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ON THE POSITIVE SOLUTIONS OF THE DIFFERENTIAL EQUATION $u'' - u^p = 0$

BY

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Abstract. In this paper we work with the ordinary equation $u'' - u^p = 0$ and obtain some interesting phenomena concerning blow-up, blow-up rate, life-span, zeros, critical points and the asymptotic behavior at infinity of solutions to this equation.

Introduction. In our papers [1, 2, 3] we studied the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, and we found some interesting results on blow-up, blow-up rate and the estimates for the life-span of solutions, but no information on the singular set. Here we want to deal with the particular cases in lower dimensional wave equations. We hope that the experiences gained here will allow us to deal with more general lower dimension later.

Consider stationary, one-dimensional semilinear wave equation

$$\begin{cases} u'' - u^p = 0,\ p \in (0,1)\,, \\ u(0) = 0 = u'(0)\,. \end{cases}$$

After some computations one can find that the equation has infinite many

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solutions given by

$$u_c(t) = \begin{cases} 0, & t \in [0, c], \\ c_p (t-c)^{\frac{1}{1-p}}, & t > c, \end{cases}$$

where $c_p = (1-p)^{2/(1-p)} (2p+2)^{1/(1-p)}$. Thus, in particular, the solutions of the above equation in general are not unique. It is clear that these functions u^p, $p \geq 1$, $u \geq 0$ are locally Lipschitz, and by the standard theory, the local existence and uniqueness of classical solutions is applicable to the equation

$$\begin{cases} u'' - u^p = 0, \ p \in (1, \infty), \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{1}$$

Our study is motivated by the research on Chinese calligraphy. Neglecting the friction force of the paper on which a calligrapher creates his work through a handwritings brush (in Chinese, maue bie) with mass $m(t)$ at time t, the displacement $u(t)$ of the brush on reispaper (rice paper) at time t is governed by the Newtons' second law of motion with the force $F(t)$

$$(m(t) u')'(t) = F(t). \tag{0.1.1}$$

Normally, the force $F(t)$ depends on the displacement $u(t)$ [4][1], that is $F(t) = F(u(t))$. Experimentally, the change rate of the force is proportional to the change rate of displacement [4], that is, there is a real p so that

$$\frac{\frac{dF(t)}{dt}}{F(t)} = p \frac{\frac{du(t)}{dt}}{u(t)}.$$

By some calculation we find the form of the force $F(u(t)) = cu(t)^p$ for some constant c.

[1]In the Han-Dynasty the famous calligrapher Tsai-Iung had already this opinion.

Note that in the normal cases, and particularly for the beginner, the mass of their handwriting brushes vary with time due to the strength of hand and the intake of ink. For simplicity, we may assume that the mass depends upon the time periodically, in piecewise time interval; in other words,

$$m(t) = \begin{cases} m_1 - k_1 t, & 0 \leq t \leq t_1, \\ m_2 - k_2 t, & t_1 < t \leq t_2, \\ \vdots & \\ m_n - k_n t, & t_{n-1} < t \leq t_n, \end{cases}$$

where k_i and m_i are positive constants depend on the authors writingsusages.

To some calligraphers the mass of their brushes play no roll, and thus the mass of that brushes are all the same, in another words, $m(t) = m$ for some constant m, therefore, the equation (0.1.1) becomes

$$u''(t) = \frac{c}{m} u(t)^p . \tag{0.1.2}$$

If we set $v(t) = (m/c)^{1/(p-1)} u(t)$, then the equation (0.1.2) becomes $v''(t) = v(t)^p$, in the form of (0.1). Thus, the model of problem (0.1) describes a calligrapher with force u^p creating his works in real action. The initial values u_0 and u_1 are non-negative. For $p > 1$, the null solution $u(t) \equiv 0$, $u_0 = 0 = u_1$, corresponds to routine, uninspired works. When one is in an outburst of enthusiasm for the writing, then in a short time there were some burned-curled-like curve would be created; in other words, for $E_u(0) < 0$ or $E(0) > 0$ and $u_1 > 0$, there exists a finite number T^* such that $u(t)^{-1} \to 0$ as $t \to T^*$, c.f. Theorem 3 and 4.

From the observations, when the characteristic p of the calligrapher is smaller than 1, then their works could be good controlled or in some sense “nachmacht” (duplicated); mathematically, $u(t) \leq k(t \pm c)^{\theta}$, $\theta > 0$.

These above-mentioned phenomena will be analyzed in the present paper mathematically bases on the model of the form (0.1).

We discuss the problem (0.1) in two parts, $p > 1$ and $p < 1$.

Part A. $p > 1$.

Notation and Fundamental Lemmas. For a given solution $u(t)$ of (0.1) we set

$$E_u(0) = u_1^2 - \frac{2}{p+1} u_0^{p+1}, \quad J_u(t) = u(t)^{-\frac{p-1}{2}}.$$

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ is said to have a blow-up rate q if g exists only in finite time, that is, there is a finite number T^* such that the following is valid

$$\lim_{t \to T^*} g(t)^{-1} = 0 \tag{0.2}$$

and that there exists a non-zero $\beta \in \mathbb{R}$ with

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta, \tag{0.3}$$

in this case β is called the blow-up constant of g.

Since the solutions for the equation (0.1) is unique, we can rewrite $J_u(t) = J(t)$ and $E_u(t) = E(t)$. From some elementary calculations we obtain the following Lemma 1.

Lemma 1. *Suppose that u is the solution of* (0.1), *then we have*

$$E(t) = u'(t)^2 - \frac{2}{p+1} u(t)^{p+1} = E(0), \tag{0.4}$$

$$(p+3)\, u'(t)^2 = (p+1)\, E(0) + \left(u^2(t)\right)'', \tag{0.5}$$

$$J''(t) = \frac{p^2-1}{4} E(0)\, J(t)^{\frac{p+3}{p-1}} \tag{0.6}$$

and

$$J'(t)^2 = J'(0)^2 - \frac{(p-1)^2}{4} E(0)\, J(0)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{4} E(0)\, J(t)^{\frac{2(p+1)}{p-1}}. \tag{0.7}$$

The following Lemma is easy to prove so we omit the arguments.

Lemma 2. *If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T} \int_0^{g(t)} h(t,r)\, dr$ exists, then*

$$\lim_{t\to T} \int_0^{g(t)} h(t,r)\, dr = \int_0^{g(T)} h(T,r)\, dr.$$

I. Estimates for the life-spans. To estimate the life-span of the solution of the equation (0.1), we separate this section into three parts, $E(0) < 0, E(0) = 0$ and $E(0) > 0$. Here the life-span T of u means that u is the solution of problem (0.1) and the existence interval of u is $[0,T)$ so that the problem (0.1) has the solution $u \in \bar{C}^2(0,T)$ and u make sense only in this interval $[0,T)$.

I.1. $\boldsymbol{E(0) \le 0}$. In this subsection we deal with the case that $E(0) < 0$ and $E(0) = 0$, $u_0u_1 > 0$. The case that $E(0) = 0$ and $u_0u_1 \le 0$ will be considered in section 3 and section 4. We have the following result.

Theorem 3. *If T is the life-span of u and u is the positive solution of the problem (0.1) with $E(0) < 0$, then T is finite. Further, for $u_0u_1 \ge 0$ we have*

$$T \le T_1^*(u_0,u_1,p) = \frac{2}{p-1} \int_0^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}; \tag{1.1.1}$$

for $u_0u_1 < 0$,

$$T \leq T_2^*(u_0,u_1,p) = \frac{2}{p-1}\left(\int_0^k \frac{dr}{\sqrt{k_1+E(0)\,r^{k_2}}} + \int_{J(0)}^k \frac{dr}{\sqrt{k_1+E(0)\,r^{k_2}}}\right), \tag{1.1.2}$$

where $k_1 := \frac{2}{p+1}$, $k_2 := \frac{2p+2}{p-1}$ *and* $k := \left(\frac{2}{p+1}\frac{-1}{E(0)}\right)^{\frac{p-1}{2p+2}}$.
Furthermore, if $E(0)=0$ *and* $u_0u_1 > 0$, *then*

$$T \leq T_3^* := \frac{2}{p-1}\frac{u_0}{u_1}. \tag{1.1.3}$$

Proof. Under the condition, $E(0) < 0$, we know immediately that $u_0^2 > 0$; otherwise we get $u_0^2 = 0$, that is, $u_0 = 0$, then $E(0) = u_1^2 \geq 0$; and this contradicts to $E(0) < 0$. In this situation we divide the proof of the Theorem into two cases, $u_0u_1 \geq 0$ and $u_0u_1 < 0$.

(i) $u_0u_1 \geq 0$. By identity (0.5) we find that

$$\begin{cases} 2uu'(t) \geq 2u_0u_1 - (p+1)E(0)\,t, & \forall t \geq 0, \\ u^2(t) \geq u_0^2 + 2u_0u_1t - \frac{p+1}{2}E(0)\,t^2, & \forall t \geq 0. \end{cases} \tag{1.1.4}$$

From identity (0.7), $u_0u_1 \geq 0$ and the fact $J'(t) = -\frac{p-1}{2}u(t)^{-\frac{p-1}{2}}u'(t) < 0$, it follows that

$$J'(t) = -\frac{p-1}{2}\sqrt{k_1+E(0)\,J(t)^{k_2}} \leq J'(0), \quad \forall t \geq 0, \tag{1.1.5}$$

where $k_1 = u_0^{-p-1}u_1^2 - E(0)\,u_0^{2-\frac{p+1}{2}} = \frac{2}{p+1}$ and

$$J(t) \leq u_0^{-\frac{p-1}{2}} - \frac{p-1}{2}u_0^{-\frac{p+1}{2}}u_1t, \qquad \forall t \geq 0.$$

Thus, there exists a finite number $T_1^*(u_0, u_1, p) \le \frac{2}{p-1}\frac{u_0}{u_1}$ such that $J(T_1^*(u_0, u_1, p)) = 0$ and $u(t) \to \infty$ for $t \to T_1^*(u_0, u_1, p)$. This means that the life-span T of u is finite, that is, $T \le T_1^*(u_0, u_1, p)$. Now we estimate this life-span $T_1^*(u_0, u_1, p)$.

By identity (1.1.5) and the fact that $J(T_1^*(u_0, u_1, p)) = 0$ we find that

$$\int_{J(t)}^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}} = \frac{p-1}{2} t, \quad \forall t \ge 0, \tag{1.1.6}$$

and hence we get the estimate (1.1.1).

(ii) $u_0 u_1 < 0$. For brevity, we only prove existence of critical point $t_0(u_0, u_1, p)$ of u, that is, $u'(t_0(u_0, u_1, p)) = 0$ and compute it later in section III. By inequality (1.1.4), $u_0 u_1 < 0$ and the convexity of u^2 we can find a unique finite number $t_0(u_0, u_1, p)$ such that

$$\begin{cases} u(t)\, u'(t) < 0 & \text{for } t \in (0, t_0(u_0, u_1, p)), \\ uu'(t_0(u_0, u_1, p)) = 0, & \\ uu'(t) > 0 & \text{for } t > t_0(u_0, u_1, p), \end{cases} \tag{1.1.7}$$

and $u(t_0(u_0, u_1, p))^2 > 0$. If not, then $u(t_0(u_0, u_1, p)) = 0$, thus

$$E(0) = E(t_0(u_0, u_1, p)) = u'(t_0(u_0, u_1, p))^2 \ge 0;$$

yet this is in contradiction with $E(0) < 0$.

Thus we conclude that

$$u^2(t) > 0, \quad \forall t \ge 0.$$

Hence we get $u'(t_0(u_0, u_1, p)) = 0$,

$$E(0) = -\frac{2}{p+1} u(t_0(u_0, u_1, p))^{p+1}$$

and

$$J\left(t_0\left(u_0,u_1,p\right)\right)^{k_2}=\frac{2}{p+1}\frac{-1}{E\left(0\right)}.$$

After arguments similar to the step (i), there exists a $T_2^*\left(u_0,u_1,p\right)$ such that the life-span T of u is bounded by $T_2^*\left(u_0,u_1,p\right)$, that is, $T\leq T_2^*\left(u_0,u_1,p\right)$. On the analogy of the above argumentation, using (1.1.7) and (0.7) we get

$$(1.1.8)\quad \begin{cases} J'\left(t\right)=-\dfrac{p-1}{2}\sqrt{k_1+E\left(0\right)J\left(t\right)^{k_2}}, & \forall t\geq t_0\left(u_0,u_1,p\right), \\ J'\left(t\right)=\dfrac{p-1}{2}\sqrt{k_1+E\left(0\right)J\left(t\right)^{k_2}}, & \forall t\in\left[0,t_0\left(u_0,u_1,p\right)\right]. \end{cases}$$

Therefore we have

$$(1.1.9)\quad \begin{cases} \displaystyle\int_{J(t)}^{J(t_0)}\frac{dr}{\sqrt{k_1+E\left(0\right)r^{k_2}}}=\frac{p-1}{2}\left(t-t_0\right), & \forall t\geq t_0, \\ \displaystyle\int_{J(0)}^{J(t_0)}\frac{dr}{\sqrt{k_1+E\left(0\right)r^{k_2}}}=\frac{p-1}{2}t_0. & \end{cases}$$

where $t_0=t_0\left(u_0,u_1,p\right)$. Utilizing (1.1.9) and the fact that $J\left(t_0\left(u_0,u_1,p\right)\right)^{k_2}=\frac{2}{p+1}\frac{-1}{E(0)}$ and $J\left(T_2^*\left(u_0,u_1,p\right)\right)=0$ we obtain the estimate

$$(1.1.10)\quad T_2^*\left(u_0,u_1,p\right)=t_0\left(u_0,u_1,p\right)+\frac{2}{p-1}\int_0^k\frac{dr}{\sqrt{k_1+E\left(0\right)r^{k_2}}}.$$

This estimate (1.1.10) is equivalent to (1.1.2).

(iii) $E\left(0\right)=0$. Now we prove $(1.1.3)$. By identity (0.6) in Lemma 1 and $E\left(0\right)=0$ we get $J''\left(t\right)=0\quad\forall t\geq 0$. From the positiveness of u_0u_1, it follows that $J'\left(0\right)<0$ and

$$J\left(t\right)=u_0^{-\frac{p-1}{2}}-\frac{p-1}{2}u_0^{-\frac{p+1}{2}}u_1t,\quad \forall t\geq 0.$$

Thus we conclude that

$$u(t) = u_0 \left(1 - \frac{p-1}{2}\frac{u_1}{u_0}t\right)^{-\frac{2}{p-1}}, \quad \forall t \geq 0. \tag{1.1.11}$$

Therefore the estimate (1.1.3) follows.

I.2. $E(0) > 0,\ u_0 \geq 0$.

In this subsection we consider two cases $E(0) > 0$, $u_0 > 0$ and $E(0) > 0$, $u_0 = 0$, $u_1 > 0$.

We have the following blow-up result.

Theorem 4. *Suppose that*

(i) $u_0 > 0$ *or*

(ii) $u_0 = 0$ *and* $u_1 > 0$.

Then the life-span T of the positive solution u of the problem (0.1) *with $E(0) > 0$ is finite, that is, u is only a local solution of* (0.1).

Further, in case of (i) *we have the estimates*

$$T \leq T_4^*(u_0, u_1, p) = \frac{2}{p-1}\int_0^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}, \quad u_1 \geq 0; \tag{1.2.1}$$

in the case of (ii)

$$T \leq T_5^*(u_0, u_1, p) = \frac{2}{p-1}\int_0^{\infty} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}. \tag{1.2.2}$$

Proof. i) $u_0 > 0$. By identity (0.6) in Lemma 1 we obtain

$$\begin{cases} k_3 J''(t) = (k_3 J(t))^q, \\ k_3 J(0) = k_3 u_0^{-\frac{p-1}{2}}, \\ k_3 J'(0) = \dfrac{1-p}{2} k_3 u_0^{-\frac{p+1}{2}} u_1, \end{cases} \tag{1.2.3}$$

where $k_3 := \left(\frac{p^2-1}{4} E(0)\right)^{\frac{p-1}{4}}$ and $q := \frac{p+3}{p-1}$.

Now we set

$$\tilde{E}(t) := k_3^2 J'(t)^2 - \frac{2}{q+1}(k_3 J(t))^{q+1},$$

after some calculations we see that $\tilde{E}(t)$ is a constant and

$$\tilde{E}(t) = \tilde{E}(0) = \frac{(p-1)^2}{4} k_3^2 u_0^{-p-1} \left(u_1^2 - E(0)\right) \tag{1.2.4}$$

From the condition that $u_0 > 0$ and the definition of $E(0)$ it follows that

$$0 < \tilde{E}(t) = \frac{(p-1)^2}{2(p+1)} k_3^2 u^2(t)^{-\frac{p+3}{2}} u(t)^{p+3} = \frac{(p-1)^2}{2(p+1)} k_3^2,$$

thus

$$u(t)^{p+1} > 0, \quad \forall t \geq 0. \tag{1.2.5}$$

By identity (0.5) in Lemma 1 we find that

$$u(t)\, u'(t) = u_0 u_1 + E(0)\, t + \frac{p+3}{p+1} \int_0^t u(r)^{p+1}\, dr, \quad \forall t \geq 0 \tag{1.2.6}$$

and so

$$u(t)\, u'(t) \geq u_0 u_1 + E(0)\, t, \quad \forall t \geq 0. \tag{1.2.7}$$

Thus, for the case $u_0 u_1 \geq 0$, using the same arguments as in the proof of Theorem 4 we get the conclusions (1.2.1) in Theorem 5.

Now let us show $u_0u_1 \geq 0$. For $u_0u_1 < 0$, from (1.2.7) it follows that $u(t)\,u'(t) \geq 0$ for large t. Suppose that $\bar{t}_0$ is the first number such that $u(t)\,u'(t) = 0$. Using identity (0.5) in Lemma 1 we get

$$(1.2.6.1)\quad u(t)\,u'(t) = E(0)(t - \bar{t}_0) + \frac{p+3}{p+1}\int_{\bar{t}_0}^{t} u(r)^{p+1}\,dr \geq 0, \quad \forall t \geq \bar{t}_0.$$

Hence we find that

$$(1.2.8)\qquad \begin{cases} uu'(t) < 0 \quad for \quad t \in (0, \bar{t}_0), \\ uu'(\bar{t}_0) = 0, \\ uu'(t) > 0 \quad for \quad t > \bar{t}_0, \end{cases}$$

and $u(\bar{t}_0) > 0$; if not, then $u(\bar{t}_0) = 0$, this is in contradiction with $(1.2.5)$. Hence we get

$$(1.2.9)\qquad u'(\bar{t}_0) = 0.$$

Therefore, by (1.2.5) we obtain that

$$(1.2.10)\qquad (p+1)\,E(0) = -2u(\bar{t}_0)^{p+1} < 0.$$

The identity (1.2.10) and the condition $E(0) > 0$ are in contradiction; therefore we get the assertion that $u_1 \geq 0$.

ii) By $u_0 = 0$ and (1.2.6) we find

$$(1.2.11)\qquad u(t)\,u'(t) = E(0)\,t + \frac{p+3}{p+1}\int_{0}^{t} u(r)^{p+1}\,dr, \quad \forall t \geq 0.$$

We claim that $uu'(t) > 0$ for every $t > 0$. If not, then according to the positiveness of u_1 there exists $\tilde{t} > 0$ such that $u(\tilde{t})\,u'(\tilde{t}) = 0$. Let $\tilde{T}$ be the first non-zero so that $u\left(\tilde{T}\right)u'\left(\tilde{T}\right) = 0$, then $u(t) > 0$ in $\left(0, \tilde{T}\right)$. By (1.2.6) again we get

$$0 = uu'\left(\tilde{T}\right) = E\left(0\right)\tilde{T} + \frac{p+3}{p+1}\int_0^{\tilde{T}} u\left(r\right)^{p+1} dr.$$

This is therefore in contradiction with $E(0) > 0$; hence $u(t)\,u'(t) > 0\ \forall t > 0$ and $J'(t) < 0\ \forall t > 0$. Using (0.6) in Lemma 1 for each $\check{t} > 0$ we conclude that

$$J'\left(t\right) = -\sqrt{J'\left(\check{t}\right)^2 - \frac{\left(p-1\right)^2}{4}E\left(0\right)\left(J\left(\check{t}\right)^{\frac{2p+2}{p-1}} - J\left(t\right)^{\frac{2p+2}{p-1}}\right)}, \qquad \forall t \geq \check{t} \tag{1.2.12}$$

and simultaneously

$$\lim_{\check{t}\to 0} J'\left(\check{t}\right)^2 - \frac{\left(p-1\right)^2}{4}u_1^2 J\left(\check{t}\right)^{\frac{2p+2}{p-1}} = \frac{\left(p-1\right)^2}{2\left(p+1\right)},$$

thus by (1.2.12), the estimate (1.2.2) follows.

I.3. Some properties concerning $T_1^*(u_0, u_1, p)$. In principle, $T_1^*(u_0, u_1, p)$ depends on three variables u_0, u_1 and p. Set $c_{k,p} := \frac{(p+1)u_1^2}{2u_0^{p+1}}$, then

$$T_1^*\left(u_0, u_1, p\right) = \frac{\sqrt{2p+2}}{p-1}u_0^{-\frac{p-1}{2}}\left(1 - c_{k,p}\right)^{-\frac{p-1}{2p+2}}\int_0^{\left(1-c_{k,p}\right)^{\frac{p-1}{2p+2}}} \frac{dr}{\sqrt{1 - r^{\frac{2p+2}{p-1}}}}.$$

It is evident that

$$\lim_{p\to\infty} T_1^*\left(u_0, u_1, p\right) = 0,\ \lim_{p\to\infty} T_1^*\left(u_0, u_1, p\right) = \infty.$$

For convenience, we consider the case $u_1 = 0$,

$$T_1^*\left(u_0, 0, p\right) = \frac{\sqrt{\pi}}{\sqrt{2p+2}}u_0^{-\frac{p-1}{2}}\frac{\Gamma\left(\frac{p-1}{2p+2}\right)}{\Gamma\left(\frac{p}{p+1}\right)}.$$

Using Maple we get the graphs of $T_1^*(u_0, 0, p)$ below:

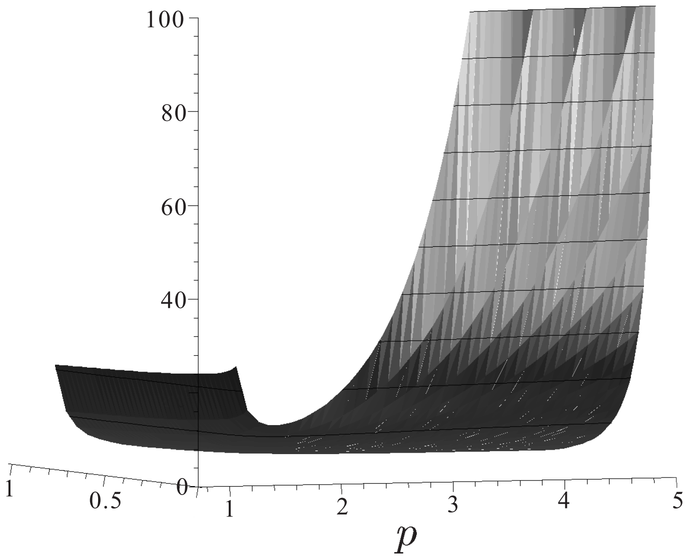

Figure 1. Graph of $T_1^*(u_0, 0, p)$, $u_0 \in (0, 1)$, $p \in [1, 5]$.

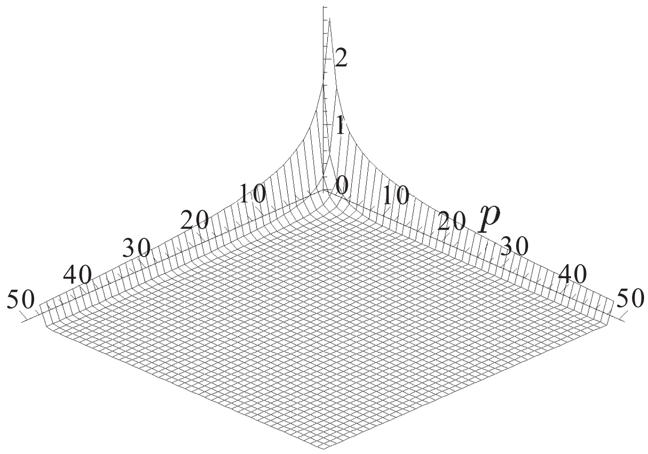

Figure 2. Graph of $T_1^*(u_0, 0, p)$, $u_0 \in [1, 50]$, $p \in [1, 50]$.

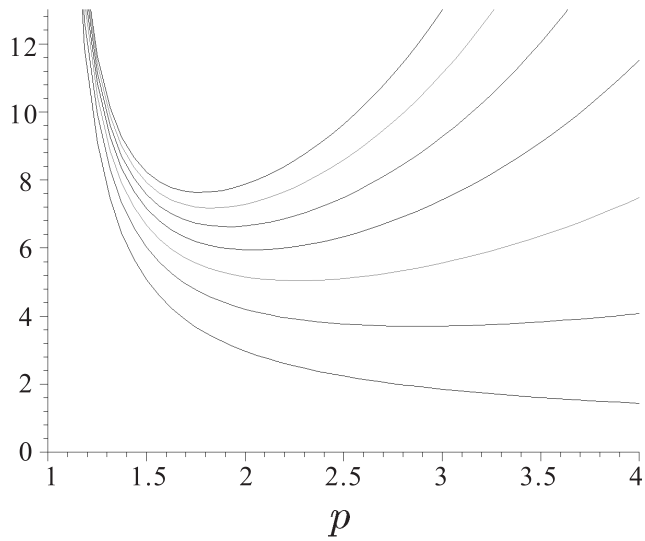

Figure 3. Graphs of $T_1^*(u_0, 0, p)$, $u_0 \leq 1$.

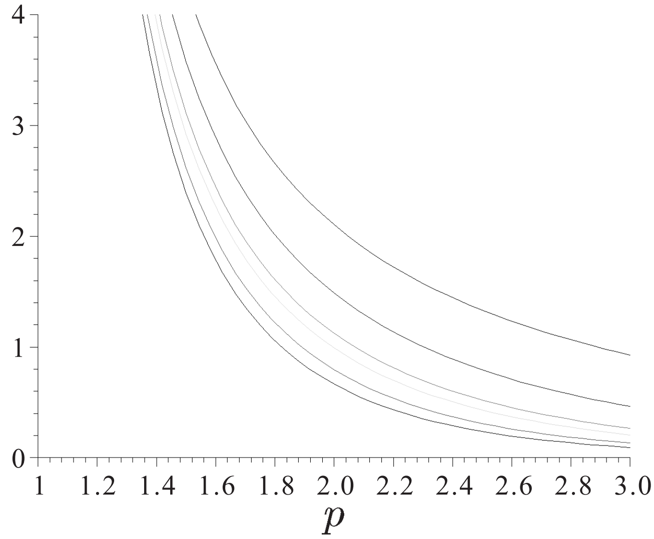

Figure 4. Graph of $T_1^*(u_0, 0, p)$, $u_0 > 1$.

The above pictures show the properties of $T_1^*(u_0, 0, p)$:

(1) there exists a constant u_0^* such that $T_1^*(u_0, 0, p)$ is monotone decreasing

in p for $u_0 \in [u_0^*, 1)$;

(2) there is a p_0 such that $T_1^*(u_0, 0, p)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(3) $T_1^*(u_0, 0, p)$ is differentiable in its variables and

(4) for $u_0 > 1$ the life-span $T_1^*(u_0, 0, p)$ is decreasing in p.

We now show the validity of statements (3) and (4) using the monotonicity of $T_1^*(1, 0, p)$ for $u_0 \neq 0$. To prove (1) and (2) we must establish the existence of u_0^* with $\frac{\partial}{\partial p} T_1^*(u_0, 0, p) \leq 0$ for $1 > u_0 \geq u_0^*$, that is,

$$\begin{aligned} 0 \quad \leq \quad & \frac{p-1}{p+1}(p+3) \int_0^1 \left(1 - r^{2\frac{p+1}{p-1}}\right)^{-1/2} dr \\ & +4 \int_0^1 \left(1 - r^{2\frac{p+1}{p-1}}\right)^{-3/2} r^{2\frac{p+1}{p-1}} \ln r \, dr \\ & + (p-1)^2 (\ln u_0) \int_0^1 \left(1 - r^{2\frac{p+1}{p-1}}\right)^{-1/2} dr, \end{aligned}$$

thus the existence of u_0^* can be obtained provided

$$\frac{p-1}{p+1}(p+3)\left(r^{2\frac{p+1}{p-1}} - 1\right) - 4 \ln r \; > 0, \quad \forall r > 1.$$

After some calculations it is easy to get the above assertion.

To grasp the property of the life-span $T_1^*(u_0, u_1, p)$ is very difficult, but for fixed initial data we want to know how the life-span varies with p, so now we consider the life-span $T_1^*(0.6, 0.2, p)$ and list the following tables as below.

p	$T_1^*(0.6, 0.2, p)$
1.001	2001. 5
1.004	501. 42
1.008	251. 42
1.012	168. 08

p	$T_1^*(0.6, 0.2, p)$
2	3. 4135
2.5	2. 7698
3	2. 4659
3.6497	2. 2644

After some computations we get

$$T_1^*(u_0,u_1,p)$$
$$= \frac{\sqrt{2p+2}}{p-1}\left(u_0^{p+1}-\frac{p+1}{2}u_1^2\right)^{-\frac{p-1}{2p+2}}\int_0^{\left(1-\frac{p+1}{2}u_0^{-p-1}u_1^2\right)^{\frac{p-1}{2p+2}}}\frac{dr}{\sqrt{1-r^{\frac{2p+2}{p-1}}}}.$$

By the experience in studying the life span $T_1^*(u_0,0,p)$, we consider the properties of the life-span $T_1^*(u_0,u_1,p)$ with $u_0u_1\ge 0$ in three cases:

Case 1: $0<u_0^{p+1}-(p+1)u_1^2/2<1$. In this situation we find that

(i) for fixed u_1,

(5) there exists a constant u_0^* depending on u_1 such that $T_1^*(u_0,u_1,p)$ is monotone decreasing in p for $u_0\ge u_0^*$,

(6) there is a p_0 so that $T_1^*(u_0,u_1,p)$ decreases in $(1,p_0)$ and increases in (p_0,∞) provided $u_0\in[0,u_0^*)$;

(ii) for fixed u_0, the life-span $T_1^*(u_0,u_1,p)$ decreases in u_1^2.

Case2: $u_0^{p+1}-(p+1)u_1^2/2>1$. The life-span $T_1^*(u_0,u_1,p)$ decreases in p.

Case 3: $u_0^{p+1}-(p+1)u_1^2/2=1$. On the surface

$$\left\{(u_0,u_1,p)\in\mathbb{R}^3\,\middle|\,u_0^{p+1}-(p+1)u_1^2/2=1,\ p>1\right\}$$

we find that

$$T_1^*(u_0,u_1,p)=T_1^*(u_0,p)=\frac{\sqrt{2p+2}}{p-1}\int_0^{u_0^{-(p-1)/2}}\frac{1}{\sqrt{1-r^{2(p+1)/(p-1)}}}dr$$

and $T_1^*(u_0,p)$ is monotone decreasing in u_0 and in p.

II. Blow-up rate and blow-up constant. In this section we study the blow-up rate and blow-up constant for $u^2,(u^2)'$ and $(u^2)''$ under the conditions in section 1. We have the following results.

Theorem 5: *If u is the positive solution of the problem* (0.1) *with one of the following properties that*

(i) $E(0) < 0$

or

(ii) $E(0) = 0, \; u_0 u_1 > 0$

or

(iii) $E(0) > 0, \; u_0 > 0$

or

(iv) $E(0) > 0, \; u_0 = 0, \; u_1 > 0$

Then the blow-up rate of u is $2/(p-1)$, and the blow-up constant of u is $\sqrt[p-1]{2(p-1)^{-2}(p+1)}$, that is, for $m \in \{1,2,3,4,5,6\}$

$$\lim_{t \to T_m^*(u_0,u_1,p)} (T_m^*(u_0,u_1,p) - t)^{\frac{2}{p-1}} u(t) = 2^{\frac{1}{p-1}} (p+1)^{\frac{1}{p-1}} (p-1)^{-\frac{2}{p-1}}. \tag{2.1.1}$$

The blow-up rate of u' is $(p+1)/(p-1)$, and the blow-up constant of $u'$is $2^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}(p-1)^{-\frac{p+1}{p-1}}$, that is, for $m \in \{1,2,3,4,5,6\}$.

$$\begin{aligned} &\lim_{t \to T_m^*(u_0,u_1,p)} u'(t) (T_m^*(u_0,u_1,p) - t)^{\frac{p+1}{p-1}} \\ = \;& 2^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} (p-1)^{-\frac{p+1}{p-1}}. \end{aligned} \tag{2.1.2}$$

The blow-up rate of u'' is $2p/(p-1)$, and the blow-up constant of $u''$is $2^{\frac{p}{p-1}}(p+1)^{\frac{p}{p-1}}(p-1)^{-\frac{2p}{p-1}}$, that is, for $m \in \{1,2,3,4,5,6\}$

$$\begin{aligned} &\lim_{t \to T_m^*(u_0,u_1,p)} u''(t) (T_m^*(u_0,u_1,p) - t)^{\frac{2p}{p-1}} \\ = \;& 2^{\frac{p}{p-1}} (p+1)^{\frac{p}{p-1}} (p-1)^{-\frac{2p}{p-1}}. \end{aligned} \tag{2.1.3}$$

Proof. i) Under this condition, $E(0) < 0, \; u_0 u_1 \geq 0$ by (1.1.1) and

(1.1.6) we get

$$\int_0^{J(t)} \frac{1}{T_1^*(u_0,u_1,p)-t} \frac{dr}{\sqrt{k_1+E(0)\,r^{k_2}}} = \frac{p-1}{2}, \quad \forall t \geq 0. \tag{2.1.4}$$

By Lemma 4 and (2.1.4) we obtain

$$\lim_{t\to T_1^*(u_0,u_1,p)} \frac{1}{\sqrt{k_1}} \frac{J(t)}{T_1^*(u_0,u_1,p)-t} = \frac{p-1}{2}. \tag{2.1.5}$$

This identity (2.1.5) is equivalent to (2.1.1) for $m=1$.

For $E(0)<0,\ u_0u_1<0$ using (1.1.9) we have also

$$\int_0^{J(t)} \frac{dr}{\sqrt{k_1+E(0)\,r^{k_2}}} = \frac{p-1}{2}\left(T_2^*(u_0,u_1,p)-t\right), \quad \forall t \geq t_0(u_0,u_1,p). \tag{2.1.6}$$

From the Lemma 4 and (2.1.6), the estimate (2.1.1) for $m=2$ follows.

Utilizing the identities (1.1.5) and (1.1.8) we find

$$\lim_{t\to T_m^*(u_0,u_1,p)} J'(t) = -\frac{p-1}{\sqrt{2p+2}}, \quad m=1,2. \tag{2.1.7}$$

Therefore, by (2.1.7) we have for $m=1,2$

$$\begin{aligned} &\lim_{t\to T_m^*(u_0,u_1,p)} \left(u^2\right)'(t)\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{p+3}{p-1}} \\ =\ & 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}}, \end{aligned} \tag{2.1.8}$$

and thus, for $m=1,2$

$$\begin{aligned} &\lim_{t\to T_m^*(u_0,u_1,p)} u'(t)^2\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}} \\ =\ & 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{2p+2}{p-1}}. \end{aligned} \tag{2.1.9}$$

Through (0.5) and (2.1.9) for $m = 1, 2$, we obtain the estimate

$$
\begin{aligned}
&\lim_{t\to T_m^*(u_0,u_1,p)} \left(u^2\right)''(t)\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}} \\
=\;& (p+3)\lim_{t\to T_m^*} u'(t)^2\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}}, \\
&\lim_{t\to T_m^*(u_0,u_1,p)} 2u(t)\,u''(t)\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}} \\
=\;& (p+1)\lim_{t\to T_m^*} u'(t)^2\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}} \\
=\;& 2^{\frac{2p}{p-1}}(p+1)^{\frac{p+1}{p-1}}(p-1)^{-\frac{2p+2}{p-1}}
\end{aligned}
\tag{2.1.10}
$$

and

$$\lim_{t\to T_m^*(u_0,u_1,p)} u''(t)\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p}{p-1}} = 2^{\frac{p}{p-1}}(p+1)^{\frac{p}{p-1}}(p-1)^{-\frac{2p}{p-1}}$$

Thus the estimate (2.1.3) for $m = 1, 2$ is proved.

ii) For $E(0) = 0,\ u_0u_1 > 0$, for $m = 3$, using identity (1.1.11) we get

$$u^2(t) = u_0^{2\frac{p+3}{p-1}}\left(\frac{p-1}{2}u_0u_1\right)^{-\frac{4}{p-1}}\left(T_m^*(u_0,u_1,p)-t\right)^{-\frac{4}{p-1}}, \quad \forall t \geq 0. \tag{2.1.11}$$

Therefore the estimates $(2.1.1), (2.1.2)$ and $(2.1.3)$ for $m = 3$ follow from $(2.1.11)$.

iii) The estimates $(2.1.1), (2.1.2)$ and $(2.1.3)$ for $m = 4, 5$ are similar to the above arguments (i) in the proof of this Theorem.

Now we consider the property of the blow-up constants K_1, K_2 and K_3. We have

$$
\begin{aligned}
K_1(p) &= 2^{\frac{1}{p-1}}(p+1)^{\frac{1}{p-1}}(p-1)^{-\frac{2}{p-1}}, \\
K_2(p) &= 2^{\frac{p}{p-1}}(p+1)^{\frac{1}{p-1}}(p-1)^{-\frac{p+1}{p-1}}, \\
K_3(p) &= 2^{\frac{p}{p-1}}(p+1)^{\frac{p}{p-1}}(p-1)^{-\frac{2p}{p-1}}.
\end{aligned}
$$

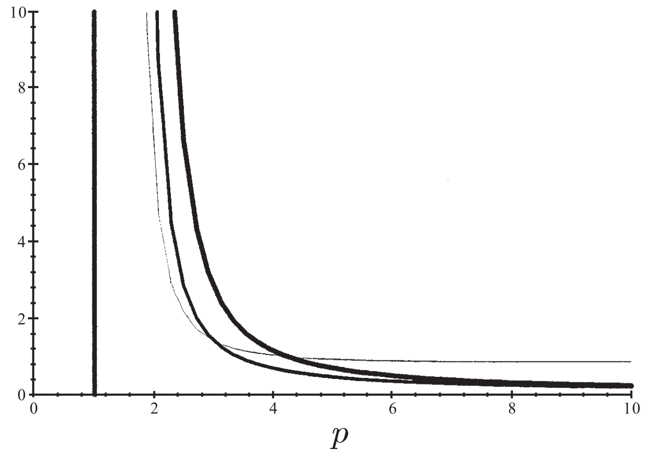

Figure 5. Graph of $K_1\,(p)$, $K_2\,(p)$, $K_3\,(p)$

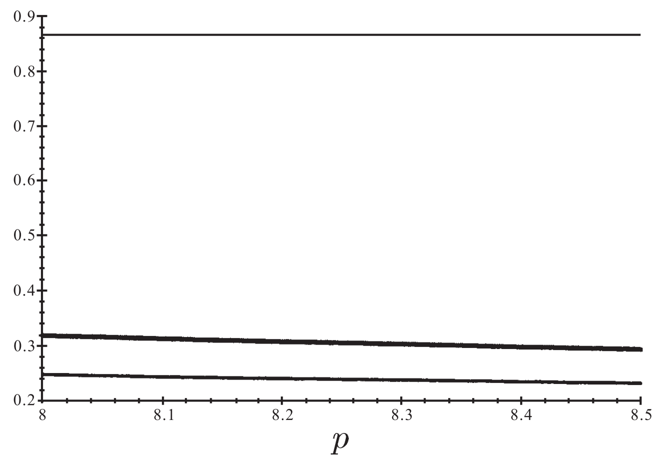

Figure 6. Graph of $K_1\,(p)$, $K_2\,(p)$, $K_3\,(p)$

We see that the graphs, $K_i\,(p)$, $i = 1, 2, 3$ are all decreasing in $p \in (1, p_1)$; and $K_i\,(p)$ tends to zero for $i = 2, 3$ and $K_1\,(p)$ tends to 1, as p tends to infinity. The monotonicity of these functions can be obtained after showing the following inequalities:

$$\frac{d}{dp}K_1(p) = (2p+2)^{\frac{1}{p-1}}(p-1)^{-\frac{2}{p-1}-2}\left(\ln\frac{(p-1)^2}{2p+2} - \frac{p+3}{p+1}\right) \leq 0, \quad p \in (1, p_1)$$

where $p_1 \sim 9.2203$,

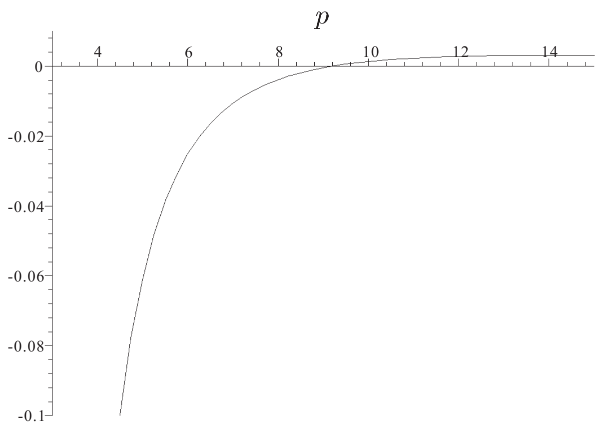

Figure 7. Graph of $\frac{d}{dp}K_1(p)$.

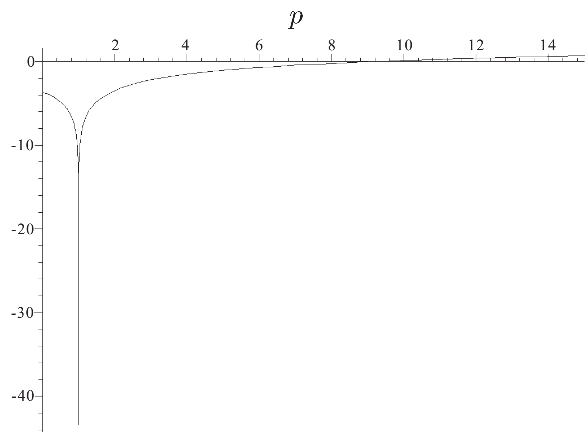

Figure 8. Graph of $\ln\frac{(p-1)^2}{2p+2} - \frac{p+3}{p+1}$.

$$p + \ln(2p+2) + \frac{2}{p+1} \geq 2\ln(p-1), \qquad \forall p > 1.$$

The above inequality is easy to prove, we omit the arguments.

III. Uniqueness on *p* and extension. In practical the characteristic index $p(t)$ depends on the characteristic (at time t) of the calligrapher himself only, in other words, when two "Werke" are similar to each other, then the correspondent characteristic $p(t)$ must very close, in mathematics, the fact can be easily solved to the scalar constant $p(t) = p$, we write it below.

Theorem 6. *Suppose that u and v are the positive solutions of the following equations respectively*

$$u''(t) = u(t)^p, v''(t) = v(t)^q \tag{3.1}$$

with $u(t) \neq 0 \neq v(t)$ for each $t \geq 0$. If they have the same rate of displacement, that is,

$$u'(t)/u(t) = v'(t)/v(t), \tag{3.2}$$

then they posses the same characteristic, this means, $p = q$.

Proof. According to the condition (3.2), we have

$$\frac{u(t)^{p+1} - u'(t)^2}{u(t)^2} = \frac{v(t)^{q+1} - v'(t)^2}{v(t)^2}.$$

Using (3.2) again, then

$$u(t)^{p-1} = v(t)^{q-1}.$$

This together with (3.2) we obtain the assertion.

For $E(0) = 0,\ u_0 u_1 < 0$, it is easy to see that

$$u(t) = u_0^{\frac{p+3}{p-1}} \left(u_0^2 - \frac{p-1}{2} u_0 u_1 t\right)^{-\frac{2}{p-1}},\ \forall t \in (0, T).$$

Hence we find the limit $\lim_{t\to\infty} u(t) = 0$ and

$$\lim_{t\to\infty} t^{\frac{2}{p-1}} u(t) = u_0^{\frac{p+3}{p-1}} \left(\frac{p-1}{-2} u_0 u_1\right)^{-\frac{2}{p-1}}.$$

The following Theorem is a direct application of Theorem 4, Theorem 6 and we omit the proof.

Theorem 7. *If $u \in PC^2(\mathbb{R}^+)$, that is, $u \in C^2(\bigcup_{i=0}^{\infty}(T_i, T_{i+1}) \cup (T_\infty, \infty))$ where $T_0 = 0, T_{i+1} \geq T_i$ and $T_\infty = \lim_{i\to\infty} T_i$, is a piecewise solution of the problem of (0.1) with $E(t) < 0$ for the continuous points of E. Then for $T_\infty = \infty$, the discontinuous points of u can be got at the blow-up points $\bar{T}_m^*(u_0, u_1, p), m \in \mathbb{N}$ of $u^2(t)$ and $\bar{T}_m^*(u_0, u_1, p)$ are given by*

$$\bar{T}_1^*(u_0,u_1,p)$$

$$(3.3) := \begin{cases} 2T_1^*(u_0,u_1,p) & \text{if } u_0u_1 \geq 0 \text{ and } uu'(T_1^{*+}(u_0,u_1,p)) \geq 0, \\ (T_1^* + T_2^*)(u_0,u_1,p) & \text{if } u_0u_1 < 0 \text{ and } uu'(T_1^{*+}(u_0,u_1,p)) \geq 0, \\ 2T_2^*(u_0,u_1,p) & \text{if } u_0u_1 < 0 \text{ and } uu'(T_1^{*+}(u_0,u_1,p)) \geq 0 \end{cases}$$

and

$$\bar{T}_{m+1}^*(u_0,u_1,p)$$

$$(3.4) \quad := \begin{cases} \left(\bar{T}_m^* + T_1^*\right)(u_0,u_1,p) & \text{if } uu'\left(\bar{T}_m^{*+}(u_0,u_1,p)\right) \geq 0, \\ \left(\bar{T}_m^* + T_2^*\right)(u_0,u_1,p) & \text{if } uu'\left(\bar{T}_m^{*+}(u_0,u_1,p)\right) < 0, \end{cases}$$

where $uu'\left(\bar{T}_m^{*+}(u_0,u_1,p)\right) := \lim_{t\to T_{m+7}^{*+}} \frac{u^2(t) - u\left(\bar{T}_m^*(u_0,u_1,p)\right)^2}{t - \bar{T}_m^*(u_0,u_1,p)}$.

Further we have the blow-up rate at $\bar{T}_m^*(u_0,u_1,p)$ of u^2 is $4/(p-1)$, and the blow-up constant of u^2 is $\sqrt[p-1]{4(p-1)^{-4}(p+1)^2}$, that is, for $m \in \mathbb{N}$

$$(3.5) \quad \lim_{t\to T_m^*} \left(\bar{T}_m^*(u_0,u_1,p) - t\right)^{\frac{4}{p-1}} u^2(t) = 2^{\frac{2}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{4}{p-1}}.$$

The blow-up rate of $(u^2)'$ at $\bar{T}_m^*(u_0,u_1,p)$ is $(p+3)/(p-1)$, and the blow-up constant of $(u^2)'$ is $2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}$, that is, for $m \in \mathbb{N}$

$$\begin{aligned} &\lim_{t\to T_m^*(u_0,u_1,p)} \left(\bar{T}_m^*(u_0,u_1,p)-t\right)^{\frac{p+3}{p-1}}\left(u^2\right)'(t) \\ &= 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}}. \end{aligned} \tag{3.6}$$

The blow-up rate of $\left(u^2\right)''$ *at* $\bar{T}_m^*(u_0,u_1,p)$ *is* $(2p+2)/(p-1)$*, and the blow-up constant of* $\left(u^2\right)''$ *is* $2^{\frac{2p}{p-1}}(p+1)^{\frac{8}{p-1}}(p-1)^{-\frac{2p+8}{p-1}}(p+3)$*, that is, for* $m \in \mathbb{N}$

$$\begin{aligned} &\lim_{t\to T_m^*(u_0,u_1,p)} \left(u^2\right)''(t)\left(T_m^*(u_0,u_1,p)-t\right)^{\frac{2p+2}{p-1}} \\ &= \left(\frac{2}{p-1}\right)^{\frac{2p}{p-1}}(p+3)\left(\frac{p+1}{p-1}\right)^{\frac{2}{p-1}}. \end{aligned} \tag{3.7}$$

Part B. Positive solution for $p < 1$. Before the study of the properties of solutions for the differential equation (0.1) we collect some results on the situation that $E_u(0) = 0$.

(1) For $u_0 > 0$ and $u_1 > 0$, we have

$$u(t) = \left(u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}}$$

and

$$t^{\frac{2}{p-1}}u(t) \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} \qquad \text{as} \quad t\to\infty.$$

(2) For $u_0 > 0$ and $u_1 < 0$, the solutions of (0.1) can be given as

$$u_c(t) = \begin{cases} \left(u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}} & t\in[0,T_0] \\ 0 & t\in[T_0,T_0+c] \\ \left(\frac{(1-p)^2}{2p+2}\right)^{\frac{1}{1-p}}(t-T_0-c)^{\frac{2}{1-p}} & t\geq T_0+c \end{cases}$$

where c is any positive real number and $T_0 = \sqrt{\frac{p+1}{2} u_0^{1-p}}$, and also

$$t^{\frac{2}{p-1}} u(t) \to \left(\frac{1-p}{2} \sqrt{\frac{2}{p+1}} \right)^{\frac{2}{1-p}} \qquad \text{as} \quad t \to \infty.$$

IV. $\boldsymbol{E_u(0) > 0}$. In this section we discuss the case $E_u(0) > 0$ and we have the following result concerning the zero point and asymptotic behavior at infinity of the solutions for the equation (0.1) :

Theorem 8. *Suppose that T^* is the life-span of u which is a positive solution of problem* (0.1) *with $E_u(0) > 0$ and $u_0 > 0$. Then for*

(1) $u_1 < 0$, *there exists a constant Z_0 so that $T^* \le Z_0$ and* $\lim_{t \to Z_0} u(t) = 0$, $\lim_{t \to Z_0} u'(t) = -\sqrt{E_u(0)}$ *and* $\lim_{t \to Z_0} u'''(t)^{-1} = 0$. *Moreover,*

$$Z_0 = \int_0^{u_0} \frac{dr}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}}, \tag{4.1}$$

$$\lim_{t \to Z_0^-} u'''(t)(t - Z_0)^{1-p} = p E_u(0)^{\frac{p}{2}}; \tag{4.2}$$

(2) $u_1 > 0$,

$$\lim_{t \to \infty} u(t) t^{-\frac{2}{1-p}} = \left(\frac{1-p}{2} \sqrt{\frac{2}{p+1}} \right)^{\frac{2}{1-p}}. \tag{4.3}$$

Proof. (1) For $u_1 < 0$, after some calculations we obtain

$$\begin{aligned} u'(t) &= -\sqrt{E_u(0) + \frac{2}{p+1} u(t)^{p+1}} \\ &\le -\sqrt{\frac{2}{p+1} u(t)^{p+1}}, \qquad \forall t \in [0, T^*) \end{aligned} \tag{4.4}$$

and

$$u(t) \le \left(u_0^{\frac{1-p}{2}} - \frac{1-p}{2} t \right)^{\frac{2}{1-p}}, \qquad \forall t \in [0, T^*);$$

thus there exists a constant Z_0 so that $T^* \leq Z_0$ and $\lim_{t\to Z_0} u(t) = 0$.

By (4.4) we conclude that $\lim_{t\to Z_0^-} u'(t) = -\sqrt{E_u(0)}$ and

$$\begin{aligned} t &= \int_{u(t)}^{u_0} \frac{dr}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}}, \quad \forall t \in [0, T^*), \\ Z_0 &= \lim_{t\to Z_0} \int_{u(t)}^{u_0} \frac{dr}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}} \\ &= \int_0^{u_0} \frac{dr}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}} \end{aligned}$$

and

$$\lim_{t\to Z_0^-} u'''(t)(t - Z_0)^{1-p} = p \lim_{t\to Z_0^-} \left(\frac{u(t)}{t - Z_0}\right)^{p-1} u'(t) = pE_u(0)^{\frac{p}{2}}.$$

Therefore (4.1) and (4.2) are proved.

(2) For $u_1 > 0$ we have

$$u'(t) = \sqrt{E_u(0) + \frac{2}{p+1} u(t)^{p+1}} \geq \sqrt{\frac{2}{p+1} u(t)^{p+1}}, \quad \forall t \geq 0,$$

$$u(t)^{\frac{1-p}{2}} \geq u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}\, t, \qquad \forall t \geq 0. \tag{4.5}$$

On the other hand,

$$u'(t) \leq \sqrt{\frac{2}{p+1}} \left(u(t) + \left(\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}}\right)^{\frac{p+1}{2}}, \quad \forall t \geq 0,$$

$$\begin{aligned} &\left(u(t) + \left(\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}}\right)^{\frac{1-p}{2}} \\ \leq\ &\left(u_0 + \left(\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}}\right)^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}\, t \quad \forall t \geq 0. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), the estimate (4.3) follows.

V. $E_u(0) < 0$. In this section we discuss the case $E_u(0) < 0$. Similar to the above arguments proving Theorem 8 we have the following result on critical point and asymptotic behavior at infinity of the solutions for the equation (0.1) :

Theorem 9. *Suppose that u is a positive solution of problem* (0.1) *with $E_u(0) < 0$ and $u_0 > 0$. Then*

$$\lim_{t\to\infty} u(t)\, t^{-\frac{2}{1-p}} = \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}. \tag{5.1}$$

Moreover, for $u_1 < 0$, there exists a constant Z_1 so that $\lim_{t\to Z_1} u'(t) = 0$ and

$$Z_1 = \sqrt[p+1]{\frac{p+1}{2}}\,(-E_u(0))^{\frac{1-p}{2p+2}} \int_1^{\left(\frac{p+1}{-2}E_u(0)\right)^{\frac{-1}{p+1}} u_0} \frac{dr}{\sqrt{r^{p+1}-1}}. \tag{5.2}$$

Remark. We do not know whether the solutions under the circumstance in Theorem 9 is analytic or not.

Through Theorems 3 through 7 may be summarized for $p > 1$, in the following tables

$E(0)$	$E(0) < 0$	$E(0) = 0$
T	(i) $u_0u_1 \geq 0, T \leq T_1^*(u_0, u_1, p)$ (ii) $u_0u_1 < 0, T \leq T_2^*(u_0, u_1, p)$	(i) $u_0u_1 > 0, T \leq T_3^*$ (ii) $u_0u_1 < 0, T = \infty$ (iii) $u_0u_1 = 0, T = \infty, u \equiv 0.$
R_1, K_1	$\frac{4}{p-1}, K1(p)$	$\frac{4}{p-1}, K1(p)$
R_2, K_2	$\frac{p+3}{p-1}, K2(p)$	$\frac{p+3}{p-1}, K2(p)$
R_3, K_3	$\frac{2p+2}{p-1}, K3(p)$	$\frac{2p+2}{p-1}, K3(p)$

$E(0)>0$	$\hat{E}(0)>0$	$\hat{E}(0)=0, u_1>0$
T	$T \leq T_4^*(u_0,u_1,p)$	$T \leq T_5^*(u_0,u_1,p)$
R_1, K_1	$\frac{4}{p-1}, K1(p)$	$\frac{4}{p-1}, K1(p)$
R_2, K_2	$\frac{p+3}{p-1}, K2(p)$	$\frac{p+3}{p-1}, K2(p)$
R_3, K_3	$\frac{2p+2}{p-1}, K3(p)$	$\frac{2p+2}{p-1}, K3(p)$

Where $T := Life - span$ of u, $E(0) = Energy$, $R_1 = blow - up$ rate of a, $K_1 = blow - up$ constant of a; $R_2 = blow - up$ rate of a', $K_2 = blow - up$ constant of a'; $R_3 = blow - up$ rate of a'', $K_3 = blow - up$ constant of a''; $\hat{E}(0) := u_0^2 u_1^2 - 4u_0^2 E(0)$.

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