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四階非線性雙曲型方程之研究
ON NONLINEAR HYPERBOLIC EQUATIONS OF FOURTH ORDER

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中文摘要

我們利用能量法去探討四階非線性雙曲型方程附帶動態邊界條件的大域解的存在與不存在性的問題。根據初值能量的正負號，我們估計出爆炸時間的上界。

關鍵詞：動態邊界條件，爆炸時間

On Nonlinear Hyperbolic Equations of Fourth Order

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Abstract

The nonexistence of global solutions of some nonlinear hyperbolic equations of fourth order with dynamic boundary conditions is discussed by using the energy method. The upper bounds for the blow up time are given respectively according to the sign of the initial energy.

1. Introduction

In this paper we shall study the fourth order hyperbolic initial boundary value problems of the following types :

$$(\mathrm{P1})\quad \begin{cases} u_{tt} + \Delta^2 u + f(u) = 0, & & (t,x) \in (0,T) \times \Omega \\ \frac{\partial \Delta u}{\partial \nu} = 0, & \Delta u + a(x)\frac{\partial u_t}{\partial \iota} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) = u_0(x), & u_t(0,x) = u_1(x), & x \in \Omega \end{cases}$$

$$(\mathrm{P2})\quad \begin{cases} u_{tt} + \Delta^2 u + f(u) = 0, & & (t,x) \in (0,T) \times \Omega \\ \Delta u = 0, & a(x)u_t = \frac{\partial \Delta u}{\partial \nu}, & (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) = u_0(x), & u_t(0,x) = u_1(x), & x \in \Omega \end{cases}$$

$$(\mathrm{P3})\quad \begin{cases} u_{tt} + \Delta^2 u + f(u) = 0, & & (t,x) \in (0,T) \times \Omega \\ u = 0, & \Delta u + a(x)\frac{\partial u_t}{\partial \iota} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) = u_0(x), & u_t(0,x) = u_1(x), & x \in \Omega \end{cases}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$, $T > 0$ is an arbitrary real number, ν is the outer normal and $a(x)$ is a smooth nonnegative function defined on the boundary $\partial\Omega$.

There is a sizable literature on the collapse in a finite time of nonlinear equations of hyperbolic type of second order. The nonexistence of the global solutions of quasilinear equations with no damping terms(dissipation) in the boundary conditions was investigated by Lions [11], Glassey [4], Levine [9], Kalantarov and Ladyzhenskaya [7], Tsai and Li [14], Georgiev and Todorova [3], and Levine and Serrin [10].

A number of dynamical problems of continum mechanics are described by initial boundary value problems for partial differential equations with a dissipative term on the boundary. Recently, many articles have appeared in which the inital boundary value problems for wave equations with a dissipative term on all or part of the boundary of the spatial domains are investigated. Problems of this type for second order equations are found in Aliev [1] and Jacques [5] and for fourth or higher order equations are given in Lasiecka [8], Maksudov and Aliev [12], Kirane , Kouachi and Tatar [6], Park.[13] and Can, Park and Aliev [2].

Two methods are being used by many authors. One is from the lemma in [7], the other one is from the concavity lemma in [9]. We shall consider the hyperbolic equations of fourth order equations with dynamic boundary conditions by a slightly different approach from them. We promote their results to the cases $E(0) \geq 0$.

2. Preliminaries

Let

$$F(u) = \int_0^u f(s)ds,$$

$$K_0 = \int_{\partial\Omega} a(x) \left(\frac{\partial u_0}{\partial \nu}\right)^2 dx$$

and

$$M_0 = \int_{\partial\Omega} a(x) u_0^2 dx.$$

We give some assumptions :

(A1) $f : R \to R$ such that for each $v \in H_0^1(\Omega)$ we have $vf(v) \in L^1(\Omega)$ and $F(v) \in L^1(\Omega)$.

(A2) $f : M1 \equiv C^1(0, T, H_0^2(\Omega)) \cap C^2(0, T, L^2(\Omega)) \to M2 \equiv L^1(0, T, L^2(\Omega))$ satisfies a local Lipschitz condition, that is, there exists a constant λ depending on B such that

$$\|f(u) - f(v)\|_{M2} \leq \lambda(B) \|u - v\|_{M1}$$

for $u, v \in M1$ with $\|u\|_{M1}, \|v\|_{M1} \leq B$.

(A3) there exists a positive constant $\delta > 0$ such that

$$(2 + 4\delta) F(s) - sf(s) \geq 0$$

for all $s \in R$.

3. Main Results

In this section we shall discuss the existence of global and nonglobal solution. Define the energy function along the solutions of the problem (P1) by

$$E(t) = \frac{1}{2}\int_{\Omega}\left[u_t^2 + (\Delta u)^2\right]dx + \int_{\Omega} F(u)dx$$

Theorem 1 : If $F(s) \geq 0$ for all $s \in R$, we get the global existence of (P1).

Remark : when $f(s) = |s|^{p-1}s$ for $p > 1$ or $f(s) = s^p$ for odd p, we get global existence.

Let

$$A(t) = \int_{\Omega} u^2 dx + \int_0^t \int_{\partial\Omega} a(x)\left(\frac{\partial u}{\partial \nu}\right)^2 dxds$$

and

$$J(t) = \{A(t) + (T_1 - t)K_0\}^{-\delta}, \quad t > 0$$

for some T_1 larger than the blow up time T^* .

Theorem 2 : Assume that (A3) holds and that either one of the following

(i) $E(0) < 0$,

(ii) $E(0) = 0$ and $A'(0) > K_0$,

(iiI) $E(0) > 0$ and $A'(0) > r\left[A(0) + (4+8\delta)E(0) + 4(\delta+1)K_0\right] + K_0$,

holds, then the solution u blows up at time T^*. and the upper bound for T^* are given by the following :

(i) If $E(0) < 0$, then

$$T^* \leq T_1 = t^* - \frac{J(t^*)}{J'(t^*)}$$

Furthermore, if $J(t^*) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$, then we have

$$T^* \leq T_1 = t^* + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t^*)}$$

where $t^* = \max\left\{\frac{A'(0)-K_0}{(4+8\delta)E(0)}, 0\right\}$, $b = 8\delta^2 E(0)$,and $a = J'(t^*)^2 - bJ(t^*)^{2+\frac{1}{\delta}}$.

(ii) If $E(0) = 0$, then

$$T^* \leq T_1 = -\frac{J(0)}{J'(0)}.$$

(iii) If $\frac{(A'(0)-K_0)^2 J(0)^{\frac{1}{\delta}}}{8\delta^2} > E(0) > 0$, then

$$T^* \leq T_1 = 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left\{1 - \left[1 + cJ(0)\right]^{\frac{-1}{2\delta}}\right\}$$

where $b = 8\delta^2 E(0), a = J'(0)^2 - bJ(0)^{2+\frac{1}{\delta}}$, $c = \left(\frac{a}{t}\right)^{2+\frac{1}{\delta}}$.

Remark : We can get the same result for the problem (P3) which has a different boundary condition as that of (P1) by using similar arguments as above.

Define the energy function along the solutions of the problem (P2) by

$$E(t) = \frac{1}{2}\int_{\Omega}\left[u_t^2 + (\Delta u)^2\right]dx + \int_{\Omega} F(u)dx$$

Let

$$G(t) = \int_{\Omega} u^2 dx + \int_0^t \int_{\partial\Omega} a(x)u^2 dx ds,$$

and let

$$Y(t) = \{\, G(t) + (T_2 - t)M_0\}^{-\delta}$$

for some T_2 larger than the blow up time T^*.

Theorem 3 : Assume that (A3) holds and that either one of the following conditions

(i) $E(0) < 0$,

(ii) $E(0) = 0$ and $G'(0) > M_0$,

(iii) $E(0) > 0$ and $G'(0) > r\,[\, G(0) + (4 + 8\delta)\, E(0) + 4(\delta + 1)M_0] + M_0$

holds, then the solution u of (P2) blows up at time T^*. and the upper bound for T^* are given by the following :

(i) if $E(0) < 0$, then

$$T^* \leq T_2 = t' - \frac{Y(t')}{Y'(t')}.$$

Furthermore, if $Y(t') < \min\left\{1, \sqrt{\frac{\alpha}{-\beta}}\right\}$, then we have

$$T^* \leq T_2 = t' + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - Y(t')}$$

where $t' = \max\left\{\frac{G'(0) - M_0}{(4+8\delta)E(0)}, 0\right\}, \beta = 8\delta^2 E(0), \alpha = Y'(0)^2 - \beta Y(0)^{2+\frac{1}{\delta}}$.

(ii) if $E(0) = 0$, then

$$T^* \leq T_2 = -\frac{Y(0)}{Y'(0)}.$$

(iii) when $\frac{(G'(0) - M_0)^2 Y(0)^{\frac{1}{\delta}}}{8\delta^2} > E(0) > 0$,

$$T \leq T_2 = 2^{\frac{3\delta+1}{2\delta}} \frac{\delta\gamma}{\sqrt{\alpha}} \left\{1 - [1 + cY(0)]^{\frac{-1}{2\delta}}\right\}$$

where $\beta = 8\delta^2 E(0), \alpha = Y'(0)^2 - \beta Y(0)^{2+\frac{1}{\delta}}, \gamma = \left(\frac{\alpha}{\beta}\right)^{2+\frac{1}{\delta}}$.

Example : Consider the nonlinear Euler-Bernoulli equation

$$\begin{cases} u_{tt} + \Delta^2 u + \mu u^p = 0, & p > 1 & (t,x) \in (0,T) \times \Omega \\ u = 0, & \Delta u + a(x)\frac{\partial u_t}{\partial \nu} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) = u_0(x), & u_t(0,x) = u_1(x), & x \in \Omega \end{cases}$$

If $\mu \geq 0$, the problem is globally solvable and if $\mu < 0$, the problem has a blow-up solution for some initial functions .

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