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FAULT DIAMETER FOR SUPERCUBES*

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ABSTRACT

Assume that N and s are positive integers with $2^s < N \leq 2^{s+1}$. It is claimed by Auletta, Rescigno, and Scarano that the fault diameter of the supercube with N nodes, is exactly $s+1$ if $N \notin \{2^{s+1}-1, 2^{s+1}-2, 2^s+2^{s-1}+1\}$, and $s+2$ otherwise. In this paper, we will argue that the above claim is not correct. Instead, we will show that the fault diameter of the supercube with N nodes is $s+2$ if $N \in \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$.

Keywords: interconnection networks, hypercube, supercube, fault diameter

1. Introduction and notations

Hypercube topology has been studied extensively as an interconnection network for parallel machines because of advantages like high bandwidth and low message latency [6]. One major constraint of the hypercube topology is that the number of nodes in the network must be 2^s for some positive integer s and as such cannot be defined for any number of nodes. Incomplete hypercube topology proposed in [3] removed this restriction. However, the incomplete hypercube has serious limitations from the fault-tolerance perspective. A single node failure may disconnect the network. In [7], Sen proposed a family of networks, called supercubes and denoted by S_N. Each S_N contains exactly N nodes. If N satisfies the relation $2^s < N \leq 2^{s+1}$, then S_N is a supergraph of the hypercube with 2^s nodes. Later, more studies has been on the investigation of the topological properties of supercubes extending results known for the hypercube to the supercube [1,8,9]. This indicates that the performance of the supercube is almost the same as the hypercube which is of the approximate size.

The fault diameter [4] is an important measure for interconnection networks. Due to the non-symmetric property and the variation in the number of vertices, it is difficult to get the precise value of fault diameter for supercubes. Assume

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that $2^s < N \leq 2^{s+1}$. In [8], it is proved that the fault diameter of S_N is at most $s+3$. Later, it is claimed in [1] that the fault diameter of S_N is exactly $s+1$ if $N \notin \{2^{s+1}-1, 2^{s+1}-2, 2^s+2^{s-1}+1\}$, and $s+2$ otherwise. In this paper, we will argue that the above assertion is not correct. Instead, we will show that the fault diameter of the supercube with N node is $s+2$ if $N \in \{2^{s+1}-2^i+1 \mid 0 \leq i \leq s-1\}$.

Now, we will formally introduce the definition of supercubes and some graph terminologies used in this paper. Most of the graph and interconnection network definitions used in this paper are standard (see e.g., [5]). Let $G = (V, E)$ be a finite, undirected graph. Throughout this paper, node and vertex are used interchangeably to represent the element of V. Edge and link are used interchangeably to represent the element of E. For a vertex u, $N(u)$ denotes the *neighborhood* of u which is the set $\{v \mid (u,v) \in E\}$. Let u, v be two nodes of G. The *distance* between u and v, denoted by $d_G(u,v)$, is the length of the shortest path between them. The *diameter* of G, denoted by $D(G)$, is the maximum distance between any two nodes in G. The *connectivity* of G, denoted by $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with $\kappa(G) = \kappa$. It follows from Menger's theorem that there are k *internal node-disjoint* (abbreviated as *disjoint*) *paths* joining any two vertices u and v when $k \leq \kappa$. Let F be a subset of V which is referred as a *faulty set*. $G - F$ denotes the subgraph induced by $V - F$. We use $d_k(G)$ to denote the largest diameter of $G - F$ for any faulty set F with $|F| \leq k$. Obviously, $d_k(G) = \infty$ if $k \geq \kappa$. The *fault diameter* of a graph G is defined as $d_{\kappa-1}(G)$. Obviously, we have $D(G) \leq d_{\kappa-1}(G)$.

Throughout this paper, we assume that N and s are positive integers with $2^s < N \leq 2^{s+1}$. Let $u = u_{(s)}u_{(s-1)}\cdots u_{(1)}u_{(0)}$ and $v = v_{(s)}v_{(s-1)}\cdots v_{(1)}v_{(0)}$ be two $(s+1)$-bit strings. The *Hamming distance* between u and v, denoted by $h(u,v)$, is the number of i, $0 \leq i \leq s$, such that $u_{(i)} \neq v_{(i)}$. The $(s+1)$*-dimensional hypercube* consists of all the $(s+1)$-bit strings as its vertices and two vertices u and v are adjacent if and only if $h(u,v) = 1$. Hence each vertex of the $(s+1)$-dimensional hypercube is labelled with a unique integer k with $0 \leq k \leq 2^{s+1}-1$. Then the N-node supercube graph can be constructed from an $(s+1)$-dimensional hypercube as below. For each node u with $N \leq u \leq 2^{s+1}-1$, merging nodes u and $u-2^s$ in the $(s+1)$-dimensional hypercube into a single node labeled as $u-2^s$ and leaving other nodes in the $(s+1)$-dimensional hypercube unchanged, we obtain an N-node supercube.

More precisely, let $S_N = (V, E)$ be a *supercube*. The vertex set V consists of N vertices which are labeled from 0 to $N-1$. Then, each vertex u $(0 \leq u \leq N-1)$ can be expressed as an $(s+1)$-bit string $u_{(s)}u_{(s-1)}\cdots u_{(1)}u_{(0)}$ such that $u = \sum_{i=0}^{s} u_{(i)}2^i$. In other words, an $(s+1)$-bit string $u_{(s)}u_{(s-1)}\cdots u_{(0)}$ is a node of S_N if and only if $u \leq N-1$. We use $\bar{u}$ to denote the string $\bar{u}_{(s)}\bar{u}_{(s-1)}\cdots \bar{u}_{(1)}\bar{u}_{(0)}$ and use u^k to denote the string $u_{(s)}u_{(s-1)}\cdots u_{(k+1)}\bar{u}_{(k)}u_{(k-1)}\cdots u_{(0)}$. The vertex set V is partitioned into three subsets V_1, V_2, and V_3, where $V_3 = \{u \mid u \in V,\ u_{(s)} = 1\}$, $V_2 = \{u \mid u \in V,\ u_{(s)} = 0, \text{ and } u^s \notin V\}$, and $V_1 = \{u \mid u \in V,\ u_{(s)} = 0, \text{ and } u^s \in V\}$. The edge set E is the union of E_1, E_2, E_3, and E_4, where $E_1 = \{(u,v) \mid u, v \in V_1 \cup V_2$ and

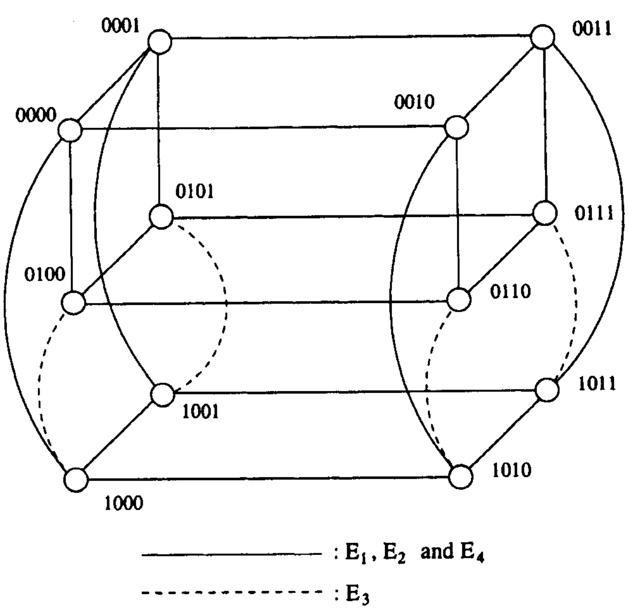

Fig. 1 The supercube with 12 nodes.

$h(u,v) = 1\}$, $E_2 = \{(u,v) \mid u,v \in V_3$ and $h(u,v) = 1\}$, $E_3 = \{(u,v) \mid u \in V_3$, $v \in V_2$ and $h(u,v) = 2\}$, and $E_4 = \{(u,v) \mid u \in V_3, v \in V_1$, and $h(u,v) = 1\}$. As an example, a supercube with 12 nodes is shown in Figure 1. In this figure, edges in E_1, E_2, and E_4 are indicated by solid lines and edges in E_3 are indicated by dashed lines. Let $Z^0 = V_1 \cup V_2$ and $Z^1 = V_3$. Obviously, Z^0 induces an s-dimensional hypercube.

It is proved in [7] that $\kappa(S_N)$ is s if $2^s < N < 2^s + 2^{s-1}$, and $s+1$ if $2^s + 2^{s-1} \leq N \leq 2^{s+1}$. In [1], it is claimed that the fault diameter $d_{\kappa-1}(S_N)$ of S_N is $s+1$ if $N \notin \{2^{s+1}-1,\ 2^{s+1}-2,\ 2^s + 2^{s-1} + 1\}$ and $s+2$ otherwise. However, this result is not true in some cases. For example, we consider the case S_{29} shown in Figure 2. The connectivity of S_{29} is 5. Let $u = 01100$ and $v = 00011$ be two nodes of S_{29}. Assume that the faulty set of S_{29} is $F = \{00100, 01000, 01110, 01101\}$ which is indicated by darkened nodes. Then use breadth first search rooted at u, $d_{S_N-F}(u,v) = 6$. Thus $d_{\kappa-1}(S_N) \geq 6$ and the result obtained in [1] is incorrect. In this paper, we will show that $d_{\kappa-1}(S_N)$ is $s+2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$. In the rest of this paper, we assume that $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$. Note that $2^s + 2^{s-1} < N \leq 2^{s+1}$.

2. Some basic properties of supercubes

In this section, we present some basic properties of supercubes. For any two vertices $u = u_{(s)}u_{(s-1)}\cdots u_{(0)}$ and $v = v_{(s)}v_{(s-1)}\cdots v_{(0)}$, we define $h'(u,v)$ to be $|\{i \mid u_{(i)} \neq v_{(i)}, 0 \leq i \leq s-1\}|$. The following properties will be useful in deriving

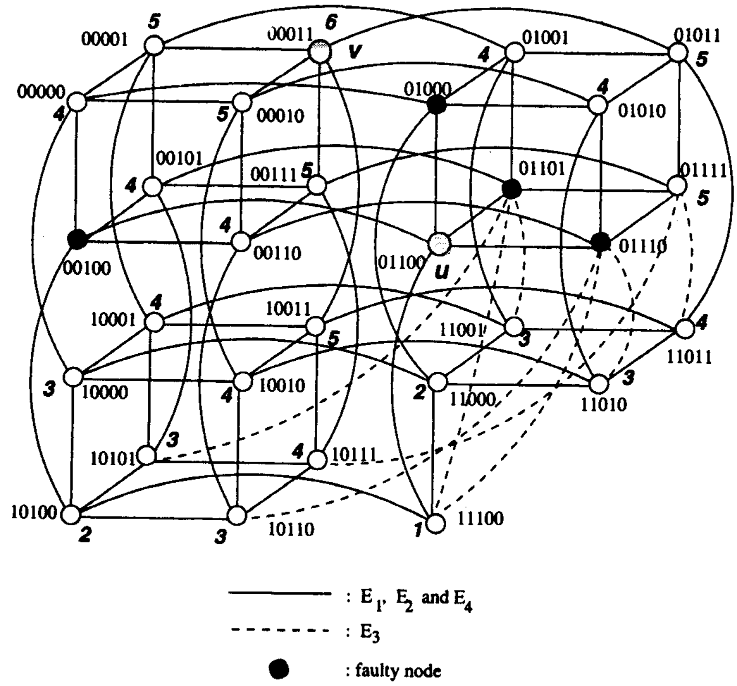

Fig.2 S_{29} with faulty set F={00100, 01000, 01110, 01101}

$d_{\kappa-1}(S_N)$ for $N \in \{2^{s+1} - 2^i + 1 \mid 0 \le i \le s-1\}$.

Lemma 1. $d_{S_N}(u,v) \ge h'(u,v)$. Hence $d_{S_N}(u,v) \ge h(u,v) - 1$. Moreover, $d_{S_N}(u,v) \ge h(u,v)$ if $u \vee v < N$.

Proof. It is observed that any edge (x,y) in $E_1 \cup E_2 \cup E_4$ satisfies $h(x,y) = 1$, and any edge (x,y) in E_3 satisfies (1) $h(x,y) = 2$, (2) $x_{(s)} \ne y_{(s)}$, and (3) $x \vee y \ge N$. Hence it is easy to see that we need at least one edge to change the ith bit with $u_{(i)} \ne v_{(i)}$, $0 \le i \le s-1$, in any path joining u to v. Hence, $d_{S_N}(u,v) \ge h'(u,v)$. Thus $d_{S_N}(u,v) \ge h(u,v) - 1$ for any u, $v \in S_N$.

Now we consider the case that $u \vee v < N$. Suppose that $d_{S_N}(u,v) = h(u,v) - 1$. Thus, there exists a path P in S_N joining u to v of length $h(u,v) - 1$. Obviously, P contains an E_3 edge, denoted by (x,y). Without loss of generality, we may write P as $u \to \ldots \to x$, $y \to \ldots \to v$ such that $(x,y) \in E_3$ and $y^s \notin S_N$. Since $u \vee v < N$, there exists some index j, $0 \le j \le s-1$, such that $u_{(j)} = v_{(j)}$ and $y_{(j)} \ne u_{(j)}$. Thus the length of P is at least $h'(u,v) + 1 \ge h(u,v)$. The result is contradictory to the assumption, hence the lemma is proved. □

Lemma 2. [6] For any two nodes u and v in the n-dimensional hypercube, let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices such that $u_{(\alpha_i)} \ne v_{(\alpha_i)}$ and $\{\beta_i\}_{i=0}^{n-h(u,v)-1}$ be the decreasing sequence of indices such that $u_{(\beta_i)} = v_{(\beta_i)}$. For $0 \le i \le h(u,v) - 1$, we set the sequence P_i as u, $x_{i,1} = u^{\alpha_{0+i \bmod h(u,v)}}$,

$x_{i,2} = x_{i,1}^{\alpha_{1+i \bmod h(u,v)}}$, ..., $x_{i,h(u,v)} = x_{i,h(u,v)-1}^{\alpha_{h(u,v)-1+i \bmod h(u,v)}} = v$; and for $0 \leq j \leq n - h(u,v) - 1$, we set $P_{j+h(u,v)}$ as u, u^{β_j}, $x_{0,1}^{\beta_j}$, $x_{0,2}^{\beta_j}$, ..., $x_{0,h(u,v)-1}^{\beta_j}$, v^{β_j}, v. Then P_0, P_1, ..., P_{n-1} form n disjoint paths joining u to v.

Lemma 3. Assume that both u and v are in either Z^i for $i = 0, 1$. Let $\{\alpha_i\}_{i=0}^{h(u,v)-1}$ be the decreasing sequence of indices such that $u_{(\alpha_i)} \neq v_{(\alpha_i)}$. There exist $h(u,v)$ disjoint paths $P_0, P_1, \ldots, P_{h(u,v)-1}$ in S_N joining u to v with the following properties: (1) The length of each P_i is at most $h(u,v)$; (2) for any node p of these paths, $p_{(j)} = u_{(j)} = v_{(j)}$ where $j \notin \{\alpha_i\}_{i=0}^{h(u,v)-1}$ and $j \neq s$; and (3) if both u and v are in Z^1 then there exists at least one path P_i in Z^1 such that for each internal node q of P_i we have $q^s \notin P_j$ for $0 \leq j \leq h(u,v) - 1$.

Proof. We are going to construct $h(u,v)$ disjoint paths, P_0, P_1, ..., $P_{h(u,v)-1}$, joining u to v such that the length of each path is at most $h(u,v)$. Then the proof of this lemma follows the construction.

Case 1 Both u and v are in Z^0. Since the subgraph of S_N induced by Z^0 is isomorphic to the s-dimensional hypercube, the paths P_0, P_1, ..., $P_{h(u,v)-1}$ described in Lemma 2 satisfy our requirement.

Case 2 Both u and v are in Z^1. Without loss of generality, we assume that $u > v$. Then $u_{(\alpha_0)} = 1$, $v_{(\alpha_0)} = 0$, and $u_{(j)} = v_{(j)}$ for all $\alpha_0 < j \leq s$. Let P_0 be u, $x_{0,1} = u^{\alpha_0}$, $x_{0,2} = x_{0,1}^{\alpha_1}$, ..., $x_{0,h(u,v)} = x_{0,h(u,v)-1}^{\alpha_{h(u,v)-1}} = v$. Thus, for any $x_{0,i}$ with $1 \leq i \leq h(u,v)$, $x_{0,i_{(\alpha_0)}} < u_{(\alpha_0)}$ and $x_{0,i_{(j)}} = u_{(j)}$ for $\alpha_0 < j \leq s$. Therefore, $x_{0,i} < u$ for $1 \leq i \leq h(u,v)$. That is, all the internal nodes of P_0 are less than u. Thus P_0 is in Z^1. For $1 \leq i \leq h(u,v) - 1$, let Q_i be the sequence u, $y_{i,1} = u^{\alpha_{0+i \bmod h(u,v)}}$, $y_{i,2} = y_{i,1}^{\alpha_{1+i \bmod h(u,v)}}$, ..., $y_{i,h(u,v)} = y_{i,h(u,v)-1}^{\alpha_{h(u,v)-1+i \bmod h(u,v)}} = v$. Then we construct the path P_i from Q_i by setting P_i to be u, $x_{i,1}$, $x_{i,2}$, ..., $x_{i,h(u,v)} = v$ where $x_{i,j} = y_{i,j}$ if $y_{i,j} \in S_N$ and $x_{i,j} = y_{i,j}^s$ otherwise. Obviously each internal node q of P_0 is in Z^1 and is not in any P_j for $1 \leq j \leq h(u,v) - 1$. Thus these paths, P_0, P_1, ..., $P_{h(u,v)-1}$, satisfy our requirement. □

Lemma 4. Let u and v be two nodes in S_N where $u \in Z^0$ and $v \in Z^1$, Let $S(u,v)$ denote the set $\{i \mid u_{(i)} = v_{(i)}\}$. Then there exist $h(u,v)$ disjoint paths $P_0, P_1, \ldots, P_{h(u,v)-1}$ in S_N joining u to v with the following properties: (1) The length of each path is at most $h(u,v) + 1$; (2) $q_{(j)} = u_{(j)}$ for each internal node q in Z^0 of these paths where $j \in S(u,v)$; (3) exactly one path, $q_0 = u$, q_1, ..., $q_{h(u,v)-2}$, v^s, v, is of length $h(u,v)$ with all its internal nodes in Z^0 and (4) if there exists any internal node q in Z^0 of P_0, P_1, ..., $P_{h(u,v)-1}$ satisfying that $p_{(j)} \neq u_{(j)}$ with $j \in S(u,v)$, then $q_{h(u,v)-2}^s$ is not in P_0, P_1, ..., $P_{h(u,v)-1}$.

Proof. We are going to construct $h(u,v)$ disjoint paths, P_0, P_1, ..., $P_{h(u,v)-1}$, joining u to v such that the length of each path is at most $h(u,v) + 1$. The proof of this Lemma can be established as follows.

Because $v \in Z^1$, we have $v^s \in Z^0$. Let $\{\alpha_i\}_{i=0}^{h(u,v)-2}$ be the decreasing sequence of indices such that $u_{(\alpha_i)} \neq v_{(\alpha_i)}^s$. Hence $u_{(\alpha_i)}^s \neq v_{(\alpha_i)}$ for every i. Since Z^0 induces an s-dimensional hypercube, from Lemma 2 there are $h(u,v^s)$ internal node disjoint paths, $Q_0, Q_1, \ldots, Q_{h(u,v)-2}$, in Z^0 joining u to v^s such that the length of each Q_i is $h(u,v^s) = h(u,v) - 1$. We may assume without loss of generality that $t_i = (v^s)^{\alpha_i}$ is the last internal node which is adjacent to v^s in Q_i. Let Q'_i be the subpath of Q_i joining u to t_i. Now we will construct $P_0, P_1, \ldots, P_{h(u,v)-1}$ as follows.

We first set $P_{h(u,v)-2}$ to be $u \stackrel{Q_{h(u,v)-2}}{\longrightarrow} v^s, v$, i.e., appending the edge (v^s, v) to the path $Q_{h(u,v)-2}$. Thus, the path $P_{h(u,v)-2}$ satisifies the third requirement (3) in this lemma. For $0 \le i \le h(u,v) - 3$, we let P_i be $u \stackrel{Q'_i}{\longrightarrow} t_i, t_i^s, v$ if $t_i^s \in S_N$, and $u \stackrel{Q'_i}{\longrightarrow} t_i, v$ otherwise. For $0 \le i \le h(u,v) - 3$, it is obvious that $(t_{h(u,v)-2})^s = v^{\alpha_{h(u,v)-2}} \notin P_i$ and P_i is of length at most $h(u,v)$. And for each internal node q in P_i, $q_{(j)} = u_{(j)}$ for $j \in S(u,v)$. Then we construct the path $P_{h(u,v)-1}$ using the following two cases.

Case 1 $u^s \in S_N$: Without loss of generality, we suppose that $u^s > v$. Thus $u^s_{(\alpha_0)} = 1$ and $v_{(\alpha_0)} = 0$. Then we set the path $P_{h(u,v)-1}$ as u, $x_1 = u^s$, $x_2 = x_1^{\alpha_0}$, $x_3 = x_2^{\alpha_1}$, $\ldots$, $x_{h(u,v)-1} = x_{h(u,v)-2}^{\alpha_{h(u,v)-3}}$, $x_{h(u,v)} = x_{h(u,v)-1}^{\alpha_{h(u,v)-2}} = v$. Because $u^s_{(\alpha_0)} = 1$ and $x_{i(\alpha_0)} = 0$ for $2 \le i \le h(u,v)$, $x_1 > x_i$. Therefore, all internal node q of $P_{h(u,v)-1}$ are in Z^1 and $q_{(j)} = u_{(j)}$ for $j \in S(u,v)$. And $P_{h(u,v)-1}$ is of length $h(u,v)$. Obviously, these paths, $P_0, P_1, \ldots, P_{h(u,v)-1}$, satisfy our requirement.

Case 2 $u^s \notin S_N$: In the case, $u_{(s-1)} = 1$ because $u^s \ge N > 2^s + 2^{s-1}$.

We first consider the case that $v_{(s-1)} = 0$. Obviously, $\alpha_0 = s - 1$. Since $u_{(\alpha_0)} = 1$ and $u^s \notin S_N$, we have $(u^s)^{\alpha_0} < 2^s + 2^{s-1} < N$ and $(u, (u^s)^{\alpha_0}) \in E(S_N)$ by definition of supercube. Let $P_{h(u,v)-1}$ be u, $x_1 = (u^s)^{\alpha_0}$, $x_2 = x_1^{\alpha_1}$, $x_3 = x_2^{\alpha_2}$, $\ldots$, $x_{h(u,v)-2} = x_{h(u,v)-3}^{\alpha_{h(u,v)-3}}$, $x_{h(u,v)-1} = x_{h(u,v)-2}^{\alpha_{h(u,v)-2}} = v$. Since $x_{i(s-1)} = x_{i(\alpha_0)} = 0$ for $1 \le i \le h(u,v) - 1$, we have $x_i < 2^s + 2^{s-1} < N$. Hence, each internal node q of $P_{h(u,v)-1}$ is in S_N and $q_{(j)} = u_{(j)}$ for $j \in S(u,v)$. Obviously, $P_0, P_1, \ldots, P_{h(u,v)-1}$ are disjoint and satisfy our requirement.

Now we consider the case that $v_{(s-1)} = 1$. Note that $v_{(s-1)} = u_{(s-1)} = 1$ and $s - 1 \in S(u,v)$. It is observed that $p_{(j)} = u_{(j)}$ with $j \in S(u,v)$ for each internal node p in $P_0, P_1, \ldots, P_{(h(u,v)-2)}$. Thus v^{s-1} is not in any P_i since $s - 1 \in S(u,v)$ and $v^{s-1}{}_{(s-1)} \ne v_{(s-1)}$. Since $(u^s)^{s-1}$, $(v^s)^{s-1} < 2^s + 2^{s-1} \le N - 1$ and $u^s \notin S_N$, then $(u^s)^{s-1}, v^{s-1} \in S_N$ and $(u, (u^s)^{s-1}) \in E(S_N)$. From Lemma 3, there exists a path W of length $h((u^s)^{s-1}, v^{s-1}) = h(u,v) - 1$ in Z^1 joining $(u^s)^{s-1}$ to v^{s-1}. Obviously, $t^s_{h(u,v)-2} = v^{\alpha_{h(u,v)-2}} \notin W$. Because $q_{(s-1)} \ne u_{(s-1)}$ for each node q of path W, W is disjoint from $P_0, P_1, \ldots, P_{h(u,v)-2}$. Then we set the path $P_{h(u,v)-1}$ as u, $(u^s)^{s-1} \stackrel{W}{\longrightarrow} v^{s-1}, v$. Obviously, the length of $P_{h(u,v)-1}$ is $h(u,v) + 1$. Since $t^s_{h(u,v)-2}$ is not in any P_i, statement (4) holds. Thus, $P_0, P_1, \ldots, P_{h(u,v)-1}$ satisfy our requirement.

Hence, the theorem is proved. □

3. Fault diameter of supercubes

In this section, we discuss $d_{\kappa-1}(S_N)$ where $N \in \{2^{s+1} - 2^i + 1 \mid 0 \le i \le s - 1\}$.

In this case, $\kappa(S_N) = s + 1$. First, we will prove $d_{\kappa-1}(S_N) \geq s + 2$.

Lemma 5. $d_{\kappa-1}(S_N) \geq s + 2$ for $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$.

Proof. Let u be the node which is labelled by $N-1$ and $v = \overline{u}$ where $h(u,v) = s+1$ and $h(u^s, v) = s$. Assume that $F = N(u^s) \cap Z^0$. Hence $|F| = s$ and $u^s, v \in Z^0$. Let $P : u^s = x_0, x_1, \ldots, x_k = v$ be any path in $S_N - F$ joining u^s to v. Obviously, $x_1 = u$. Since $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$, $u = \overbrace{11\ldots1}^{s-i+1}\overbrace{00\ldots0}^{i}$ and $v = \overbrace{00\ldots0}^{s-i+1}\overbrace{11\ldots1}^{i}$. Obviously, $x_2 = u^j$ where $i \leq j \leq s - 1$ because u^0, u^1, $\ldots$, u^{i-1} are greater than u and not in S_N. Since $x_2 \vee v < N$ and $h(x_2, v) = s$, $d_{S_N - F}(x_2, v) \geq s$ derived from Lemma 1. Hence the length of P is at least $s + 2$. Therefore $d_{\kappa-1}(S_N) \geq s + 2$. □

Now we will discuss the upper bound of $d_{S_N-F}(u,v)$ for any two vertices $u, v \in S_N - F$ in supercubes. We will discuss this problem into three cases depending on $h(u,v)$. We first consider the case that $h(u,v) = s + 1$.

Lemma 6. Let F be any faulty set with $|F| \leq s$ and $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$. Then $d_{S_N-F}(u,v) \leq s + 2$ for any $u, v \in S_N - F$ with $h(u,v) = s + 1$.

Proof. Since $h(u,v) = s + 1$, we may assume without loss of generality that $u \in Z^0$ and $v \in Z^1$. From Lemma 4, there exist $s + 1$ disjoint paths of length at most $h(u,v) + 1 = s + 2$ joining u to v. Thus, $d_{S_N-F}(u,v) \leq s + 2$ can be derived from the construction. Hence, the lemma is proved. □

Now we discuss the second case that $h(u,v) = s$ in the following lemma.

Lemma 7. Let F be any faulty set of S_N with $|F| \leq s$ and $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$. Let u and v be any two nodes in $S_N - F$ with $h(u,v) = s$. Then $d_{S_N-F}(u,v) \leq s + 2$.

Proof. Let t be the only index such that $u_{(t)} = v_{(t)}$. We consider the following three cases.

(1) Both u and v are in Z^0: From Lemma 3, there exist s disjoint paths, $P_0, P_1, \ldots, P_{s-1}$, of length s joining u to v in Z^0. Since $\kappa(S_N) = s+1$, we may set the adjacent node of u and v in Z^1 to be p and q respectively. Then $h(p,q) \leq h(u,v) = s$. By (3) of Lemma 3, there exists one path, denoted by W, of length $h(p,q)$ in Z^1 joining p to q. We set P_s as u, $p \xrightarrow{W} q$, v. Obviously P_0, P_1, $\ldots$, P_s are disjoint and of length no more than $s + 2$. Since $|F| \leq s$, $d_{S_N-F}(u,v) \leq s + 2$ holds true.

(2) Both u and v are in Z^1: Without loss of generality, we assume that $u > v$. By Lemma 3, there exist s disjoint paths, P_0, P_1, $\ldots$, P_{s-1}, of length s from u to v such that one of them is in Z^1. Let $P_0 : u, x_1, x_2, \ldots, x_{s-1}, v$ be such a path in Z^1. From statement (3) of Lemma 3, $x_i^s \notin P_j$ for $0 \leq j \leq s - 1$. Then we set P_s as $u, u^s, x_1^s, x_2^s, \ldots, x_{s-1}^s, v^s, v$. Therefore, P_0, P_1, $\ldots$, P_s are $s + 1$ disjoint paths of length at most $s + 2$ from u to v. Since $|F| \leq s$, $d_{S_N-F}(u,v) \leq s + 2$.

(3) u is in Z^0 and v is in Z^1: Let P_0, P_1, $\ldots$, P_{s-1} be the s disjoint paths of length at most $s + 1$ claimed in Lemma 4. From statement (2) of Lemma 4, each

internal node q in Z^0 of these paths has $q_{(t)} = u_{(t)}$. If any P_i is fault-free, then $d_{S_N-F}(u,v) \leq s+1$ follows. Hence, we consider the case that each P_i contains exactly one faulty node. Let $P_0 : u = x_0, x_1, x_2, \ldots, x_{s-1} = v^s, x_s = v$ be the path satisfying statement (3) of Lemma 4 such that $x_i \in Z^0$ for $0 \leq i \leq s-1$. Let the faulty node of P_0 be x_b where $1 \leq b \leq s-1$. We will find a fault-free path depending on the location of x_b.

Suppose that $x_b \neq x_{s-1}$. We construct P_0' as $u, x_1, x_2, \ldots, x_{b-1}, x_{b-1}^t, x_b^t, x_{b+1}^t, x_{b+1}, x_{b+2}, \ldots, x_s = v$. Since $x^t_{b-1(t)} \neq u_{(t)}$, $x^t_{b(t)} \neq u_{(t)}$, and $x^t_{b+1(t)} \neq u_{(t)}$, all nodes of P_0' are disjoint from those in P_i for $1 \leq i \leq s-1$. Thus, P_0' forms a fault-free path of length no more than $s+2$. Thus, $d_{S_N-F}(u,v) \leq s+2$ follows.

Now we consider that $x_b = x_{s-1}$.

Assume that there exists any internal node p of P_i, $0 \leq i \leq s-1$, satisfying $p_{(t)} \neq u_{(t)}$. Then we have $x^s_{s-2} \notin P_i$ for $0 \leq i \leq s-1$ by statement (4) of Lemma 4. We modify P_0 as $u = x_0, x_1, x_2, \ldots, x_{s-2}, x^s_{s-2}, v$ if $x^s_{s-2} \in S_N$; and as $u = x_0, x_1, x_2, \ldots, x_{s-2}, v$ otherwise. Therefore, we get a fault-free path with length no more than s.

Now we assume that each internal node p of P_i for $0 \leq i \leq s-1$ satisfies $p_{(t)} = u_{(t)}$. Obviously, $(x^t_{s-2}, x^t_{s-1}) \in E_1$ because $(x_{s-2}, x_{s-1}) \in E_1$. We construct P_0' as $u, x_1, x_2, \ldots, x_{s-2}, x^t_{s-2}, x^t_{s-1}, (x^t_{s-1})^s = v^t, v$ if $(x^t_{s-1})^s \in S_N$; and $u, x_1, x_2, \ldots, x_{s-2}, x^t_{s-2}, x^t_{s-1}, v$ otherwise. Since $x^t_{s-2(t)} \neq u_{(t)}$, $x^t_{s-1(t)} \neq u_{(t)}$, and $(x^t_{s-1})^s{}_{(t)} \neq u_{(t)}$, P_0' is disjoint from $P_1, P_2, \ldots, P_{s-1}$. Therefore, P_0' is a fault-free path of length no more than $s+2$.

Thus, $d_{S_N-F}(u,v) \leq s+2$ holds true. □

Now we discuss the last case that $h(u,v) \leq s-1$ in the following lemma.

Lemma 8. Let F be any faulty set of S_N with $|F| \leq s$. Let u and v be any two nodes in $S_N - F$ with $h(u,v) \leq s-1$. Then $d_{S_N-F}(u,v) \leq s+2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s-1\}$.

Proof. Without loss of generality, we assume that $u > v$. Here we consider the following three conditions.

(1) Both u and v are in Z^0. Since the subgraph of S_N induced by Z^0 is isomorphic to the s-diamensional hypercube. According to Lemma 2, in Z^0 there exist s disjoint paths, $P_0, P_1, \ldots, P_{s-1}$, of length at most $h(u,v)+2 \leq s+1$ joining u to v. All these paths are in Z^0. We will propose the $(s+1)$-th path P_s with length at most $s+2$ from u to v such that all internal nodes are in Z^1. We first claim that $(u^s)^{s-1}$ is a neighboring node in Z^1 of u if $u^s \notin S_N$. Assume that $u^s \notin S_N$. Since $u^s \geq N > 2^s + 2^{s-1}$, we have $u^s{}_{(s-1)} = 1$. Thus $(u^s)^{s-1} < 2^s + 2^{s-1} < N$ and hence $(u^s)^{s-1} \in Z^1$. Moreover, $(u, (u^s)^{s-1}) \in E_3$ by definition of S_N.

Now we choose one adjacent node in Z^1 for u and v respectively. We set p to be u^s if $u^s \in S_N$, and $(u^s)^{s-1}$ otherwise. Similarly, we set q to be v^s if $v^s \in S_N$, and $(v^s)^{s-1}$ otherwise. We have $h(p,q) \leq h(u,v)+1 \leq s$. By (3) of Lemma 3, there exists one path, denoted by W, in Z^1 of length $h(p,q)$ from p to q. We set P_s as $u, p \xrightarrow{W} q, v$. Obviously P_s is of length at most $s+2$. Since $|F| \leq s$, we have

$d_{S_N-F}(u,v) \leq s+2$.

(2) Both u and v are in Z^1: We let P_0, P_1, ..., P_{s-1} be the s sequences of binary strings defined by Lemma 2. Any P_i might not be able to form a path of S_N. However, we can modify these sequences to form s disjoint paths, Q_1, Q_2, ..., Q_{s-1}, joining u to v.

For $0 \leq i \leq h(u,v)-1$, we set Q_i as $u = z_{i,0}$, $z_{i,1}$, ..., $z_{i,h(u,v)} = v$ such that $z_{i,j} = x_{i,j}$ if $x_{i,j} \in S_N$ and $z_{i,j} = x^s_{i,j}$ otherwise with $1 \leq j \leq h(u,v)-1$. Similarly, for $h(u,v) \leq i \leq s-1$, we set Q_i as $u = z_{i,0}$, $z_{i,1}$, ..., $z_{i,h(u,v)+2} = v$ such that $z_{i,j} = x_{i,j}$ if $x_{i,j} \in S_N$ and $z_{i,j} = x^s_{i,j}$ otherwise with $1 \leq j \leq h(u,v)+1$. By the definition of supercubes, Q_0, Q_1, ..., Q_{s-1} form s disjoint paths of length no more than $h(u,v)+2 \leq s+1$ joining u to v.

Without loss of generality, we assume that $u > v$. By the same reason as in (3) of Lemma 3, our construction scheme of Q_0 demonstrates that (1) Q_0 is in Z^1, (2) Q_0 is of length $h(u,v)$, and (3) $p^s \in Z^0$ and p^s is not in any Q_k for $0 \leq k \leq s-1$ where p is any node of Q_0. Then, we construct another path Q_s as u, u^s, $z^s_{0,1}$, $z^s_{0,2}$, ..., $z^s_{0,h(u,v)-1}$, v^s, v. Now, Q_0, Q_1, ..., Q_s form $s+1$ disjoint paths of length no more than $s+1$. Since $|F| \leq s$, $d_{S_N-F}(u,v) \leq s+2$ ensues.

(3) u is in Z^0 and v is in Z^1: In this case both u and v^s are in Z^0. From Lemma 2, there exist s disjoint paths, $P_0, P_1, \ldots, P_{s-1}$, of length at most $h(u,v^s)+2$ joining u to v^s in Z^0. Let P_i be the path joining u to v through $(v^s)^i$ for $0 \leq i \leq s-1$, and let P'_i be the subpath of P_i that joining u to $(v^s)^i$. Then for $0 \leq i \leq s-1$, we set Q_i as $u \xrightarrow{P'_i} (v^s)^i, v^i, v$ if $v^i \in S_N$, and $u \xrightarrow{P'_i} (v^s)^i, v$ otherwise. Obviously, $Q_0, Q_1, \ldots, Q_{s-1}$ form s disjoint paths of length at most $h(u,v^s)+3 \leq s+1$ from u to v. Obviously, $d_{S_N-F}(u,v) \leq s+1$ if any Q_i is fault-free. Hence, we should concentrate our attention on the case that each Q_i has exactly one faulty internal node.

Suppose that the faulty node of Q_i is v^i for some i. It is observed from our construction that v^s is not in any Q_k for $0 \leq k \leq s-1$. We can modify Q_i into $u \xrightarrow{P'_i} (v^s)^i, v^s, v$ and obtain a fault-free path of length at most $s+1$.

Now we consider the case that the faulty node of Q_i is not v^i for every i, that is, P'_i is faulty. Note that P'_i is in Z^0 for $0 \leq i \leq s-1$. Thus $F \subset Z^0$. In the following, we construct a fault-free path of length less than $s+2$ joining u to v with all internal nodes in Z^1. Then $d_{S_N-F}(u,v) \leq s+2$ ensues.

Suppose that $u^s \in S_N$. According to Lemma 3, there exists a path R in Z^1 joining u^s to v of length $h(u^s,v) = h(u,v)-1$. Hence u, $u^s \xrightarrow{R} v$ is a fault-free path of length at most $h(u,v) \leq s-1$.

Suppose that $u^s \notin S_N$. Then $u_{(s-1)} = 1$ because $u \geq N > 2^s + 2^{s-1}$. We have $(u^{s-1})^s \in S_N$ because $(u^{s-1})^s < 2^s + 2^{s-1} < N$. And $(u,(u^s)^{s-1}) \in E_3$. From Lemma 3, there exists a path S of length no more than $h((u^s)^{s-1},v) \leq h(u,v)$ joining $(u^s)^{s-1}$ to v. Hence u, $(u^s)^{s-1} \xrightarrow{S} v$ is a fault-free path of length at most $h(u,v)+1 \leq s$.

Thus, $d_{S_N-F}(u,v) \leq s+2$. □

Combining Lemmas 5, 6, 7, and 8, we get the following theorem.

Theorem 1. $d_{\kappa-1}(S_N) = s + 2$ if $N \in \{2^{s+1} - 2^i + 1 \mid 0 \leq i \leq s - 1\}$.

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