3 Learning process

We consider that the exchange rate movements are influenced by the interest rate changes between two countries. Hence the exchange rates are correlated with the cross country interest rate movements. The standard Brownian motions Z are defined on a probability space $(\Omega, P, \mathcal{F})$ with a standard filtration $\mathcal{F} = \{\mathcal{F}_t : t \leq T\}$. The investor's information structure is summarized by the filtration $\mathcal{F}^I$ generated by the joint processes of signals $I(t) = (e(t), L(t))$ and $\mathcal{F}_t^I \subset \mathcal{F}_t$. Processes of dZ and β are adapted to $\mathcal{F}_t$, but not to $\mathcal{F}_t^I$, because the investor does not directly observe β. Let $b_t \equiv E(\beta|\mathcal{F}_t^I)$ and $v_t \equiv E((\beta - b_t)(\beta - b_t)^\top|\mathcal{F}_t^I)$ denote the conditional mean and variance of the investor's estimate. The investor is assumed to have a Gaussian prior probability distribution over β, with mean b_0 and variance v_0.

Following Xia (2001), Liptser and Shiryayev (2001), the distribution of β conditional on $I(t) = (e(t), L(t))$ is also Gaussian with mean b_t and variance v_t (subscript t is dropped later). To gain more intuition for the continuous time Bayesian updating rule, we concentrate on a specific simplification of the model and leave the general case and details of derivation to Appendix A.

Assume that there is one predictive variable that follows an Ornstein–Uhlenbeck (O-U) process:

$$dL = \kappa(\overline{L} - L)dt + \sigma_L dZ_L.$$

Then, we can rewrite the exchange rate process as

$$\frac{de}{e} = (\overline{\mu}_e + \beta(L - \overline{L}))dt + \sigma_e dZ_e.$$

We also assume that the coefficient β follows an O-U process,

$$d\beta = \vartheta(\overline{\beta} - \beta)dt + \sigma_\beta dZ_\beta.$$

Conditional on the investor's filtration F_t^I, the exchange rate follows the stochastic process

$$\frac{de}{e} = (\overline{\mu}_e + b(L - \overline{L}))dt + \sigma_e d\widehat{Z}_e,$$

here

$$\begin{pmatrix} d\widehat{Z_d}(t) \\ d\widehat{Z_f}(t) \\ d\widehat{Z_e}(t) \end{pmatrix} = \begin{pmatrix} \sigma_{r_d} & 0 & 0 \\ 0 & \sigma_{r_f} & 0 \\ 0 & 0 & \sigma_e(t) \end{pmatrix}^{-1} \left[\begin{pmatrix} dr_d(t) \\ dr_f(t) \\ \frac{de(t)}{e(t)} \end{pmatrix} - \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \\ \overline{\mu}_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ r_d(t) - b_d & r_f(t) - b_f \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) dt \right],$$

and the posterior mean

$$
\begin{pmatrix} db_1(t) \\ db_2(t) \end{pmatrix} = \begin{pmatrix} \vartheta_1(\overline{b_1} - b_1(t)) \\ \vartheta_2(\overline{b_2} - b_2(t)) \end{pmatrix} dt + \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right]
$$

$$
\times \begin{bmatrix} \sigma_{r_d}^2 & 0 & 0 \\ 0 & \sigma_{r_f}^2 & 0 \\ 0 & 0 & \sigma_e^2(t) \end{bmatrix}^{-1}
$$

$$
\times \left[\begin{pmatrix} dr_d(t) \\ dr_f(t) \\ \frac{de(t)}{e(t)} \end{pmatrix} - \left(\begin{pmatrix} a_d(b_d - r_d(t)) \\ a_f(b_f - r_f(t)) \\ \overline{\mu}_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ r_d(t) - b_d & r_f(t) - b_f \end{pmatrix} \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \right) dt \right]
$$

and posterior variance

$$
\begin{aligned}
\begin{pmatrix} dv_{11}(t) & dv_{21}(t) \\ dv_{12}(t) & dv_{22}(t) \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_{\beta_1}^2 & 0 \\ 0 & \sigma_{\beta_2}^2 \end{pmatrix} \\
&- \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right] \\
&\times \begin{bmatrix} \sigma_{r_d}^2 & 0 & 0 \\ 0 & \sigma_{r_f}^2 & 0 \\ 0 & 0 & \sigma_e^2(t) \end{bmatrix}^{-1} \\
&\times \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{11}(t) & v_{21}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 & r_d(t) - b_d \\ 0 & 0 & r_f(t) - b_f \end{pmatrix} \right]^{\intercal}
\end{aligned}
$$

We suppose the state that the learning process will be with steady convergence, in other words, the variation of the estimation parameter will not change to count with new observed value. Thus $dv_{ij}(t) = 0, t \in [t, T]$, $i, j = 1, 2$ and v_{ij} do not need to be considered a state variable in portfolio choice problem.