3. Methodology

3.1 Reverse Optimization

We start with the equations for “reverse optimization.” Throughout this section, K is used to represent the number of views and N is used to represent the number of funds in the portfolio.

$$U = w^T \prod - (\lambda / 2) w^T \Sigma w \tag{3.1}$$

U : the fund manager’s utility, and this is the objective function of portfolio optimization.

w : the vector of weights invested in each fund (N x 1 column vector)

$\prod$: the vector of equilibrium excess returns for each asset (N x 1 column vector)

λ : the risk aversion coefficient

Σ : the covariance matrix of excess returns (N x N matrix)

The fund manager’s utility U is a concave function, so it will have a single global maximum. If we maximize the utility with no constraints, there is a closed form solution. We find the exact solution by taking the first derivative of (3.1) with respect to the weights and setting it to 0. We can achieve (3.2)

$$\prod = \lambda \Sigma w_{mkt} \tag{3.2}$$

where

w_{mkt} : the market capitalization weight (N x 1 column vector) of the funds

The Black-Litterman model uses “equilibrium” returns as a neutral starting point. Equilibrium returns are the set of returns that clear the market. The equilibrium returns are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using formula (3.2). The risk-aversion coefficient (λ) characterizes the expected risk-return tradeoff. It is the rate at which an investor will forgo expected return for less variance. In the reverse optimization process, the risk aversion coefficient acts as a scaling factor for the reverse optimization estimate of excess returns; the weighted reverse optimized excess returns equal the specified market risk premium. More excess return per unit of risk (a larger lambda) increases the estimated excess returns. We can find λ by multiplying both sides of (3.2) by w^t and replacing vector terms with scalar terms. And then we can achieve (3.3):

$$\lambda = \frac{(E(r) - r_f)}{\sigma^2} \tag{3.3}$$

$E(r)$: the total return on the market portfolio ($E(r) = \prod + r_f$)

r_f : the risk free rate

σ^2 : the variance of the market portfolio ($\sigma^2 = w^t \sum w$)

Since we get λ through (3.3), we can arrive at $\prod$ when we plug w, λ and $\sum$ into Eq. (3.2) and then generate the equilibrium fund returns.

3.2 Specifying the views

We will describe the process of specifying the investors' views of estimated returns. We define the combination of the investors' views as the prior distribution. Firstly, by construction we will require each view to be unique and uncorrelated with the other views. This will give the prior distribution the property that the covariance matrix will be diagonal, with all off diagonal entries equal to 0. Besides, we will require views to be fully invested, either the sum of weights in a view is zero (relative view) or is one (an absolute view).

We will represent the investors' k views on n assets using the following matrices

1. P is a K×N matrix of the asset weights within each view. For a relative view the sum of the weights will be 0, for an absolute view the sum of the weights will be:

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix} \tag{3.4}$$

2. Ω is a K×K matrix the covariance of the views. Ω is diagonal as the views are required to be independent and uncorrelated. Ω is also known as the confidence in the investor's views. The ith diagonal element of Ω is represented as w_i.

3. Q is a k×1 matrix of the returns for each view.

Given these matrices we can formulate the prior distribution mean and variance in portfolio space as:

$$PE(r) \sim N(\ Q, \Omega\ ) \tag{3.5}$$

There are three main ways to calculate Ω. Firstly, we actually compute the variance of the view. This is most easily done by defining a confidence interval around the return. Secondly, we can just assume that the variance of the view will be proportional to the variance of the assets, just as the variance of the sampling distribution is. He and Litterman (1999) use this method, and we can use the variance of the view computed from the sampling distribution:

$$\Omega = diag(P(\tau \Sigma)P^T) \tag{3.6}$$

This specification of the variance, or uncertainty, of the views essentially equally weights the investor's views and the market equilibrium weights. By including τ in the expression, the final solution becomes less dependent on the specific value of τ as well. Several authors have specified the confidence matrix as τΩ in order to manage this interaction.

Idzorek (2004) introduces the third method. He allows the specification of the view confidence in terms of the percentage move of the weights from no views to total certainty in the view.

We use Idzorek's method because this method is more intuitive.. The greater the level of confidence (certainty) in the expressed views, the closer the new return vector will be to the views. If the investor is less confident in the expressed views, the new return vector should be closer to the implied equilibrium return vector (Π).

3.3 The Black-Litterman Formula

Applying Bayes theory to the problem of blending the sampling and prior distributions, we can create a new posterior distribution of the asset returns. Given Eq.(3.5) and Eq.(3.6) we can apply Bayes Theorem to derive the equation for the posterior distribution of asset returns.

$$E(R) \propto N([(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q][(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}, ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}) \quad (3.7)$$

E(R): the new combined return vector (Nx1)

τ : a scalar

Σ :the covariance matrix of excess returns (NxN matrix)

P : a matrix that identifies the assets involved in the views(KxN)

Ω : a diagonal covariance matrix of error terms from the expressed views (KxK)

Π : the implied equilibrium return vector

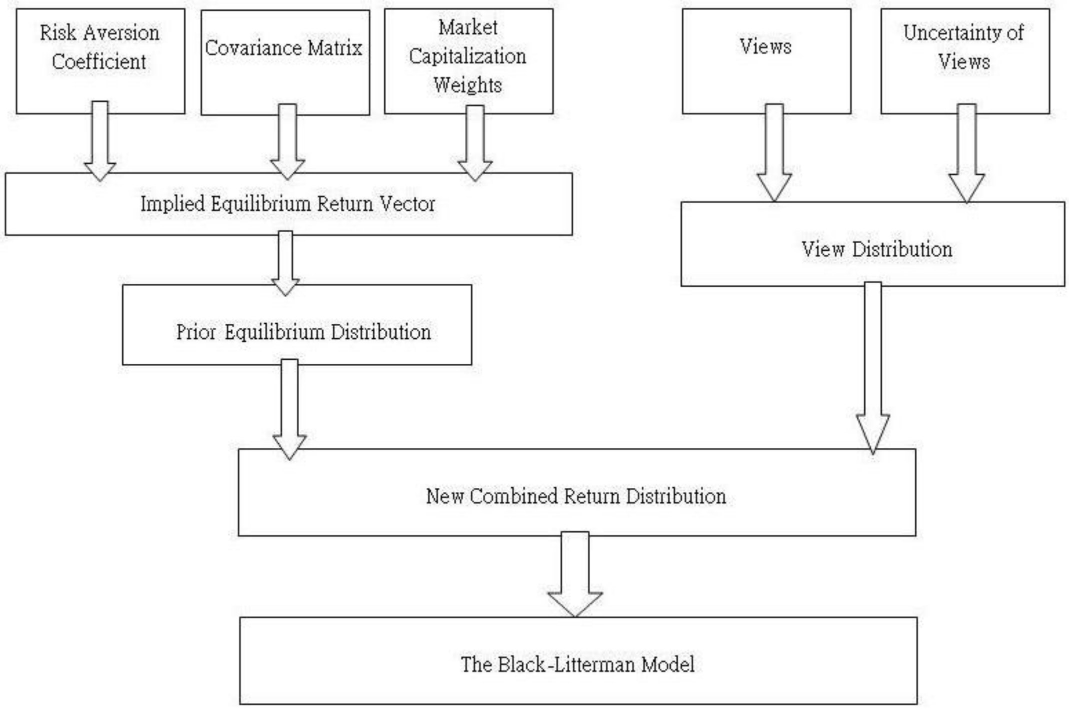
Q: the view vector

This is the famous Black-Litterman formula. We can represent the same formula for the mean returns in an alternative way:

$$E(R) = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q] \tag{3.8}$$

Eq.(3.8) is the new (posterior) combined return vector, and with all of the inputs and then entered into Eq.(3.8) new combined return vector is derived. The new recommended weights (w^*) can be calculated by solving Eq.(3.4) without constraints or Eq.(3.1) with constraints. This process is presented as Figure 3-1:

Figure 3-1. The process of deriving the New Combined Return Vector (E[R])



3.4 Summary

The Black-Litterman model combines views of the investor and the market equilibrium on the expected return of assets in one formula. This formula should be a better approximation of the expected returns. These expected returns, or more precisely the estimator of the expected return, could then be used in a mean-variance optimizer.

The Black-Litterman model can be summarized by the following points:

1. The market consists of N assets. Each asset has a return and variance. The return of asset i is denoted by r_i. The expected return of asset i becomes $E(r_i)$. For a portfolio that consists of n assets, the return of each asset in the portfolio is captured by the vector of returns. The vector of returns also has an expected value, $E(r)$. The expected return $E(r)$ is an unknown and normally distributed random variable and is assumed to have mean u and variance $\tau \sum$.
2. The first source of information about E(r) is the equilibrium returns u. The equilibrium returns are found by Eq.(3.2) $\prod = \lambda \sum w_{mkt}$.
3. The second source of information are the K views of the investor. The views are expressed as $PE(r) = Q + \varepsilon$.
4. Combination of these two sources of information leads to E(R) being normally distributed with mean $[(\tau \sum)^{-1} + P^T \Omega^{-1} P]^{-1} [(\tau \sum)^{-1} \prod + P^T \Omega^{-1} Q]$ and variance $[(\tau \sum)^{-1} + P^T \Omega^{-1} P]^{-1}$.
5. The mean $[(\tau \sum)^{-1} + P^T \Omega^{-1} P]^{-1} [(\tau \sum)^{-1} \prod + P^T \Omega^{-1} Q]$ and variance $[(\tau \sum)^{-1} + P^T \Omega^{-1} P]^{-1}$ can be used in a mean-variance optimization process to obtain a mean-variance efficient portfolio.