行政院國家科學委員會專題研究計畫 成果報告

2維半線性波方程式解的暴破集研究
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 96-2115-M-004-003-
執 行 期 間 ： 96年08月01日至97年07月31日
執 行 單 位 ： 國立政治大學應用數學學系

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處 理 方 式 ： 本計畫涉及專利或其他智慧財產權，2年後可公開查詢

中 華 民 國 97年10月29日

Blow-up set of positive solutions for the semilinear wave equations $u_{tt} - \triangle u = u^p$

in two space dimension with non-positive energy

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1 Introduction:

Consider the initial value problem for the semilinear wave equation of the type

$$u_{tt} - \triangle u = -g(u) \quad in \quad [0, T) \times \mathbb{R}^2, \tag{1.1}$$

$$u(0, \cdot) = u_0, \;\; u_t(0, \cdot) = u_1 \tag{1.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a real valued function, the initial data are given sufficiently smooth functions and $u_t = \frac{\partial u}{\partial t}$, $\triangle$ is the Laplace operator. The linear case $g(u) = mu$, where m is a constant, corresponds to the classical Klein Gordon equation in relativistic particle physics; the constant m is interpreted as the mass and is assumed to be nonnegative generally. To model also nonlinear phenomena like quanti- zation, in the 1950s equations of (1.1) type with nonlinearities like $g(u) = mu + u^3$, $m \geq 0$, were proposed as models in relativistic quantum mechanics with local interaction; see for instance Schi [28] and Segal [29]. Solutions could be considered as real or complex valued functions. In the latter case it was assumed that the nonlinearity commutes with the phase; that is, $g(e^{i\varphi}u) = e^{i\varphi}g(u)$ for $\varphi \in \mathbb{R}$ and that $g(0) = 0$. In this case, g may be expressed $g(u) = uf(u^2)$, which gives the study of equation (1.1) [3]. Here, for simplicity, we confine ourselves to the study of real-valued solutions of equation (1.1). In spinor fields u, the scalar equation (1.1) also was considered in space dimensions $n \geq 3$; see [29].

Various other models involving nonlinearity g depending also on u_t and ∇u, have been studied. The "δ -model" involves an equation of type (1.1) for vector-valued functions subject to a certain (nonlinear) constraint. In this case $g(u) = u\left(|u_t|^2 - |\nabla u|^2\right)$, $u = (u_1, u_2, \cdots, u_n)$, and u is assumed to satisfy the condition $|u|^2 = u_1^2 + \cdots + u_n^2 = 1$; for some results on this problem see Shatah [30] and the references. We restrict our study to nonlinearity depending only on u. The above-mentioned examples suggest we assume that $g(0) = 0$ and that g satisfies the following for all u in $\mathbb{R}$

$$g(u) = c\left(1 + |u|^{p-2}\right)|u| \qquad for \;\; some \;\; p \geq 2, c \in \mathbb{R}, \tag{1.3}$$

Following Strauss [13, Theorem 3.1], we assume that g satisfies the conditions

$$G(u) \geq c\,|u|^2 \qquad for \;\; some \;\; c \in \mathbb{R} \tag{1.4}$$

and

$$G(u)/g(u) \to \infty \quad as \quad |u| \to \infty, \;\; G(u) = \int_0^u g(v)\,dv, \tag{1.5}$$

(1.4) and (1.5) include the linear case (with no sign condition) or, more generally, the case of local Lipschitz nonlinearity. In the super-linear case; that is, if $|g(u)|/|u| \to \infty$ as $|u| \to \infty$, the conditions (1.4), (1.5) should be regarded as a coerciveness condition. In fact, in this case finite propagation speed ≤ 1 and conservation of energy imply locally uniform a-priori bounds in L^2 for solutions of (1.1) in terms of the initial data.

Contrastively, in the noncoercive case it is easy to construct solutions of (1.1) with smooth initial data that blow up in finite time; for instance, for any $\alpha > 0$ the function $u(t,x) = (1-t)^{-1/m}$ solves the equation

$$u_{tt} - \triangle u = -\alpha(1+\alpha)\,u\,|u|^{2m}, \quad m \in \mathbb{N}$$

and blows up at $t = 1$. Modifying the initial data off $\{x : |x| \leq 2\}$, say, we even posses a singular solution with C^∞-data having compact support. (See John [23] for a blow-up result for a similar equation.) Thus, conditions like (1.3)-(1.5) seem natural if we are interested in global solutions.

$$g(u) = m\,|u|^{q-2}\,u + |u|^{p-2}\,u, \quad m \geq 0, 2 \leq q < p. \tag{6}$$

For nonlinearity of this kind the answer to the existence problem for (1.1), (1.2) in a striking way depends on the space dimension n and on the exponent p. In particular, in the physically interesting case $n = 3$, global existence for $p < 6$ can be established, while the same question for $p > 6$ so far has eluded all research attempts. In fact, the apparent existence of a "critical power" for (1.1) and recent advances on elliptic problems involving critical nonlinearity prompted the interest in the u^5 -Klein Gordon equation. "Critical powers" very often come into play in nonlinear problems through Sobolev embedding. In particular, $p = 6$ is the critical power for the Sobolev embedding $W^{1,2}_{loc}(\mathbb{R}^3) \hookrightarrow L^p_{loc}(\mathbb{R}^3)$. (In $n-$dimension the critical power for this embedding is $p = 2n/(n-2)$.) Moreover, they very often arise naturally from the requirements of scale invariance, that is, whenever "intrinsic" notions are involved. An example of such a problem is the Yamabe problem concerning the existence of conformal metrics with constant scalar curvature on a given (compact) Riemannian manifold. Through the work of Trudinger, Aubin, and—finally—Schoen this problem has been completely solved and it has become apparent that at the critical power properties like "compactness of the solution set" depend crucially on global aspects of the problem; in this case, on the topological and differentiable structure of the manifold. See Lee and Parker [24] for a survey of the Yamabe problem.

Incidentally, for nonlinear wave equations (or nonlinear Schrödinger equations $iu_t - \triangle u + |u|^{p-2}\, u = 0$) there appear to be many "critical powers", depending on what aspect of the problem we consider: global existence, scattering theory,$\cdots$; see Strauss [13, p.14]. As regards global existence, it remains to be seen whether the critical power represents only a technical barrier or, in fact, defines the dividing line between qualitatively different regimes of behavior of (1.1), (1.2). We conclude this introduction with a short overview of the existence results in the case of a pure power

$$u_{tt} - \triangle u + |u|^{p-2}\, u = 0, \quad p > 2. \tag{1.7}$$

The sub-critical case. For $n = 3,\ p < 6$ global existence and regularity was established by Jörgens [3] in 1961. Jörgens also was able to show local (small time) existence of regular solutions to (1.7), (1.2) for arbitrarily large p. Moreover, he was able to reduce the problem of existence of global, regular solutions to (1) to (local) estimates of the $L^\infty-$norms of solutions. These results were generalized to higher dimensions; however, such extensions have been very hard to obtain. While Jörgens' work relies on the classical representation formula for the 3$-$dimensional wave equation, this method fails in higher dimensions $n > 3$. The fundamental solution to the wave equation no longer is positive; moreover, it carries derivatives transverse to the wave cone. Nevertheless, at least for $n \leq 9$, the existence results of Pecher [26], Brenner-von Wahl [19] now cover the full sub-critical range $p < 2n/(n-2)$. Regular solutions are unique.

Global weak solutions. On the other hand, by a suitable approximation and using energy estimates, for all $p > 2,\ n \geq 3$ it is possible to construct global weak solutions, satisfying (1.7) in a distributional sense; see Segal [29], Lions [25]. In this case, it even suffices to assume that the initial data $u_0, u_1 \in L^2_{loc}(\mathbb{R}^n)$

with $u_0 \in L^p_{loc}(\mathbb{R}^n)$ and distributional derivative $\nabla u_0 \in L^2_{loc}(\mathbb{R}^n)$ Energy estimates immediately give uniqueness of weak solutions in case $p \leq \frac{2n-2}{n-2}$; see Browder [1]. However, this range is well below the critical Sobolev exponent $p = 2n/(n-2)$. In order to improve the range of admissible exponents, more sophisticated tools were developed, based, in particular, on the $L^p - L^q-$estimates for the wave operator by Strichartz [31]; see also Brenner [18].

In their simplest version, these estimates allow to prove uniqueness of solutions to (1.7),(1.2) for $p \leq \frac{2(n+1)}{n-1}$, the Sobolev exponent in $(n+1)$ space dimensions. In fact, uniqueness can be established for $p < 2n/(n-2)$; see Ginibre-Velo [20]. In this case, moreover, the unique solution can be shown to be "strong", that is, to possess second derivatives in L^2 and to satisfy the energy identity [20]. The critical case in dimension $n = 3$, global existence of C^2 -solutions in the critical case $p = 6$ was first obtained by Rauch [27], assuming the initial energy

$$E(u(0)) = \int_{\mathbb{R}^3} \left(\frac{|u_1|^2 + |u_0|^2}{2} + \frac{|u_0|^6}{6} \right)(x)\, dx$$

to be small. In 1987, also for "large" data global C^2 -solutions were shown to exist by this author [32] in the radially symmetric case $u_0(x) = u_0(|x|), u_1(x) = u_1(|x|)$. Finally, Grillakis [22] in 1989 was able to remove the latter symmetry assumption, yielding the following result:

For any $u_0 \in C^3(\mathbb{R}^3)$, $u_1 \in C^2(\mathbb{R}^3)$ there exists a unique solution u in $C^2(\mathbb{R}^3 \times [0,\infty))$ to the Cauchy problem

$$u_{tt} - \triangle u + u^5 = 0, \tag{1.8}$$

$$u(0,\cdot) = u_0, u_t(0,\cdot) = u_1. \tag{1.9}$$

Related partial regularity results independently have been obtained by Kapitanskii [34] in 1989. Uniqueness holds among C^2 -solutions. The proof proceeds via a priori estimates. The classical representation formula crucially enters. It seems unlikely that regularity or uniqueness of weak solutions to (1.8), (1.9) can be established in a similar way. Research on the critical case in higher dimensions is in progress; however, to this moment the results on this subject still seem incomplete. Advances in these questions may require eliminating the use of the wave kernel. The super-critical case. We observe that for sufficiently small initial data the existence of global regular solutions, for instance, to the equation $u_{tt} - \triangle u + u^5 + u|u|^{p-2} = 0 \;\; in \;\; [0,\infty) \times \mathbb{R}^3$, for any $p > 2$ can be deduced as a corollary to Rauch's result. Various qualitative properties of solutions in the super-critical case have been studied [8], [33]. Other open problems concern scattering theory, involving, in particular, decay estimates for solutions of (1.1) (see Ginibre-Velo [21]), or existence and regularity results for initial-boundary value problems.

2 Fundamental Lemma

In this study, in another topic, we want to seek the blow-up set of positive solutions for the 2−dimensional semilinear wave equation

$$u_{tt} - \triangle u = u^p \quad in \quad [0,T) \times \mathbb{R}^2 \tag{2.1}$$

with initial values

$$u(0,\cdot) = u_0 \in H^2\left(\mathbb{R}^2\right) \cap H_0^1\left(\mathbb{R}^2\right), \tag{2.2}$$

$$u_t(0,\cdot) = u_1 \in H_0^1\left(\mathbb{R}^2\right), \tag{2.3}$$

where$p > 1$ and $u^p := |u|^{p-1} u$, that is, the superlinear case. We will use the following notations:

$$a_u(t) := \int_{\mathbb{R}^2} u^2(t,x,y)\, dxdy,$$

$$E_{p,u}(t) := \int_{\mathbb{R}^2} \left(u_t^2 + u_x^2 + u_y^2 - \frac{2}{p+1} |u|^{p+1} \right)(t,x,y)\, dxdy.$$

For a Banach space X and $0 < T \le \infty$ we set

$$C^k(0,T,X) = \text{Space of } C^k - \text{ functions} : [0,T] \to X,$$

$$H\ 1 := C^1\left(0,T,H_0^1\left(\mathbb{R}^2\right)\right) \cap C^2\left(0,T,L^2\left(\mathbb{R}^2\right)\right),$$

$$H\ 2 := C^2\left(0,T;\ H_0^1\left(\mathbb{R}^2\right)\right) \cap C^1\left(0,T;\ H^2\left(\mathbb{R}^2\right) \cap H_0^1\left(\mathbb{R}^2\right)\right).$$

We have the following lemma (see for example [10] , [11])

Lemma 2.1: Suppose that $u \in H\ 2$ is the solution of equation (2.1) satisfying (2.2) and (2.3), then

$$E_{p,u}(t) = E_p(u(0)) = \int_{\mathbb{R}^2} \left(\frac{|u_1|^2 + |u_0|^2}{2} - \frac{|u_0|^{p+1}}{p+1} \right)(x,y)\, dxdy$$

is a constant and $a_u(t)$ is differentiable in t.

Lemma 2.2: *Suppose that $u \in H\ 1$ is a positive weakly solution of (2.1) with $a_u(0) > 0$ and $E_p(u(0)) < 0$. Then*

(i) for $a_u'(0) \ge 0$, we have $a_u'(t) > 0\ \ \forall t > 0$.

(ii) for $a_u'(0) < 0$, there exists a constant $t_5 > 0$ with $a_u'(t) > 0\ \ \forall t > t_5$, $a_u'(t_5) = 0$ and

$$t_5 \le t_6 := \frac{-a_u'(0)}{(p-1)(\delta^2 - E)},$$

where δ is the positive root of the equation

$$\frac{2}{p+1}\lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E_p(u(0)) = 0.$$

Lemma 2.3: Suppose that $u \in H\ 2$ is the solution of equation (2.1) satisfying (2.2),(2.3) with $E_p(u(0)) < 0$, then u blows up in finite time.

3 Main Result

According to the above results concerning blow-up solution, we want to seek the (set of) blow-up point(s) and the blow-up rate and blow-up constant of the solution for the semilinear wave equation $\Box u = u^p$ with smooth initial values, for instance, u_0, u_1 are both in $C_0^\infty(\mathbb{R}^2)$ but the blow-up set have many kinds of definition we mention some of it as follows. We consider the sets

$$S := \left\{(t_0, x_0) \in \mathbb{R}^3 \,\middle|\, u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (t_0, x_0)\right\},$$

$$S_{T^*} := \left\{x_0 \in \mathbb{R}^2 \,\middle|\, u(t,x)^{-2} \to 0, \quad \text{for} \quad (t,x) \to (T^*, x_0)\right\},$$

$$S_{T^*,\ L^q} := \left\{x_0 \in \mathbb{R}^2 \,\middle|\, \begin{array}{c} \lim_{t\to T^*}\left(\int_{B_r(x_0)} |u|^q(t,x)\,dx\right)^{-1} = 0 \\ \lim_{t\to T^*}\left(\int_{\mathbb{R}^2 - B_r(x_0)} |u|^q(t,x)\,dx\right)^{-1} > 0 \end{array} \text{for each } r > 0\right\},$$

where $B_r(x_0) = \left\{x \in \mathbb{R}^2 : |x - x_0| \le r\right\}$. We call $S,\ S_{T^*}$ and $S_{T^*,\ L^q}$ the blow-up set, blow-up set at time T^* and the blow-up set in the sense of L^q of u.

For a general Banach space X generated by functions defined in $\mathbb{R}^n$, $n \in \mathbb{N}$ and $u(t,\cdot)$ defined in $[0, T^*)$ and ranged in X , we denote the blow-up set of u at time T^* by

$$B_{T^*}, n(u, X) := \left\{x \in \mathbb{R}^n : \begin{array}{c} \|u(t,\cdot)\|_{B_{r,n}(x)}^{-1} \to 0 \quad as \quad t \to T^* \\ \lim_{t\to T^*} \|u(t,\cdot)\|_{\mathbb{R}^n - B_{r,n}(x)}^{-1} > 0 \end{array} \text{for each } r > 0\right\}$$

where $B_{r,n}(x_0) = \left\{x \in \mathbb{R}^n : \|x - x_0\|_{\mathbb{R}^n} \le r\right\}$, $\|u(t,\cdot)\|_{B_{r,n}(x_0)}$:=the norm of $u(t,\cdot)$ restricted in $B_{r,n}(x_0)$ and $\|u(t,\cdot)\|_{\mathbb{R}^n - B_{r,n}(x_0)}$:=the norm of $u(t,\cdot)$ restricted in $\mathbb{R}^n - B_{r,n}(x_0)$.

We have the main results on blow-up set

Theorem 1: *Suppose that* $u \in H\ 2$ *is the solution of equation* (2.1) *satisfying* (2.2),(2.3) *with* $E_p(u(0)) < 0$, *then there exist* u_0 *and* u_1 *so that* $\|S\| = 0 = \|S_{T^*}\|$, *where* $\|M\|$ *is the Lebsgue measure of* M *in* $\mathbb{R}^n, n = 2$ *or* 3.

Theorem 2 : *Suppose that* $u \in H\ 2$ *is the solution of equation* (2.1) *satisfying* (2.2),(2.3) *with* $E_p(u(0)) < 0$, *then there exist* u_0 *and* u_1 *so that* $\|S\| > 0$.

The proofs to the above results are based on the solution structure and we do not want to give the tedious detailed argumentations. These two results are although fundamental but give some information on blow-up set, particular singularity of the solution to the semilinear wave equation.

4 Remarks

The following problems are far from our study but they are very important although it seems very fundamental:

(1) What are the sets S, S_{T^*} and $S_{T^*,\ L^q}$?

(2) There is a real number q so that $\|S_{T^*,\ L^q}\| > 0$? If so, how large are these sets?

(3) What are the blow-up rate of u in the neighborhoods of S, S_{T^*} and $S_{T^*,\ L^q}$?

(4) What are the blow-up constants of u in the neighborhoods of S, S_{T^*} and $S_{T^*,\ L^q}$?

To study the above hard problems we concentrate on the properties later, first, we should know the measure of sets S, S_{T^*} and $S_{T^*,\ L^q}$ for some $q \geq 1$.

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