatrix} -2 & 4 & -1 & -4 \\ 4 & -8 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} -10 \\ -4 \\ 6 \\ 18 \end{vmatrix}. \qquad (60)$$

From (60), the matrix $B=\begin{vmatrix} -2 & 4 & -1 & -4 \\ 4 & -8 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \end{vmatrix}$ is non-singular,

whose determinant value is equal to 9 and its inverse is

$$B^{-1}=\frac{1}{3}\begin{vmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 4 & -1 \\ 1 & -4 & -54 & 18 \\ -1 & 1 & 18 & -6 \end{vmatrix}. \qquad (61)$$

Premultiply both sides of (60) by B^{-1} (61) and we have

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{vmatrix} + \frac{1}{3}\begin{vmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 4 & -1 \\ 1 & -4 & -54 & 18 \\ -1 & 1 & 18 & -6 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} 4 \\ 2 \\ 2 \\ 2 \end{vmatrix}.$$

It is obvious that the basic solution, $x_1=4$, $x_2=2$, $v_1=2$, $v_2=2$, $u_1=0$, $u_2=0$, $y_1=0$, $y_2=0$, satisfies the conditions (19) (20) or (32) (33) and thus, the optimal solution to the example la is equal to (23) or (27),

$$2f=10.4+4.2+6.2+18.2,$$

or $$f=48.$$

It is clearly observed that the two-phase method requires more iterations than the simplex method for the quadratic programming problem, but the revised simplex method is more efficient.

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