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圖形的訊息傳遞問題

Message Transmission Problems of Graphs

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中文摘要

給定一個圖形G，以及集合$M=\{m(v):v\in V(G)\}$，M為一描述圖形G中各點擁有訊息之情形的集合。圖形G相對於M的的傳遞數是指，於最短時間內，讓圖形中全部點皆獲得所有種類之訊息，並將符號記為$t(G;M)$。傳遞過程中每個時間單位將受到下列限制:

(1)圖形上的每個點只能與自己相鄰的點交換訊息。

(2)兩個相鄰的點在每個單位時間裡至多只能交換一個訊息。

我們希望可以找到在最短的時間裡完成傳遞的方法，也就是讓圖形G中的每一個點都獲得所有種類之訊息，我們稱此類型問題為訊息傳遞問題。

在本論文中，給定一個圖形G，且圖形G中每個點的訊息只有一個，G中任兩點的訊息都不會相同，符號$t(G)$代表完成傳遞所需最少的時間單位。我們給定圖形的傳遞數的上界與下界，並且定出一套公式計算樹圖、完全二部圖及雙環網路圖的傳遞數。

關鍵詞:傳遞數、傳遞集、樹圖、完全二部圖、雙環網路

Message Transmission Problems of Graphs

Ming-Fen Yu

Abstract

Given a graph G together with a set $M=\{m(v):v\in V(G)\}$, the transmission number of G corresponding to M, denoted by $t(G;M)$, is the minimum number of time needed to complete the transmission , that is, to let all the vertices in G know all the messages in $\bigcup_{v\in V(G)} m(v)$, subject to the constraints that at each time unit, each vertex can interchange messages with all its neighbors, but the number of messages that two vertices can interchange at each time unit is at most one. We want to find the minimum number of time units required to complete the transmission, that is, to let all the vertices in G know all the messages. We call such a problem the message transmission problem. Given a graph G, the transmission number of G, denoted $t(G)$, is the minimum number of time units required to complete the transmission, under the condition that $|m(v)|=1$ for all v in $V(G)$ and $m(u)\neq m(v)$ for all distinct vertices u, v in $V(G)$, and $M=\{m(v):v\in V(G)\}$. In this thesis, we give upper and lower bounds for the transmission number of G, and give formulas to compute the transmission numbers of trees, complete bipartite graphs and double loop networks.

Keywords:transmission number, transmitting set, tree, complete bipartite graph, double loop network.

1 Introduction

Given a connected graph G, consider the following message transmission problem defined on G: Assume that each vertex v in G owns a set of messages $m(v)$ ($m(v)$ could be an empty set) at the beginning, and $M = \{m(v_i) : 1 \leq i \leq n\}$. At each time unit, each vertex can interchange messages with its neighbors. For $A \subseteq \bigcup_{v \in V(G)} m(v)$, we use $A^i_{\overrightarrow{uv}}$ to denote that at the ith time unit, u send all the messages in A to v, and call $A^i_{\overrightarrow{uv}}$ a *call*. A set of calls $B(G)$ is said to be a *transmitting set of G corresponding to M* (or, simply, a *transmitting set of G* if M need not to be specified) if for each $A^i_{\overrightarrow{uv}} \in B(G)$, $A \subseteq (\cup\{B : B^l_{\overrightarrow{wu}} \in B(G), 1 \leq l \leq i-1\}) \cup m(u)$. For a transmitting set $B(G)$ of G, we use $\Delta_{B(G)}$ to denote the number $\max\{i : A^i_{\overrightarrow{uv}} \in B(G)\}$, and for all v in $V(G)$ and all i, $1 \leq i \leq \Delta_{B(G)}$, we use $(m_i(v))_{B(G)}$ to denote the set $(\cup\{A : A^l_{\overrightarrow{uv}} \in B(G), 1 \leq l \leq i\}) \cup m(v)$. If $B(G)$ is the only transmitting set we considered, we use $m_i(v)$ to replace $(m_i(v))_{B(G)}$ for short.

A transmitting set $B(G)$ of G corresponding to M is called a *complete transmitting set of G corresponding to M* if $(m_{\Delta_{B(G)}}(v))_{B(G)} = \bigcup_{w \in V(G)} m(w)$ for all $v \in V(G)$. A complete transmitting set of G corresponding to M is called a *k-complete transmitting set of G corresponding to M* if $\Delta_{B(G)} = k$. If $B(G)$ is a k-complete transmitting set of G corresponding to M, then we use $b(G; M; B(G))$ to denote the number $\Delta_{B(G)}$. And we let $b(G; M) = \min\{b(G; M; B(G)) : B(G)$ is a complete transmitting set of G corresponding to $M\}$.

Chang et al.[1] consider the transmission problem under the restriction that at each time unit, each vertex can interchange messages with at most one of its neighbors, and the number of messages they can interchange is bounded by a constant k. They called such a problem a bounded-k broadcasting problem. This kind of problem can be viewed as a generalization of both the broadcasting problem and the gossiping problem. Most of the different variations of the broadcasting problem are bounded-1 broadcasting problem with special initial states. For example, the general broadcasting problem is equivalent to the bounded-1 broadcasting problem under the initial conditions $|m(v)| = 1$ for some $v \in V(G)$, and $m(u) = \emptyset$ for all $u \neq v$. Hedetniemi and Hedetniemi [9] considered the bounded-1 broadcasting problem with the initial conditions $m(v) = \{a\}$ for all $v \in S \subseteq V(G)$, and $m(u) = \emptyset$ for all $u \neq v$. They called this the *multiple originator broadcasting problem* of G. Chinn, Hedetniemi, and Mitchell [3] and Farley [5] introduced the *multiple message broadcasting problem* of G, which can be viewed as the bounded-1 broadcasting problem of G with the initial conditions $|m(v)| = l$ for some $l \geq 1$ and some v in $V(G)$, and $m(u) = \emptyset$ for all $u \neq v$. It is easy to see that bounded-k broadcasting problem of a graph G under the initial conditions $k \geq |V(G)|$, $|m(v)| = 1$ for all $v \in V(G)$, and $m(u) \cap m(v) = \emptyset$ for all $u \neq v$, is equivalent to the well-known gossiping problem of the graph G.

We consider the message transmission problem under the restriction that at each time unit, each vertex can interchange messages with all its neighbors, but the number of messages they can interchange is bound by one, in this thesis. To distinguish this with the general message transmission problem, we use $t(G; M; B(G))$ and $t(G; M)$ to replace the numbers $b(G; M; B(G))$ and $b(G; M)$, respectively. A message set $M = \{m(v) : v \in V(G)\}$ on G is called a *standard message set* if $|m(v)| = 1$ for all v in $V(G)$ and $m(u) \neq m(v)$ for all distinct vertices u, v in $V(G)$. Given a graph G, the *transmission number* of G, denoted $t(G)$, is the number $t(G; M)$, where M is the standard message set on G. We give upper and lower bounds for the transmission number of graphs in Section two, and give formulas to compute the transmission number of trees and complete bipartite graphs in Section three and four. And, in the last section, we present some results for the transmission number of double loop networks.

2 Preliminary

We give some basic properties for the transmission number of graphs in this section. From now on, when consider a transmitting set $B(G)$ of G, if $A^i_{\overrightarrow{uv}} \in B(G)$ and $j \in A$, we always assume that $j \notin (m_{i-1}(v))_{B(G)}$. And, when consider the graphs K_n and C_n, we always assume that $V(K_n) = V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $m(v_i) = \{i\}$ for all i, $0 \leq i \leq n-1$. The following lemma is easy to verify.

Lemma 1 *If H is a spanning subgraph of a graph G, then $t(G) \leq t(H)$.*

Lemma 2 *For any graph G with $|V(G)| = n$ and $|E(G)| = m$, $t(G) \geq \left\lceil \frac{n(n-1)}{m} \right\rceil$.*

Proof. The total number of messages need to be transmitted is $n(n-1)$. Since at most m messages can be transmitted at each time unit, the result follows. ■

Lemma 3 *For any graph G with $|V(G)| = n$ and $|E(G)| = m$, if $B(G)$ is a k-complete transmitting set of G such that $|\{A^i_{\overrightarrow{uv}} : A^i_{\overrightarrow{uv}} \in$*

$B(G)\}| = m$ *for all* i, $1 \leq i \leq r < k$, *and* $\sum_{j=r+1}^{k} |\{A^j_{\overrightarrow{uv}} : A^j_{\overrightarrow{uv}} \in B(G)\}| \geq (k-r-1)m+1$, *then* $B(G)$ *is an optimal transmitting set of* G, *and* $t(G) = \left\lceil \frac{n(n-1)}{m} \right\rceil$.

Proof. Since $|\{A^i_{\overrightarrow{uv}} : A^i_{\overrightarrow{uv}} \in B(G)\}| = m$ for all i, $1 \leq i \leq r < k$, and $\sum_{j=r+1}^{k} |\{A^j_{\overrightarrow{uv}} : A^j_{\overrightarrow{uv}} \in B(G)\}| \geq (k-r-1)m+1$, $n(n-1) \geq rm + (k-r-1)m + 1 = (k-1)m+1$. Hence $t(G) \geq \left\lceil \frac{n(n-1)}{m} \right\rceil \geq \left\lceil \frac{(k-1)m+1}{m} \right\rceil = k$ by Lemma 2. Therefore, since $B(G)$ is a k-complete transmitting set of G, $B(G)$ is an optimal transmitting set of G and $t(G) = k = \left\lceil \frac{n(n-1)}{m} \right\rceil$. ■

Lemma 4 *Given a graph* G *with* $|V(G)| = n$, *if* uv *is a cut-edge of* G *and* G_1, G_2 *are components of* $G - uv$ *containing* u, v, *respectively, then*

$$t(G) \geq n + \min\{\max_{w \in V(G_1)} d_{G_1}(u,w), \max_{w \in V(G_2)} d_{G_2}(v,w)\}.$$

Proof. Let $B(G)$ be an optimal transmitting set of G. Note that since uv is a cut-edge of G, every message in $\cup_{v \in V(G)} m(v)$ must pass through uv at some transmitting step. If in $B(G)$, a is the last message in $\cup_{v \in V(G)} m(v)$ that pass through uv, then there exists k, $k \geq n$, such that $\{a\}^k_{\overrightarrow{uv}} \in B(G)$ or $\{a\}^k_{\overrightarrow{vu}} \in B(G)$. If $\{a\}^k_{\overrightarrow{uv}} \in B(G)$, and x is a vertex in G_2 such that $d_{G_2}(v,x) = \max_{w \in V(G_2)} d_{G_2}(v,w)$, then there exists a vertex y and a number k', such that $\{a\}^{k'}_{\overrightarrow{yx}} \in B(G)$. Clearly, $k' \geq k + d_{G_2}(v,x)$. Thus $t(G) = \Delta_{B(G)} \geq k' \geq k + d_{G_2}(v,x) \geq n + \max_{w \in V(G_2)} d_{G_2}(v,w)$ in this case. By a similar argument, $t(G) \geq n + \max_{w \in V(G_1)} d_{G_1}(u,w)$ if $\{a\}^k_{\overrightarrow{vu}} \in B(G)$. Hence $t(G) \geq n + \min\{\max_{w \in V(G_1)} d_{G_1}(u,w), \max_{w \in V(G_2)} d_{G_2}(v,w)\}$. ■

Theorem 5 $t(K_n) = 2$ *for all* $n \geq 2$.

Proof. Consider the transmitting set $B(K_n)$ of K_n, defined by $B(K_n) = \{\{j\}^{1}_{\overrightarrow{v_j v_k}} : 0 \le j < k \le n-1\} \cup \{\{k\}^{2}_{\overrightarrow{v_k v_j}} : 0 \le j < k \le n-1\}$. Since each vertex v_i received all the messages in $\{0, 1, \dots, i-1\}$ at the first time unit, and received all the messages in $\{i+1, i+2, \dots, n-1\}$ at the second time unit, $B(K_n)$ is a 2-complete transmitting set of K_n. Thus $t(K_n) \le 2$. By Lemma 2, we also have $t(K_n) \ge \left\lceil \frac{n(n-1)}{|E(K_n)|} \right\rceil \ge 2$. Hence $t(K_n) = 2$ for all $n \ge 2$. ■

Theorem 6 *$t(G) \ge 2$ for any graph G with $|V(G)| \ge 2$. And equality holds only if $G = K_n$, $n \ge 2$.*

Proof. $t(G) \ge 2$ for any graph G with $|V(G)| \ge 2$ follows from Lemma 2. Note that the equality does not hold if $|E(G)| < \frac{n(n-1)}{2}$. Hence by Theorem 5, $t(G) = 2$ if and only if $G = K_n$, $n \ge 2$. ■

From now on, for convenience, we use the notation $[k]_n$ to denote the number $(k \bmod n)$.

Theorem 7 *$t(C_n) = n - 1$ for all $n \ge 3$.*

Proof. Consider the transmitting set $B(C_n)$ of C_n, defined by $B(C_n) = \{\{[j+i-1]_n\}^{i}_{\overrightarrow{v_j v_{[j+1]_n}}} : 1 \le i \le n-1,\ 0 \le j \le n-1\}$. Since each vertex v_j received the message $[j-i]_n$ at the ith time unit, $(m_{n-1}(v_i))_{B(C_n)} = \{0, 1, 2, \cdots, n-1\}$ for each vertex v_i. Hence $B(C_n)$ is an $(n-1)$-complete transmitting set of C_n, and so $t(C_n) \le n-1$. By Lemma 2, we also have $t(C_n) \ge \left\lceil \frac{n(n-1)}{|E(C_n)|} \right\rceil \ge n-1$. Hence $t(C_n) = n-1$ for all $n \ge 3$. ■

3 Trees

A *rooted tree* is a tree with one vertex chosen as the root. We use T_v to denote a tree T rooted at v. The *height* $h(T_v)$ of T_v is defined by $h(T_v) = \max\{d(v,w) : w \in V(T)\}$. A vertex w in T_v is said to be in *level* i, denoted $l(w) = i$, if $d(v,w) = i$, and we let $L_j(T_v) = \{u \in V(T) : d(v,u) = j\}$ for all j, $0 \leq j \leq h(T_v)$. If $v_i \in N(v)$, we use T_{v_i} to denote the subtree of T_v rooted at v_i. If u is a vertex in a rooted tree T_v and $u \neq v$, we use u_f to denote the father of u in T_v.

Given a tree T together with a message set $M = \{m(u) : u \in V(T)\}$, if T' is a subtree of T, then the message set restrict on T', denoted $M|_{T'}$, is defined by $M|_{T'} = \{m(u) : u \in V(T')\}$. If $B(T)$ is a transmitting set of T corresponding to M, we use $M_{B(T)}$ to denote the message set $\{m'(u) : u \in V(T)\}$, where $m'(u) = m_{\Delta_{B(T)}}(u)$ for all $u \in V(T)$.

Given a rooted tree T_v with $N(v) = \{v_1, v_2, \ldots, v_r\}$, together with a message set $M = \{m(u) : u \in V(T_v)\}$ on T_v. We say that M satisfies the *message decreasing property* if $m(w) \subseteq m(w_f)$ for all $w \in V(T_v)$, $w \neq v$, and $|m(u)| \geq |m(v)| - i$ for all $u \in L_i(T_v)$. And we say that M satisfies the k*-extended message decreasing property* if for each i, $1 \leq i \leq r$, $|m(v) \backslash m(v_i)| = k$, and the message set $M|_{T_{v_i}}$ on T_{v_i} satisfies the message decreasing property. A message set M is said to satisfies the k*-total message decreasing property* if M satisfies the k-extended message decreasing property and $m(v_i) \subseteq m(v)$ for all i, $1 \leq i \leq r$.

Lemma 8 *Given a rooted tree T_v together with a message set $M = \{m(u) : u \in V(T_v)\}$. If M satisfies the message decreasing property,* *then there exists a transmitting set $B(T_v)$, such that $\Delta_{B(T_v)} = 1$, $m_1(w) \subseteq m_1(w_f)$ for all $w \in V(T_v)$, $w \neq v$, and $|m_1(u)| \geq |m(v)| - i + 1$ for all $u \in L_i(T_v)$, $i \geq 1$.*

Proof. Choose $a_w \in m(w_f) \backslash m(w)$ for each $w \in V(T_v) - \{v\}$ with $m(w) \neq m(w_f)$, and let $B(T_v) = \{\{a_w\}^1_{\overrightarrow{w_f w}} : w \in V(T_v) - \{v\}$ and $m(w) \neq m(w_f)\}$. Then, clearly, $\Delta_{B(T_v)} = 1$, and $m_1(w) \subseteq m_1(w_f)$ for all $w \in V(T_v)$, $w \neq v$. Note that for each $w \in V(T_v) - \{v\}$ with $|m(w)| = |m(v)| - l(w)$, we have $|m_1(w)| = |m(w)| + 1 = |m(v)| - l(w) + 1$. Hence $|m_1(u)| \geq |m(v)| - i + 1$ for all $u \in L_i(T_v)$, $i \geq 1$. ■

Lemma 9 *Given a rooted tree T_v together with a message set $M = \{m(u) : u \in V(T_v)\}$. If M satisfies the message decreasing property, then $t(T_v; M) \leq h(T_v)$.*

Proof. We prove this by induction on the heights of the rooted trees. The conclusion clearly holds for all rooted trees of height 1. Suppose it holds for all rooted trees of height less than or equal to k, and let T_v be a rooted tree of height $k+1$. Let $N(v) = \{v_1, v_2, \ldots, v_r\}$. Since M satisfies the message decreasing property, by Lemma 8, there exists a transmitting set $B'(T_v)$, such that $\Delta_{B'(T_v)} = 1$, $(m_1(w))_{B'(T_v)} \subseteq m_1(w_f)_{B'(T_v)}$ for all $w \in V(T_v)$, $w \neq v$, and $|(m_1(u))_{B'(T_v)}| \geq |m(v)| - i + 1$ for all $u \in L_i(T_v)$, $i \geq 1$. Let $m'(w) = (m_1(w))_{B'(T_v)}$ for all $w \in V(T_v)$, and let $M_{B'(T_v)} = \{m'(u) : u \in V(T_v)\}$. Since for each i, $1 \leq i \leq r$, $h(T_{v_i}) \leq k$, and the message set $M_{B'(T_v)}|_{T_{v_i}}$ of T_{v_i} satisfies the message decreasing property, there exists a k-complete transmitting set $B_i(T_{v_i})$ of T_{v_i}. Let $B(T_v) = B'(T_v) \cup \{A^{j+1}_{\overrightarrow{wu}} : A^{j}_{\overrightarrow{wu}} \in B_i(T_{v_i})$ for some i, $1 \leq i \leq r\}$. Then, clearly, $B(T_v)$ is a $(k+1)$-

complete transmitting set of T_v. Hence $t(T_v; M) \leq k + 1$, which complete the induction. ■

Lemma 10 *Given a rooted tree T_v with $N(v) = \{v_1, v_2, \ldots, v_r\}$, if $M = \{m(u) : u \in V(T_v)\}$ is a massage set on T_v which satisfies the k-extended message decreasing property, then there exists a transmitting set $B(T_v)$ of T_v such that $\Delta_{B(T_v)} = l \leq k$, and the message set $M_{B(T_v)}$ satisfies the $(k-l)$-extended message decreasing property.*

Proof. We prove this by induction on l. The conclusion clearly holds for $l = 0$. Suppose it holds for all $0 \leq l < p$. Let $M = \{m(u) : u \in V(T_v)\}$ be a massage set on T_v which satisfies the k-extended message decreasing property. Choose $a_w \in m(w_f) \backslash m(w)$ for each $w \in V(T_v) - \{v\}$ with $m(w) \neq m(w_f)$, and let $B'(T_v) = \{\{a_w\}^1_{\overrightarrow{w_f w}} : w \in V(T_v) - \{v\} \text{ and } m(w) \neq m(w_f)\}$. Then, clearly, the message set $M' = M_{B'(T_v)}$ of T_v satisfies the $(k-1)$-extended message decreasing property. By the induction hypothesis, there exists a transmitting set $B''(T_v)$ of T_v corresponding to M', such that $\Delta_{B''(T_v)} = p - 1$, and the message set $M'_{B''(T_v)}$ satisfies the $(k - p)$-extended message decreasing property. If we let $B(T_v) = B'(T_v) \cup \{A^{j+1}_{\overrightarrow{wu}} : A^{j}_{\overrightarrow{wu}} \in B''(T_v)\}$, then clearly, $\Delta_{B(T_v)} = p$, and the message set $M_{B(T_v)}$ satisfies the $(k-p)$-extended message decreasing property. Hence the conclusion also holds for $l = p$. By the principle of mathematical induction, the conclusion holds for all $0 \leq l \leq k$. ■

Lemma 11 *Given a rooted tree T_v together with a message set $M = \{m(u) : u \in V(T_v)\}$ on T_v. If M satisfies the k-total message decreasing property and $|m(v)| = n$, then there exists a $(k+h(T_v)-1)$-complete transmitting set $B(T_v)$ of T_v corresponding to M, such that*

$|(m_{k-1+i}(w))_{B(T_v)}| \geq \min\{n,\, n-l(w)+i\}$ *and* $(m_{k-1+i}(w))_{B(T_v)} \subseteq (m_{k-1+i}(w_f))_{B(T_v)}$ *for all* $w \neq v$ *in* $V(T_v)$ *and all* i, $0 \leq i \leq h(T_v)$.

Proof. Since M satisfies the k-extended message decreasing property, by Lemma 10, there exists a transmitting set $B'(T_v)$ of T_v such that $\Delta_{B'(T_v)} = k-1$, and the message set $M_{B'(T_v)}$ satisfies the 1-extended message decreasing property. Thus, since M satisfies the k-total message decreasing property and $|m(v)| = n$, the message set M' on T_v, defined by $M' = \{m'(u) : u \in V(T_v)\}$, where $m'(u) = (m_{k-1}(u))_{B'(T_v)}$ for all u in $V(T_v)$, satisfies the message decreasing property. Hence by Lemma 9 and its proof, there exists a $h(T_v)$-complete transmitting set $B''(T_v)$ of T_v corresponding to M', such that for all $w \neq v$ in $V(T)$ and all i, $0 \leq i \leq h(T_v)$, $|(m'_i(w))_{B''(T_v)}| \geq \min\{n,\, n-l(w)+i\}$, and $(m'_i(w))_{B''(T_v)} \subseteq (m'_i(w_f))_{B''(T_v)}$. If we let $B(T_v) = B'(T_v) \cup \{A^{k-1+j}_{\overrightarrow{uw}} : A^{j}_{\overrightarrow{uw}} \in B''(T_v)\}$, then clearly, by the definition of $B'(T_v)$ and $B''(T_v)$, $B(T_v)$ is a $(k+h(T_v)-1)$-complete transmitting set of T_v corresponding to M, such that $|(m_{k-1+i}(w))_{B(T_v)}| \geq \min\{n, n-l(w)+i\}$ and $(m_{k-1+i}(w))_{B(T_v)} \subseteq (m_{k-1+i}(w_f))_{B(T_v)}$ for all $w \neq v$ in $V(T_v)$ and all i, $0 \leq i \leq h(T_v)$.
■

From now on, in convenience, when consider a rooted tree T_v with $N(v) = \{v_1, v_2, \ldots, v_r\}$, we always assume that $|V(T_{v_1})| \geq |V(T_{v_2})| \geq \cdots \geq |V(T_{v_r})|$, and let $n_i = |V(T_{v_i})|$ for all i, $1 \leq i \leq r$, $n = |V(T_v)| = 1 + n_1 + n_2 + \cdots + n_r$. For a rooted tree T_v together with a standard message set M, a complete transmitting set $B(T_v)$ of T_v is called a *good transmitting set* if it satisfies the following conditions.

(1) $\Delta_{B(T_v)} = n + h(T_v) - 1$.

(2) $|(m_i(v))_{B(T_v)}| \geq i+1$ for all i, $0 \leq i \leq n-1$.
(3) For all $w \neq v$ in $V(T_v)$ and all i, $0 \leq i \leq h(T_v)$, $|(m_{n-1+i}(w))_{B(T_v)}| \geq \min\{n,\, n-l(w)+i\}$, and $(m_{n-1+i}(w))_{B(T_v)} \subseteq (m_{n-1+i}(w_f))_{B(T_v)}$.

Lemma 12 *Every rooted tree T_v with a standard message set $M = \{m(u) : u \in V(T_v)\}$ has a good transmitting set.*

Proof. We prove this by induction on the order k of the rooted tree. The conclusion clearly holds for $k = 1, 2$. Suppose it holds for all $2 \leq k < n$, and T_v be a rooted tree with n vertices. Let $N(v) = \{v_1, v_2, \ldots, v_r\}$. For each i, $1 \leq i \leq r$, let $T^*_{v_i}$ be the subtree induced by $V(T_{v_i}) \cup \{v\}$ which is rooted at v. Since each of the rooted subtrees $T_{v_1}, T^*_{v_2}, T^*_{v_3}, \ldots, T^*_{v_r}$ has fewer than n vertices, by the induction hypothesis, there is a good transmitting set $B_1(T_{v_1})$ of T_{v_1}, and for each i, $2 \leq i \leq r$, there is a good transmitting set $B_i(T^*_{v_i})$ of $T^*_{v_i}$.

Let $m(v_1) = a_1$. For each j, $1 \leq j \leq n_1 - 1$, choose an element a_{j+1} from $m_j(v_1) - \{a_1, a_2, \ldots, a_j\}$ if $a_1, a_2, \ldots, a_j$ are all determined. Let $B_{11}(T_{v_1}) = \{A^j_{\overrightarrow{uw}} \in B_1(T_{v_1}) : 1 \leq j \leq n_1\}$. For all i, $2 \leq i \leq r$, let $B_{i1}(T^*_{v_i}) = \{A^j_{\overrightarrow{uw}} \in B_i(T^*_{v_i}) : 1 \leq j \leq n_i\}$, and let M'_i be the message set on $T^*_{v_i}$ defined by $\{m'(w) : w \in V(T^*_{v_i})\}$, where $m'(w) = (m_{n_i}(w))_{B_i(T^*_{v_i})}$ and $m'(v) = \{a_1, a_2, \ldots, a_{n_1}\}$. Since for each i, $2 \leq i \leq r$, $B_i(T^*_{v_i})$ is a good transmitting set of $T^*_{v_i}$, M'_i is a message set on $T^*_{v_i}$ which satisfies the n_1-extended message decreasing property. By Lemma 10, for each i, $2 \leq i \leq r$, there exists a transmitting set $B_{i2}(T^*_{v_i})$ of $T^*_{v_i}$ corresponding to M'_i such that $\Delta_{B_{i2}(T^*_{v_i})} = n_1 - n_i$, and the message set $(M'_i)_{B_{i2}(T^*_{v_i})}$ satisfies the n_i-extended message decreasing property. Let $B'_{i2}(T^*_{v_i}) = \{A^{n_i+j}_{\overrightarrow{uw}} : A^j_{\overrightarrow{uw}} \in B_{i2}(T^*_{v_i})\}$, and let $B'(T_v)$ be the transmitting set of T_v corresponding to M defined by $B'(T_v) = B_{11}(T_{v_1}) \cup (\cup_{i=2}^{r} B_{i1}(T_{v_i}^*)) \cup (\cup_{i=2}^{r} B'_{i2}(T_{v_i}^*)) \cup \{\{a_j\}_{\overrightarrow{v_1 v}}^{j} : 1 \leq j \leq n_1\}$. If M'' is the message set on T_v defined by $M'' = \{m''(w) : w \in V(T_v)\}$, where $m''(w) = (m_{n_1}(w))_{B'(T_v)}$, then, by the definition of $B_{11}(T_{v_1})$, $B_{i1}(T_{v_i}^*)$ and $B'_{i2}(T_{v_i}^*)$, M'' satisfies the $(n - n_1)$-total message decreasing property and $|m(v)| = n$. Hence by Lemma 11, there exists a $(n-n_1+h(T_v)-1)$-complete transmitting set $B''(T_v)$ of T_v corresponding to M'' such that $|(m_{n-n_1-1+i}(w))_{B(T_v)}| \geq \min\{n, n - l(w) + i\}$ and $(m_{n-n_1-1+i}(w))_{B(T_v)} \subseteq (m_{n-n_1-1+i}(w_f))_{B(T_v)}$ for all $w \neq v$ in $V(T_v)$ and all i, $0 \leq i \leq h(T_v)$.

Now, let $B(T_v) = B'(T_v) \cup \{A_{\overrightarrow{uw}}^{n_1+j} : A_{\overrightarrow{uw}}^{j} \in B''(T_v)\}$. Then, by the definition of $B'(T_v)$ and $B''(T_v)$, $B(T_v)$ is an α-complete transmitting set of T_v corresponding to M, where $\alpha = n_1 + (n - n_1 + h(T_v) - 1) = n + h(T_v) - 1$, $|(m_i(v))_{B(T_v)}| \geq i + 1$ for all i, $0 \leq i \leq n - 1$, and $|(m_{n-1+i}(w))_{B(T_v)}| \geq \min\{n, n - l(w) + i\}$, $(m_{n-1+i}(w))_{B(T_v)} \subseteq (m_{n-1+i}(w_f))_{B(T_v)}$ for all $w \neq v$ in $V(T_v)$ and all i, $0 \leq i \leq h(T_v)$. Hence $B(T_v)$ is a good transmitting set of T_v, and so the conclusion also holds for $k = n$. By the principle of mathematical induction, every rooted tree T_v with a standard message set $M = \{m(u) : u \in V(T_v)\}$ has a good transmitting set. ■

Theorem 13 *For any tree T with $|V(T)| = n \geq 2$.*

$$t(T) = n + \mathrm{rad}(T) - 1.$$

Proof. Choose a vertex v in the center, and consider the rooted tree T_v. By lemma 12, there exists a $(n+h(T_v)-1)$-complete transmitting set $B(T_v)$ of T_v. Hence $t(T) \leq \Delta_{B(T_v)} = n + h(T_v) - 1$. Since v is a vertex in the center, $h(T_v) = \mathrm{rad}(T)$, thus $t(T) \leq n + \mathrm{rad}(T) - 1$. To prove the lower bound, consider a vertex w in $V(T_v)$ with $d(v, w) = \mathrm{rad}(T)$, and let T_u be a rooted subtree of T_v containing w which is rooted at a vertex u in $N(v)$. Then, by Lemma 4, $t(T) \geq n + d(u,w) = n + \text{rad}(T) - 1$. Hence $t(T) = n + \text{rad}(T) - 1$ for any tree T with $|V(T)| = n \geq 2$. ■

Corollary 14 $t(P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1$ *for all* $n \geq 2$.

Combining Lemma 1 and Theorem 13, we have

Theorem 15 *For any graph* G *with* $|V(G)| \geq 2$, $t(G) \leq |V(G)| + \text{rad}(G) - 1$.

If uv is a cut-edge of G, then we use $\alpha(u,v)$ to denote the number $\min\{\max_{w \in V(G_1)} d_{G_1}(u,w), \max_{w \in V(G_2)} d_{G_2}(v,w)\}$, where G_1, G_2 are components of $G - uv$ containing u, v, respectively. By lemma 4 and Lemma 12, we have

Theorem 16 *If* G *is a graph with cut-edges, then* $t(G) = |V(G)| + \max\{\alpha(u,v) : uv \text{ is a cut-edge of } G\} - 1$.

4 Complete bipartite graphs

We study the transmission number of complete bipartite graphs in this section. For convenience, when consider the complete bipartite graph $K_{m,n}$, $m \geq n$, we always assume that $V(K_{m,n}) = \{v_i : 0 \leq i \leq m+n-1\}$, $E(K_{m,n}) = \{v_i v_j : 0 \leq i \leq m-1, m \leq j \leq m+n-1\}$, and $m(v_i) = \{i\}$ for all i, $0 \leq i \leq m+n-1$.

Theorem 17 $t(K_{m,n}) = \left\lceil \frac{(m+n)(m+n-1)}{mn} \right\rceil$ *for all* $m \geq n \geq 1$.

Proof. Let $d = \gcd(m,n)$ (the greatest common divisor of m,n), let $m - 1 = nq + r$, where $0 \leq r \leq n-1$, and let $r' = [mr]_n$. Let

$$\begin{aligned} B^1(K_{m,n}) &= \{\{j\}^1_{\overrightarrow{v_j v_{m+i}}} : 0 \leq j \leq m-1, 0 \leq i \leq n-1\}, \\ B^2(K_{m,n}) &= \{\{m+i\}^2_{\overrightarrow{v_{m+i} v_j}} : 0 \leq j \leq m-1, 0 \leq i \leq n-1\}, \end{aligned}$$

and for all l, $3 \leq l \leq q+2$, let

$$B^l(K_{m,n}) = \{\{[(l-3)n+j+i+1]_m\}^{l}_{\overrightarrow{v_{m+i}v_j}} : 0 \leq j \leq m-1, 0 \leq i \leq n-1\}.$$

Consider the following cases:

Case 1. $mr + n(n-1) \leq mn$.

In this case, let

$$\begin{aligned} & B^{q+3}(K_{m,n}) \\ = \ & \{\{[nq + i + \left\lfloor \frac{i}{m} \right\rfloor + 1]_m\}^{q+3}_{\overrightarrow{v_{m+[i+\lfloor \frac{id}{mn} \rfloor]_n} v_{[i]_m}}} : 0 \leq i \leq mr - 1\} \\ \cup \ & \{\{m + [r' + i + \left\lfloor \frac{rd}{n} \right\rfloor + \left\lfloor \frac{i}{n} \right\rfloor + 1]_n\}^{q+3}_{\overrightarrow{v_{[i]_m} v_{m+[r'+i+\lfloor \frac{rd}{n} \rfloor]_n}}} : 0 \leq i \leq n^2 - n - 1\}, \end{aligned}$$

and let $B(K_{m,n}) = \cup_{i=1}^{q+3} B^i(K_{m,n})$.

Case 2. $mr + n(n-1) > mn$.

In this case, let

$$\begin{aligned} B^{q+3}(K_{m,n}) &= \{\{[qn + j + i + 1]_m\}^{q+3}_{\overrightarrow{v_{m+i}v_j}} : 0 \leq j \leq m-1, 0 \leq i \leq r-1\}, \\ B^{q+4}(K_{m,n}) &= \{\{m + [j + i + 1]_n\}^{q+4}_{\overrightarrow{v_j v_{m+i}}} : 0 \leq j \leq n-2, 0 \leq i \leq n-1\}, \end{aligned}$$

and let $B(K_{m,n}) = \cup_{i=1}^{q+4} B^i(K_{m,n})$.

It is easy to verify that for each of the two cases above, $B(K_{m,n})$ is a complete transmitting set of $K_{m,n}$. Since $|\{A^i_{\overrightarrow{uv}} : A^i_{\overrightarrow{uv}} \in B(K_{m,n})\}| = mn$ for all i, $1 \leq i \leq q+2$, and $|\{A^{q+3}_{\overrightarrow{uv}} : A^{q+3}_{\overrightarrow{uv}} \in B(K_{m,n})\}| + |\{A^{q+4}_{\overrightarrow{uv}} : A^{q+4}_{\overrightarrow{uv}} \in B(K_{m,n})\}| \geq mn + 1$ when $mr + n(n-1) > mn$, by Lemma 3, $B(K_{m,n})$ is an optimal transmitting set of $K_{m,n}$, and $t(K_{m,n}) = \left\lceil \frac{(m+n)(m+n-1)}{mn} \right\rceil$ in either case. ■

5 Double loop network

A *double-loop network* $\overrightarrow{D_n}(a, b)$ with n being positive integer, $0 < a < n$, $0 < b < n$, and $a \neq b$ can be viewed as a directed graph with n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and $2n$ directed edges of the form $\overrightarrow{v_i v_{[i+a]_n}}$ and $\overrightarrow{v_i v_{[i+b]_n}}$, referred to as a-links and b-links. The underlying graph of the directed graph $\overrightarrow{D_n}(a;b)$ is denoted $D_n(a,b)$. We study the transmission number of $D_n(1,b)$ in this section. Note that $D_n(1,b) \simeq D_n(1,n-b)$ for all b, $2 \le b \le n-2$. Hence, when consider the graph $D_n(1,b)$, we always assume that $2 \le b \le \left\lfloor \frac{n}{2} \right\rfloor$.

For convenience, when consider the standard message set $M = \{m(v_i) : 0 \le i \le n-1\}$ of $D_n(1,b)$, we always assume that $m(v_i) = \{i\}$ for all i, $0 \le i \le n-1$. A message set $M = \{m(v_i) : 0 \le i \le n-1\}$ on $D_n(1,b)$ is said to satisfy the *p-condition* if $m(v_i) = \{0,1,2,\ldots,n-1\} \backslash \{[i+1]_n, [i+2]_n, \ldots, [i+p]_n\}$ for all i, $0 \le i \le n-1$.

Lemma 18 *If M is a message set on $D_n(1,b)$ which satisfies the p-condition, where $0 \le p \le 2b-3$, then $t(D_n(1,b);M) \le \left\lceil \frac{p}{2} \right\rceil$.*

Proof. To prove this, we only need to show that there exists a $\left\lceil \frac{p}{2} \right\rceil$-complete transmitting set of $D_n(1,b)$ corresponding to M. We prove this by induction on p. The conclusion clearly holds for $p = 0$. For $p = 1$, let $B(D_n(1,b)) = \{\{[i+2]_n\}^1_{\overrightarrow{v_i v_{[i+1]_n}}} : 0 \le i \le n-1\}$. Then clearly, $B(D_n(1,b))$ is a 1-complete transmitting set of $D_n(1,b)$ corresponding to M. Suppose the conclusion holds for all $1 \le p < k \le 2b-3$, and M is a message set on $D_n(1,b)$ which satisfies the k-condition. Let

$$\begin{aligned} B'(D_n(1,b)) &= \{\{[i+k+1]_n\}^1_{\overrightarrow{v_i v_{[i+1]_n}}} : 0 \le i \le n-1\} \\ &\cup \{\{[i+k+b-1]_n\}^1_{\overrightarrow{v_i v_{[i+b]_n}}} : 0 \le i \le n-1\}, \end{aligned}$$

and let $M' = \{m'(v_i) : 0 \le i \le n-1\}$, where $m'(v_i) = (m_1(v_i))_{B'(D_n(1,b))}$. Then, it is easy to see that M' is a message set on $D_n(1,b)$ which satisfies the $(k-2)$-condition. By the induction hypothesis, there exists a $\left\lceil \frac{k-2}{2} \right\rceil$-complete transmitting set $B''(D_n(1,b))$ of $D_n(1,b)$ corresponding to M'. If we let $B(D_n(1,b)) = B'(D_n(1,b)) \cup \{A^{l+1}_{\overrightarrow{v_iv_j}} : A^{l}_{\overrightarrow{v_iv_j}} \in B''(D_n(1,b))\}$, then, clearly, $B(D_n(1,b))$ is a $\left\lceil \frac{k}{2} \right\rceil$-complete transmitting set of $D_n(1,b)$ corresponding to M. Hence the conclusion also holds for $p = k$. By the principle of mathematical induction, the conclusion holds for all p, $0 \le p \le 2b-3$. ■

Theorem 19 *$t(D_n(1,b)) = \left\lceil \frac{n-1}{2} \right\rceil$ for all $n \ge 5$, $2 \le b \le \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let $n-1 = (2b-2)q + r$, where $0 \le r \le 2b-3$, and let $j_b = [j-1]_{b-1} + \left\lfloor \frac{j-1}{b-1} \right\rfloor (2b-2)$ for all positive integer j. For each i, $0 \le i \le n-1$, let

$$\begin{aligned} B^i(D_n(1,b)) &= \{\{i\}^{j}_{\overrightarrow{v_{[i+j_b]_n} v_{[i+j_b+1]_n}}} : 1 \le j \le (b-1)q\} \\ &\cup \{\{i\}^{j}_{\overrightarrow{v_{[i+j_b]_n} v_{[i+j_b+b]_n}}} : 1 \le j \le (b-1)q\}, \end{aligned}$$

and let $B'(D_n(1,b)) = \cup_{i=0}^{n-1} B^i(D_n(1,b))$. By the definition of $B^i(D_n(1,b))$, it is easy to see that all the vertex v_j, $j \in \{i, [i+1]_n, [i+2]_n, \ldots, [i+(2b-2)q]_n\}$, owns the message i after the ith transmission step under the transmitting set $B'(D_n(1,b))$. Thus $(m_{(b-1)q}(v_i))_{B'(D_n(1,b))} = \{i, [i+r+1]_n, [i+r+2]_n, \ldots, [i+r+(2b-2)q]_n\} = \{0,1,2,\ldots,n-1\} \backslash \{[i+1]_n, [i+2]_n, \ldots, [i+r]_n\}$ for all i, $0 \le i \le n-1$.

Now, if we let $m'(v_i) = (m_{(b-1)q}(v_i))_{B'(D_n(1,b))}$ for all i, $0 \le i \le n-1$, and let $M' = \{m'(v_i) : 0 \le i \le n-1)\}$, then the message set M' on $D_n(1,b)$ satisfies the r-condition. Hence by Lemma 18, there exists a $\left\lceil \frac{r}{2} \right\rceil$-complete transmitting set $B''(D_n(1,b))$ of $D_n(1,b)$ corresponding to M'. If we let $B(D_n(1,b)) = B'(D_n(1,b)) \cup \{A^{l+(b-1)q}_{\overrightarrow{v_iv_j}} : A^{l}_{\overrightarrow{v_iv_j}} \in B''(D_n(1,b))\}$, then, $B(D_n(1,b))$ is a complete transmitting set of $D_n(1,b)$ corresponding to M with $\Delta_{B(D_n(1,b))} = (b-1)q + \left\lceil \frac{r}{2} \right\rceil = (b-1)q + \left\lceil \frac{n-(2b-2)q-1}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$. Hence $t(D_n(1,b)) \le \left\lceil \frac{n-1}{2} \right\rceil$. Since

$|E(D_n(1,b))| = 2n$, by Lemma 2, we also have $t(D_n(1,b)) \geq \lceil \frac{n-1}{2} \rceil$. Thus $t(D_n(1,b)) = \lceil \frac{n-1}{2} \rceil$ for all $n \geq 5$, $2 \leq b \leq \lfloor \frac{n}{2} \rfloor$. ■

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