國立政治大學應用數學系

碩士學位論文

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Tropical Meromorphic Functions and Their Application on Difference Equations

熱帶亞純函數及其在差分方程之應用

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中華民國一百年七月一日

誌　謝

時間過得真快，轉眼間在政大待了三年了。有點不敢相信自己可以準時畢業，記得三月的時候還在擔心論文寫不出來，找指導教授蔡炎龍老師問說如果論文來不及寫完，能不能帶論文實習。幸好蔡老師有一篇熱帶幾何的論文可以套用Latex格式，也幸好之前碩一時有修過蔡老師開的數學程式設計專題，從中學到了如何用Latex打論文，還有beamer簡報系統，動態幾何軟體Geogebra，這些在我論文生成的過程中全部都派上了用場，此外也學到了Python，Flex等等，這麼多的幸好讓我能夠準時的完成這篇論文。感謝蔡炎龍老師帶我進入熱帶幾何這個有趣的新領域，雖然自己只能窺得一小部份，感謝陳天進老師在之前meeting還有口試時提供寶貴的意見，感謝台大數學系的助理教授劉瓊如博士特地前來參加我的論文口試。

這三年在政大過得很充實，有許多美好的回憶。一二年級時住在自強九舍D221，剛好同學陳家盛住在隔壁D222，一起拼實變期中期末考，實變資格考，一起跑步，吃大餐，唱歌，到處遊玩…… ，感謝這三年能夠認識你，讓碩士生活變得更精彩，但唯一遺憾的就是這三年我還是交不到女朋友。政大應數所有一個很棒的研究室，整個和樂融融，常常會有一些聚餐，活動等，增進師生間以及學長姐學弟妹間的感情。山上宿舍環境清幽，空氣新鮮；校內一元公車交通便利，司機親切有人情。畢業在即，卻有幾分不捨。

相對於大學時自己的茫然不上進，研究所這三年比較知道要做什麼，不僅修完所上規定的30學分，還另外修習了教育學程，而且這三年來都有在補習班打工或是家教，所得足夠支付學雜費及自己的生活所需。碩一時接了蔡老師財管系大一微積分助教，碩三時接了陳老師的高微助教，由於自己的高微底子不是很好，幾乎每個星期二晚上都準備演習課到早上，感謝盈穎一起搭檔高微助教，分擔了許多工作。雖然辛苦，但撐過之後覺得自己的分析能力提升了不少，對於論文的撰寫有很大的助益。

最後再次感謝三位口試老師在口試期間給予的意見與補充，才能讓本論文更加完善。也感謝家人支持我在當完兵之後繼續回學校就讀研究所，之後8月要在板橋高中實習半年，接著考教檢教甄，希望可以一切順遂。

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Abstract

In this thesis, we find the formula of tropical meromorphic function by giving finite number of roots and poles (with multiplicities). We also find a simple formula for tropical periodic function by giving finite number of roots and poles (with multiplicities) during a period $[0, T)$. We then discuss all cases of the tropical meromorphic solution functions of first-order linear difference equation. At last, we provide a tropical approximated function of a given continuous function. We hope it is helpful in solving the tropical meromorphic solution functions of a given difference equation.

摘　要

在這篇論文中，一個熱帶亞純函數(tropical meromorphic function)若給定有限個零根(roots)與極點(poles)還有它們的重數(multiplicities)，我們證明了這個熱帶亞純函數的存在與唯一性。另外，一個熱帶週期函數(tropical periodic function)若給定一個週期區間內的有限個零根與極點還有它們的重數，我們也找到了這個熱帶週期函數的一個簡單表達式。接著，給定一個一階線性差分方程(first-order linear difference equation)，我們討論了各種情況下的所有熱帶亞純函數解的表達式。最後，對於連續函數我們提供了一個它的熱帶近似函數，希望對於解差分方程的熱帶亞純函數解時能有所助益。

Contents

Chapter 1

Introduction

Tropical geometry is a relatively new area in mathematics, first developed in the 1980s by Imre Simon, a mathematician and computer scientist from Brazil. It is a piecewise-linear version of algebraic geometry, the algebraic structure we work on is the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with addition and multiplication defined by:

$$x \oplus y = \max\{x, y\},$$
$$x \odot y = x + y.$$

In words, the tropical sum of two numbers is their maximum, and the tropical product of two numbers is their usual sum. These two operations both are associative and commutative, and the multiplication is distributive with respect to the addition.

The identity elements for the tropical operations are $0_{\mathbb{T}} = -\infty$ for addition and $1_{\mathbb{T}} = 0$ for multiplication. Observe that such a structure is not a ring, since not all elements have tropical additive inverses. For example, the function $x \oplus 1 = 2$ has no solutions in $\mathbb{T}$.

We can define the tropical semiring in another way. For example, $x \oplus y = \min\{x, y\}$, $x \odot y = x + y$. This is min-plus tropical semiring. However, the work on min-plus tropical semiring is similar to the work on max-plus tropical semiring. We will work on max-plus tropical semiring in this thesis, and only with one variable.

For basic tropical geometry, one can see book [5], and [1], [2], [6], [7]. In [1] and [2], Grigg and Manwaring give an elementary proof of the Fundamental Theorem of Algebra for polynomials over the rational tropical semi-ring, and they provide a simple algorithm for factoring tropical polynomials of a single variable.

The main references of this thesis is [3] and [4], both are introducing tropical nevenlinna theory and applications on ultra-discrete equations, which is an equation, written in terms of addition and max operators, such that both dependent and independent variables take only discrete values. Here we do not mention the tropical nevenlinna theory, and emphasizing on tropical meromorphic function with integer slope or real slope (extended). Given finite number of roots and poles (with multiplicities respectively) of a tropical meromorphic function, we can represent the function immediately just like general rational funtion by giving roots and poles (Chapter 3). And in Chapter 4, we define a function

$$\pi_a(x) := \max\{(1-a)([x]-x), a([-x]-(-x))\}.$$

to generate the tropical T-periodic functions by giving roots and poles (with multiplicities respectively) during an period $[0, T)$. In chapter 5, we want to obtain a representation of all the extended tropical meromorphic solution functions of first-order linear difference equation $y(x+1) = ay(x) + b$, for $a, b \in \mathbb{R}$.

At last, we provide a tropical approximated function of a given continuous function $f(x)$ by

$$(f([x+1]) - f([x]))(x-[x]) + f([x]).$$

It may be helpful in dealing with ultra-discrete equations, at least for first-order linear difference equations.

Chapter 2

Background

2.1 Arithmetic

Definition 2.1. We define operators $\oplus : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}$, and $\odot : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}$ by:

$$x \oplus y := \max\{x, y\},$$
$$x \odot y := x + y.$$

Remark 2.1. For each $x \in \mathbb{T}$,
$x \oplus (-\infty) = -\infty$, hence $-\infty$ is the additive identity element.
$x \odot 0 = x$, hence 0 is the multiplicative identity element.

Example 2.1.

$2 \oplus 3 = \max\{2, 3\} = 3;\ 2 \oplus 2 = \max\{2, 2\} = 2;\ 2 \oplus (-\infty) = \max\{2, -\infty\} = 2.$
$2 \odot 3 = 2 + 3 = 5;\ 2 \odot 0 = 2 + 0 = 2;\ 2 \odot (-\infty) = 2 + (-\infty) = -\infty.$

Definition 2.2. For each integer n, define

$$x^{\odot n} := n \times x.$$

And define tropical division to be their usual subtraction:

$$x \oslash y := x - y.$$

Moreover, define

$$\bigoplus_{i=1}^{n} a_i := \max\{a_1, a_2, \cdots, a_n\}$$
$$\bigodot_{i=1}^{n} a_i := a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

The definition of semiring are given formally as the following:

Definition 2.3 (Semiring)**.** A semiring is a set S equipped with two algebraic operations, called addition and multiplication, such that:

(i) The addition and multiplicatioon are associative.

(ii) The addition is commutative.

(iii) The multiplication is distributive with respect to the addition.

Remark 2.2. $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is a semiring.

2.2 Some Equalities and Inequalities

Under these new operations, there are some funny equalities and inequalities.

(i) $(b \oslash a) \oplus (d \oslash c) = (b \odot c \oplus a \odot d) \oslash (a \odot c)$
It is an analogue of

$$\frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}.$$

(ii) $x^{\odot(1_{\mathbb{T}} \oslash n)} = 1_{\mathbb{T}} \oslash (x^{\odot n}) = -n \times x$
It is an analogue of

$$x^{\frac{1}{n}} = \frac{1}{x^n}.$$

(iii) The Freshman's dream come true in tropical arithmetic:

$$(x \oplus y)^{\odot n} = x^{\odot n} \oplus y^{\odot n}$$

(iv) Tropical Cauchy inequality holds:

$$(\bigoplus_{i=1}^{n} a_i^{\odot 2}) \odot (\bigoplus_{i=1}^{n} b_i^{\odot 2}) \geq (\bigoplus_{i=1}^{n} (a_i \odot b_i))^{\odot 2}$$

i.e.

$$\max\{a_1, \cdots, a_n\} + \max\{b_1, \cdots, b_n\} \geq \max\{a_1 + b_1, \cdots, a_n + b_n\} \quad (2.1)$$

(v) for $a_{ij} \in \mathbb{R} \cup \{-\infty\}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$,

$$\bigoplus_{j=1}^{n} (\bigoplus_{i=1}^{m} a_{ij}) = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{n} a_{ij}) = \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n} a_{ij} \quad (2.2)$$

It is an analogue of

$$\sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij}) = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij}) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} a_{ij}$$

Proof. (iv): Let $a_r = \max\{a_1, a_2, \cdots, a_n\}$, $b_s = \max\{b_1, b_2, \cdots, b_n\}$, then $a_r \geq a_k$ for each $k = 1, 2, \cdots, n$. And $b_s \geq b_k$ for each $k = 1, 2, \cdots, n$. Hence, $a_r + b_s \geq a_k + b_k$ for each $k = 1, 2, \cdots, n$. Thus, (2.1) holds. □

Remark 2.3. Let $A = \{r | a_r = \max\{a_1, a_2, \cdots, a_n\}\}$, $B = \{s | b_s = \max\{b_1, b_2, \cdots, b_n\}\}$, then the equal sign of (2.1) holds if and only if $A \cap B \neq \phi$.

Chapter 3

Tropical Meromorphic Functions

In this Chapter, we will derive the equation of tropical meromorphic functions with prescribed roots and poles. In section 3.1, we first introduce the tropical polynomials in one variable, and then tropical meromorphic functions in section 3.2, if the slope is allowed to be non-integer, it is extended tropical meromorphic functions in section 3.3.

3.1 Tropical Polynomials in One Variable

Definition 3.1. In general, the tropical polynomials in one variable are

$$f(x) = \bigoplus_{i=0}^{n} (a_i \odot x^{\odot i}) = a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot (n-1)} \oplus \cdots\cdots \oplus a_1 \odot x \oplus a_0.$$

Here the coefficients $a_n, a_{n-1}, \cdots\cdots, a_0$ are real numbers and n is a non-negative integer. When evaluating this function in classical arithmetic, we have

$$f(x) = \max\{a_n + nx, a_{n-1} + (n-1)x, \cdots\cdots, a_1 + x, a_0\}.$$

Remark 3.1. It is clear that this function $f(x)$ maps $\mathbb{T}$ to $\mathbb{T}$ has the following three important properties:

(i) $f(x)$ is continuous.

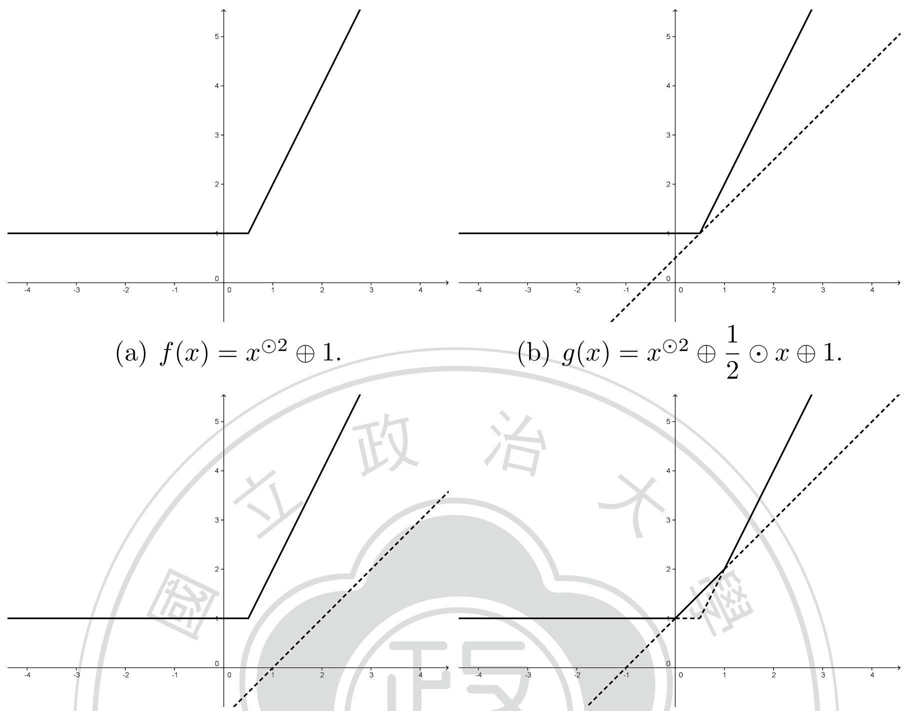
(ii) $f(x)$ is piecewise-linear, where the number of pieces is finite.

(iii) $f(x)$ is convex, i.e., $f(\frac{x+y}{2}) \leq \frac{1}{2}(f(x) + f(y))$ for all $x, y \in \mathbb{R}$.

Example 3.1. $f(x) = x^{\odot 2} \oplus 1$ is a tropical polynomial, the coefficient of $x^{\odot 2}$ is 0, and the coefficient of x is $-\infty$, it is a missing term, and the constant term is 1. We will see $f(x) = \max\{2x, 1\}$, i.e., $f(x) = 1$ if $x \leq \frac{1}{2}$; and $f(x) = 2x$ if $x > \frac{1}{2}$. (Figure 3.1(a))

Example 3.2. $g(x) = x^{\odot 2} \oplus \frac{1}{2} \odot x \oplus 1$ (Figure 3.1(b)), $h(x) = x^{\odot 2} \oplus (-1) \odot x \oplus 1$ (Figure 3.1(c)). From the graph, we see $f(x)$ in Example 3.1 and $g(x)$, $h(x)$ are different tropical polynomials which define the same function. And $p(x) = x^{\odot 2} \oplus 1 \odot x \oplus 1$ (Figure 3.1(d)) defines a different function of them. From Figure 3.1, we know that if the coefficient of x is less or equal to $\frac{1}{2}$, then they all define the same function; and if the coefficient of x is larger than $\frac{1}{2}$, then it will be a different function, it seems that $\frac{1}{2}$ is the largest coefficient of x to unchange the function. So we have the following definition.

Definition 3.2. Given two polynomials $g(x)$ and $h(x)$, if $g(c) = h(c)$ for all $c \in \mathbb{R}$, then $g(x)$ and $h(x)$ are functionally equivalent.



(a) $f(x) = x^{\odot 2} \oplus 1$. (b) $g(x) = x^{\odot 2} \oplus \frac{1}{2} \odot x \oplus 1$.

(c) $h(x) = x^{\odot 2} \oplus (-1) \odot x \oplus 1$. (d) $p(x) = x^{\odot 2} \oplus 1 \odot x \oplus 1$.

Figure 3.1:

In classical, if $g(x)$ and $h(x)$ are functionally equivalent, if and only if $g(x)$ and $h(x)$ are the same polynomials. But in tropical geometry, functional equivalence does not mean the functions obtained from the same polynomials.

Although different tropical polynomials might define the same function, we can adjust each of coefficient to be maximal, and to get a maximally represented tropical polynomial which still represent the same function. One can see section 3 in [8] to know how to derive the maximally represented tropical polynomial of a given tropical polynomial. And the similar result can be found in [2] and [1] on min-plus tropical polynomial, it is call least-coefficient tropical polynomial in [2] and [1]. Moreover, each maximally represented tropical polynomial (or least-coefficient tropical polynomial on min-plus tropical polynomial) can be factorized into a product of linear terms, it is the Fundamental Theorem of Tropical Algebra, you can see the proof in [2] on min-plus form.

3.2 Tropical Meromorphic Functions

Definition 3.3. A tropical rational function is a function of the form

$$R(x) = (a_m \odot x^{\odot m} \oplus \cdots \oplus a_1 \odot x \oplus a_0) \oslash (b_n \odot x^{\odot n} \oplus \cdots \oplus b_1 \odot x \oplus b_0)$$

where m and n are non-negative integers,and $a_i, b_j \in \mathbb{T}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$.

Definition 3.4. A continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is said to be a tropical meromorphic function on $\mathbb{R}$ if both one-sided derivatives are integers at each point $x \in \mathbb{R}$.

In this thesis, if $f(x)$ is a tropical meromorphic function on $\mathbb{R}$, we call $f(x)$ an $\mathbb{R}$-tropical meromorphic function. Note that every tropical polynomial is an $\mathbb{R}$-tropical meromorphic function, and so is tropical rational function. Moreover, the word "piecewise" implies that there are finitely many pieces in every bounded interval.

Example 3.3. $f(x) = (x^{\odot 2} \oplus 0) \oslash x$ (Figure 3.2(a)) is a tropical rational function. In fact, it is equal to $|x|$. It shows in the following.

$$|x| = x^+ + x^-$$

$$x = x^+ - x^-$$

where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}$.
Summation of the above two equalities, it gets

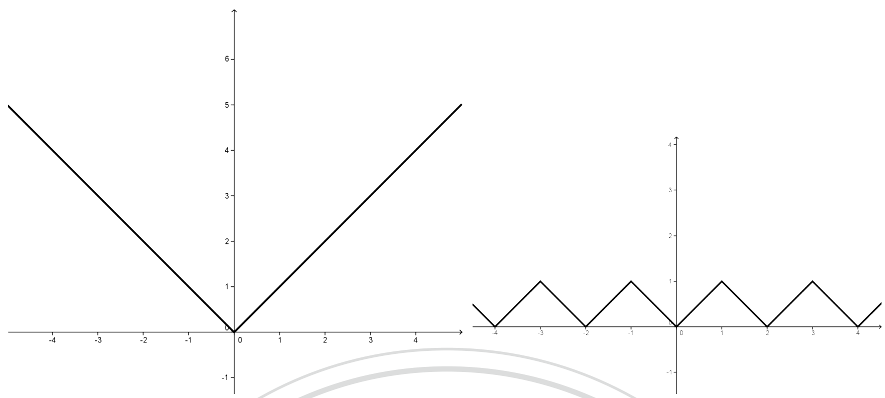
$$|x| = 2x^+ - x = 2\max\{x, 0\} - x = \max\{2x, 0\} - x = (x^{\odot 2} \oplus 0) \oslash x$$

Example 3.4. Figure 3.2(b) shows a function which is an $\mathbb{R}$-tropical meromorphic function but not a rational function.

Note that in tropical functions, those non-differentiable points are special, it comes the following definitions.

Definition 3.5. For each $x \in \mathbb{R}$, let

$$w_f(x) = \lim_{\epsilon \to 0^+} (f'(x + \epsilon) - f'(x - \epsilon))$$



(a) $f(x) = (x^{\odot 2} \oplus 0) \oslash x = |x|$.

(b) $f(x) = |x|, -1 \leq x \leq 1; f(x+2) = f(x)$ for each $x \in \mathbb{R}$..

Figure 3.2:

(i) If $w_f(x) > 0$, then x is called a root of f with multiplicity $w_f(x)$.

(ii) If $w_f(x) < 0$, then x is called a pole of f with multiplicity $-w_f(x)$.

(iii) If $w_f(x) = 0$, then x is called an ordinary point of f.

Note that all roots and poles of f constitute the support of $w_f(x)$.

Consider the function $f(x) = x \oplus a = \max\{x, a\}$, which has a root at a. From the point of view of algebra and factorization, it is perhaps more natural to think of a as the negative of the root. However, there is no subtraction in the tropical semi-ring and the definition given above is the natrual one from the point of view of geometry.

Definition 3.6. If f is tropical meromorphic on $\mathbb{R}$ and $f'(x) = m$ for all $x < x_0$, for some constant $m \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$, then we say that f is tropical meromorphic on $\mathbb{T}$. Define $w_f(-\infty) = m$ and

(i) If $m > 0$, then the point $-\infty$ is called a root of order m.

(ii) If $m < 0$, then the point $-\infty$ is called a pole of order $-m$.

(iii) If $m = 0$, then the point $-\infty$ is called an ordinary point.

If $f(x)$ is a tropical meromorphic function on $\mathbb{T}$, we call $f(x)$ an $\mathbb{T}$-tropical meromorphic function.

We know that in traditional rational functions, given roots and poles and their multiplicities, we can find a suitable rational functions.

Example 3.5. Given two roots -1 and -2 with multiplicities 2 and 5 respectively, and one pole -3 with multiplicity 7, and no other roots or poles. Then the suitable function is $f(x) = \dfrac{k(x+1)^2(x+2)^5}{(x+3)^7}$, k is a nonzero constant.

We have a similar tropical result in the following. Before it, we need some lemmas at first.

Lemma 3.1 (The Linear Property of $w_f(x)$)**.** *Suppose $f_i(x)$ are $\mathbb{R}$-tropical meromorphic functions, $i = 1, 2, \cdots, n$, and $c_1, c_2, \cdots, c_n$ are constants in $\mathbb{R}$. Let*

$$f(x) = \sum_{i=1}^{n} c_i f_i(x),$$

then

$$w_f(x) = \sum_{i=1}^{n} c_i w_{f_i}(x)$$

Proof. If x is not a root or pole of any $f_i(x)$, then $f'(x) = \sum_{i=1}^{n} c_i f_i'(x)$, hence, for any $x \in \mathbb{R}$, and small ϵ,

$$\begin{aligned}
w_f(x) &= \lim_{\epsilon\to 0}(f'(x+\epsilon) + f'(x-\epsilon)) \\
&= \lim_{\epsilon\to 0}\Big(\sum_{i=1}^{n} c_i f_i'(x+\epsilon) - \sum_{i=1}^{n} c_i f_i'(x-\epsilon)\Big) \\
&= \lim_{\epsilon\to 0}\Big(\sum_{i=1}^{n} c_i (f_i'(x+\epsilon) - f_i'(x-\epsilon))\Big) \\
&= \sum_{i=1}^{n} c_i \lim_{\epsilon\to 0}(f_i'(x+\epsilon) - f_i'(x-\epsilon)) \\
&= \sum_{i=1}^{n} c_i w_{f_i}(x)
\end{aligned}$$

□

Lemma 3.2. *Two* $\mathbb{R}$*-tropical meromorphic functions* $f(x)$ *and* $g(x)$ *satisfy the relation*

$$w_f(x) - w_g(x) \equiv 0$$

if and only if $f(x) - g(x)$ *is a linear function on* $\mathbb{R}$*. That is,* $f(x) - g(x) = Ax + B$ *for some constants* A *and* B*.*

Proof. Since $f(x)$ and $g(x)$ are $\mathbb{R}$-tropical meromorphic functions, let $h(x) = f(x) - g(x)$, then by Lemma 3.1,

$$w_h(x) = w_f(x) - w_g(x) = 0$$

Therefore, $h'(x+) = h'(x-)$ holds for any $x \in \mathbb{R}$. Then $w_h(x) \equiv 0$ implies $h'(x)$ is a constant on $\mathbb{R}$. Hence, $h(x) = f(x) - g(x)$ is a linear function on $\mathbb{R}$. Coversely, of course that the w of any linear function vanishes identically. □

Theorem 3.3. *Given* $c_1 < c_2 < \cdots < c_n \in \mathbb{T}$*, and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{Z}$*, there exist a tropical rational function* f *such that* $w_f(c_i) = \alpha_i$ *for* $i = 1, 2, \cdots, n$ *and* $w_f(x) = 0$ *for* $x \neq c_i, i = 1, 2, \cdots, n$*.(i.e.,* c_i *is a root if* $\alpha_i > 0$ *and a pole if* $\alpha_i < 0$*, and for any* $x \neq c_1, c_2, \cdots, c_n$ *is an ordinary point.)*

In fact,

$$f(x) = k \odot (\bigodot_{i=1}^{n} (x \oplus c_i)^{\odot \alpha_i})$$

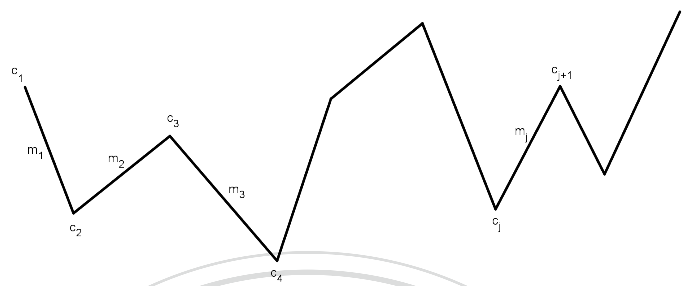
with k *a constant in* $\mathbb{R}$ *is the unique function satisfying the condition, the uniqueness is up to the constant* k*.(Note that* c_1 *might be* $-\infty$*.)*

First prove the existence of f.
Define $m_0 = 0$ and

$$m_j = \alpha_1 + \alpha_2 + \cdots + \alpha_j = \sum_{i=1}^{j} \alpha_i$$

to be the slope of $f(x)$ in (c_j, c_{j+1}), then

$$\begin{aligned} f(x) &= k \odot (\bigodot_{i=1}^{n} (x \oplus c_i)^{\odot \alpha_i}) \\ &= k + \sum_{i=1}^{n} \alpha_i \max\{x, c_i\} \\ &= k + \sum_{i=1}^{n} (m_i - m_{i-1}) \max\{x, c_i\} \end{aligned}$$



Proof.

Figure 3.3:

If $x \in (c_j, c_{j+1})$ for some $j = 1, 2, \cdots, n$, then

$$
\begin{aligned}
f(x) &= k + \sum_{i=1}^{j}(m_i - m_{i-1}) \max\{x, c_i\} + \sum_{i=j+1}^{n}(m_i - m_{i-1}) \max\{x, c_i\} \\
&= k + \sum_{i=1}^{j}(m_i - m_{i-1})x + \sum_{i=j+1}^{n}(m_i - m_{i-1})c_i \\
&= k + m_j x + \sum_{i=j+1}^{n}(m_i - m_{i-1})c_i
\end{aligned}
$$

Therefore,

$$f'(x) = m_j \quad for \quad x \in (c_j, c_{j+1}). \tag{3.1}$$

If $x = c_i$ for some $i = 2, 3, \cdots, n$, then

$$
\begin{aligned}
w_f(c_i) &= \lim_{\epsilon \to 0^+} (f'(c_i + \epsilon) - f'(c_i - \epsilon)) \\
&= \lim_{\epsilon \to 0^+} f'(c_i + \epsilon) - \lim_{\epsilon \to 0^+} f'(c_i - \epsilon)
\end{aligned}
$$

Since $c_i + \epsilon \in (c_i, c_{i+1})$ and $c_i - \epsilon \in (c_{i-1}, c_i)$, by (3.1) it follows

$$w_f(c_i) = m_i - m_{i-1} = \alpha_i$$

If $c_1 \neq -\infty$, let $c_0 = -\infty$,$-\infty$ is a ordinary point implies that $f'(x) = 0$ for all $x < x_0$ for some $x_0 \in \mathbb{R}$. Since there is no other non-ordinary point between c_0 and c_1, so

$$f'(x) = 0 \quad for \quad all \quad x < c_1. \tag{3.2}$$

Hence,

$$w_f(c_1) = \lim_{\epsilon \to 0^+} (f'(c_i + \epsilon) - f'(c_i - \epsilon)) = m_1 - 0 = \alpha_1$$

If $c_1 = -\infty$, by (3.1) it follows

$$f'(x) = m_1 \quad for \quad x \in (c_1, c_2)$$

i.e.,

$$f'(x) = m_1 \quad for \quad all \quad x < c_2, \tag{3.3}$$

then

$$w_f(-\infty) = m_1 = \alpha_1$$
$$w_f(c_1) = \alpha_1$$

It completes the proof of existence.

Now prove the uniqueness of f.
If $g(x)$ is another function satisfying $w_g(c_i) = \alpha_i$, $i = 1, 2, \cdots, n$; and $w_g(x) = 0$ for any $x \neq c_1, c_2, \cdots, c_n$. Then we will prove the uniqueness by proving that $f(x) - g(x)$ is a constant.

case 1: $c_1 \neq -\infty$

On this case,the least non-ordinary point is c_1, hence $c_0 = -\infty$ is an ordinary point, then $w_g(-\infty)$=0, it follows $g'(x) = 0$ for all $x < c_1$, and by (3.2) we know $f'(x) = 0$ for all $x < c_1$. Let $h(x) = f(x) - g(x)$, then

$$h'(x) = f'(x) - g'(x) = 0 - 0 = 0 \qquad for \qquad x < c_1. \tag{3.4}$$

And by Lemma 3.1,

$$\begin{aligned} w_h(x) &= w_f(x) - w_g(x) \\ &= \begin{cases} 0 - 0 = 0 & \text{if } x \neq c_1, c_2, \cdots, c_n \\ w_f(c_i) - w_g(c_i) = 0 & \text{if } x = c_i \text{ for some } i = 1, 2, \cdots, n. \end{cases} \end{aligned}$$

Lemma 3.2 implies $h(x) = Ax + B$, for some constant A and B. And equation (3.4) implies $A = 0$, therefore, $h(x) = f(x) - g(x)$ is a constant.

case 2: $c_1 = -\infty$. (i.e., $w_g(-\infty) = \alpha_1$)

It follows $g'(x) = \alpha_1$ for all $x < c_2$, and by (3.3), $f'(x) = \alpha_1$ for all $x < c_2$. Then $h'(x) = f'(x) - g'(x) = \alpha_1 - \alpha_1 = 0$ for all $x < c_2$. Hence, $w_h(c_1) = 0$, and $h(x)$ is a constant for all $x < c_2$.

If $x = c_i$ for $i = 1, 2, \cdots, n$, then

$$w_h(c_i) = w_f(c_i) - w_g(c_i) = \alpha_i - \alpha_i = 0$$

holds for the same reason in case 1.

And if x is an ordinary point,

$$w_h(x) = 0$$

holds for the same reason in case 1 too.

It completes the proof of uniqueness. □

It follows the next theorem immediately.

Theorem 3.4. *If a function is $\mathbb{R}$-tropical meromorphic, then it is tropical rational if and only if it has a finite number of roots and poles.*

Proof. By definition it is clear that if f is tropical rational then it has a finite number of roots and poles; conversely, if a tropical meromorphic function f has a finite number of roots and poles then f is a tropical rational function by Theorem 3.3. □

So the Figure 3.2(b) in Example 3.3 with infinite roots and poles implies that it is not a tropical rational function.

Example 3.6. Let $g(x)$ be a polynomial with root $\frac{1}{2}$ of multiplicities 2. Applying Theorem 3.3, one solution is the following polynomial:

$$\begin{aligned} g(x) &= (x \oplus \frac{1}{2})^{\odot 2} = (x \oplus \frac{1}{2}) \odot (x \oplus \frac{1}{2}) \\ &= x^{\odot 2} \oplus x \odot \frac{1}{2} \oplus \frac{1}{2} \odot x \oplus \frac{1}{2} \odot \frac{1}{2} \\ &= x^{\odot 2} \oplus \frac{1}{2} \odot x \oplus 1 \end{aligned}$$

We find that it is exactly the function in Figure 3.1(b). And since

$$\max\{2x, 1\} = \frac{\max\{2x, 1\} + \max\{2x, 1\}}{2} \geq \frac{2x + 1}{2} = x + \frac{1}{2},$$

then

$$g(x) = x^{\odot 2} \oplus 1$$

But $x^{\odot 2} \oplus \frac{1}{2} \odot x \oplus 1$ is the maximally represented tropical polynomial of $g(x)$.

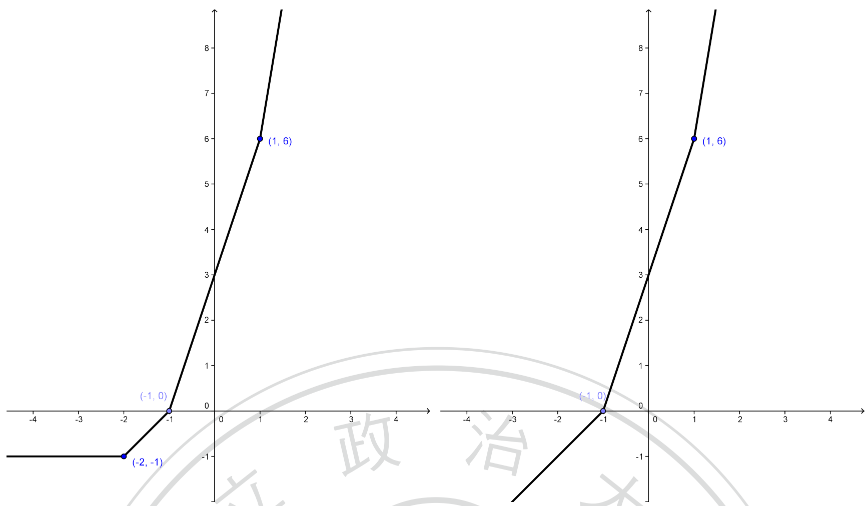
Example 3.7. $f_1(x) = (x \oplus (-2)) \odot (x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 3}$ is a tropical polynomial with roots -2, -1, 1 of multiplicities 1, 2, 3 respectively.(Figure 3.4(a)). $f_2(x) = (x) \odot (x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 3}$ is a tropical polynomial with roots $-\infty$, -1, 1 of multiplicities 1, 2, 3 respectively. (Figure 3.4(b))

Example 3.8. In Example 3.3, we have show that $|x|$ is a tropical rational function, now we can get the expression of $|x|$ in tropical rational form immediately. From the graph of $|x|$, we know $|x|$ have a pole $-\infty$ of multiplicity 1, and a root 0 of multiplicity 2. Therefore, $|x| = k \odot (x \oplus 0)^{\odot 2} \oslash (x \oplus (-\infty))$ for some constant k, $k = 0$ follows from $|0| = 0$, one can omit $-\infty$ to get the same expression

$$|x| = (x \oplus 0)^{\odot 2} \oslash x = (x^{\odot 2} \oplus 0) \oslash x$$

Example 3.9. Find the tropical rational function $f(x)$ with non-ordinary points $-\infty$, -1, 0, 1, and $w_f(-\infty) = 1$, $w_f(-1) = -2$, $w_f(0) = 2$, $w_f(1) = -2$. That is, $-\infty$, 0 are roots of multiplicities 1, 2 respectively; and -1, 1 are poles of multiplicities 2, 2 respectively. Therefore,

$$\begin{aligned} f(x) &= k \odot (x \oplus (-\infty))^{\odot 1} \odot (x \oplus (-1))^{\odot(-2)} \odot (x \oplus 0)^{\odot 2} \odot (x \oplus 1)^{\odot(-2)} \\ &= k \odot (x \oplus (-\infty)) \odot (x \oplus 0)^{\odot 2} \oslash ((x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 2}) \\ &= k \odot (x) \odot (x \oplus 0)^{\odot 2} \oslash ((x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 2}) \end{aligned}$$



(a) $f_1(x)$.   (b) $f_2(x)$.

Figure 3.4:

If we hope $f(0) = 0$, then $k = 2$.

$$f(x) = 2 \odot (x) \odot (x \oplus 0)^{\odot 2} \oslash ((x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 2})(Figure 3.5(a))$$

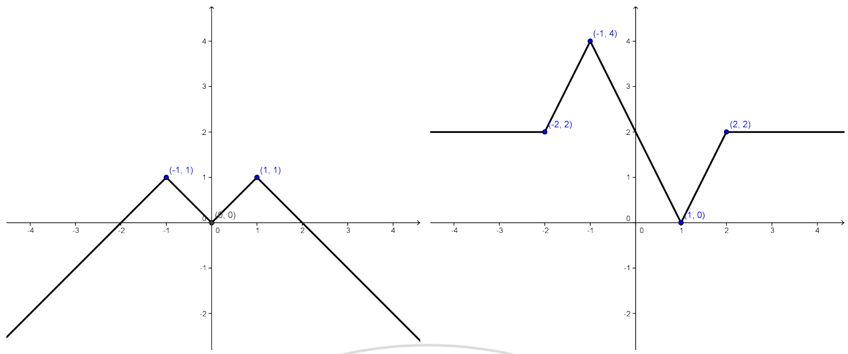
Example 3.10. Given Figure 3.5(b), find the tropical rational expression $g(x)$ of it. Since $g'(x) = 0$ for all $x < -2$, then $-\infty$ is an ordinary point, the slope change at $-2, -1, 1, 2$, and $w_g(-2) = 2-0 = 2$, $w_g(-1) = -2-2 = -4$, $w_g(1) = 2-(-2) = 4$, $w_g(2) = 0-2 = -2$.Then

$$\begin{aligned} g(x) &= k \odot (x \oplus (-2))^{\odot 2} \odot (x \oplus (-1))^{\odot(-4)} \odot (x \oplus 1)^{\odot 4} \odot (x \oplus 2)^{\odot(-2)} \\ &= (k \odot (x \oplus (-2))^{\odot 2} \odot (x \oplus 1)^{\odot 4}) \oslash ((x \oplus (-1))^{\odot 4} \odot (x \oplus 2)^{\odot 2}) \end{aligned}$$

$k = 2$ follows from $g(0) = 2$.

3.3 Extended Tropical Meromorphic Functions

If we allow a tropical meromorphic function to have real slopes, this kind of tropical meromorphic function is called an extended tropical meromorphic function. So we have the following definitions.



(a) $f(x) = 2 \odot (x) \odot (x \oplus 0)^{\odot 2} \oslash ((x \oplus (-1))^{\odot 2} \odot (x \oplus 1)^{\odot 2})$.

(b) $g(x)$.

Figure 3.5:

Definition 3.7. An extended tropical polynomial is of the form:

$$f(x) = a_n \odot x^{\odot r_n} \oplus a_{n-1} \odot x^{\odot r_{n-1}} \oplus a_1 \odot x^{\odot r_1} \oplus a_0,$$

where $r_i \in \mathbb{R}^+$ for all $i = 1, 2, \ldots, n$; $a_i \in \mathbb{T}, i = 0, 1, 2, \cdots, n$.

Definition 3.8. An extended tropical rational function is of the form:

$$f(x) = a_n \odot x^{\odot r_n} \oplus a_{n-1} \odot x^{\odot r_{n-1}} \oplus a_1 \odot x^{\odot r_1} \oplus a_0,$$

where $r_i \in \mathbb{R} \setminus \{0\}$ for all $i = 1, 2, \ldots, n$; $a_i \in \mathbb{T}, i = 0, 1, 2, \cdots, n$.

Definition 3.9. A continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is said to be a extended tropical meromorphic function on $\mathbb{R}$ if both one-sided derivatives are real at each point $x \in \mathbb{R}$

Theorem 3.3 will hold too in the extended tropical rational functions, we restate the theorem in the following after slight modifications.

Theorem 3.5. *Given $c_1 < c_2 < \cdots < c_n \in \mathbb{T}$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$, there exist a extended tropical rational function f such that $w_f(c_i) = \alpha_i$ for $i = 1, 2, \cdots, n$ and $w_f(x) = 0$ for $x \neq c_i, i = 1, 2, \cdots, n$.(i.e., c_i is a root if $\alpha_i > 0$ and a pole if $\alpha_i < 0$, and for any $x \neq c_1, c_2, \cdots, c_n$ is an ordinary point.)*

In fact,

$$f(x) = k \odot (\bigodot_{i=1}^{n} (x \oplus c_i)^{\odot \alpha_i})$$

with k a constant in $\mathbb{R}$ is the unique function satisfying the condition, the uniqueness is up to the constant k.(Note that c_1 may be $-\infty$.)

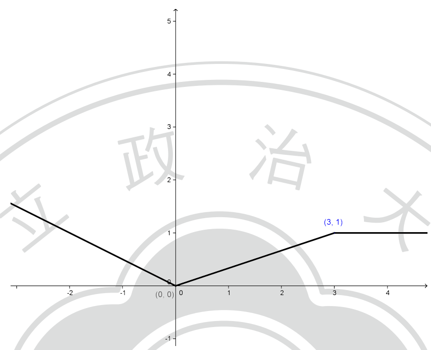


Figure 3.6: $h(x) = 1 \odot (x \oplus (-\infty))^{\odot(-\frac{1}{2})} \odot (x \oplus 0)^{\odot \frac{5}{6}} \odot (x \oplus 3)^{\odot(-\frac{1}{3})}$.

Example 3.11. Find the extended tropical rational expression $h(x)$ of Figure 3.6. Since $h'(x) = -\frac{1}{2}$ for all $x < 0$, then $-\infty$ is a pole with $w_h(-\infty) = -\frac{1}{2}$, and $w_h(0) = \frac{1}{3} - (-\frac{1}{2}) = \frac{5}{6}$, $w_h(3) = 0 - \frac{1}{3} = -\frac{1}{3}$. Therefore,

$$h(x) = k \odot (x \oplus (-\infty))^{\odot(-\frac{1}{2})} \odot (x \oplus 0)^{\odot \frac{5}{6}} \odot (x \oplus 3)^{\odot(-\frac{1}{3})}$$

$k = 1$ follows from $h(0) = 0$. we check this is really the true function of Figure 3.6 in the following.

$$h(x) = \begin{cases} 1 + (-\frac{1}{2})x + \frac{5}{6} \times 0 + (-\frac{1}{3}) \times 3 & x \leq 0 \\ 1 + (-\frac{1}{2})x + \frac{5}{6} \times x + (-\frac{1}{3}) \times 3 & 0 \leq x \leq 3 \\ 1 + (-\frac{1}{2})x + \frac{5}{6} \times x + (-\frac{1}{3}) \times x & x \geq 3 \end{cases}$$

$$= \begin{cases} -\frac{1}{2} \times x & x \leq 0 \\ \frac{1}{3} \times x & 0 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

This is exactly the function of Figure 3.6.

Chapter 4

Tropical Periodic Functions

In this chapter, we examine the extended tropical periodic meromorphic functions. It will be shortened to be called tropical periodic function here. Given an extended tropical meromorphic function f on an interval $[0, T)$, with $f(0) = \lim_{x \to T^-} f(x)$, if we extend f to be a periodic function with period T, that is, $f(x+T) = f(x)$, can we find a formula to express it? It is "Yes", let's see how to do it in the following.

In [4], Laine and Tohge consider a tropical meromorphic 1-period function defined by

$$\begin{aligned}\pi^{(a,b)}(x) &= \frac{1}{a+b}\max\{a(x-[x]), -b((x-[x]-1))\}\\ &= \{(a(x-[x])) \oplus (-b(x-[x])-1)\}^{\odot \frac{1}{a+b}}\end{aligned}$$

for arbitrary parameters $a, b \in \mathbb{R}^-$. Laine and Tohge make conclusion that any non-constant tropical meromorphic 1-periodic function $f(x)$ can be represented as an $\mathbb{R}$-linear combination of such function $\pi^{(a,b)}(x)$.

But here we provide a better function

$$\begin{aligned}\pi_a(x) &= \max\{(1-a)([x]-x), a([-x]-(-x))\}\\ &= ([x]-x)^{\odot(1-a)} \oplus ([-x]-(-x))^{\odot a}\end{aligned}$$

These functions will be used to generate the tropical periodic functions in section 4.2 and 4.3. We first prove some properties of $\pi_a(x)$ in section 4.1.

4.1 Generating Functions of Tropical Periodic Functions

Definition 4.1 (Tropical Periodic Function)**.** If $f(x)$ is an extended tropical meromorphic function on $\mathbb{R}$, and there is a real number $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, then we say $f(x)$ is an extended tropical periodic meromorphic function.

It will be shortened to be called tropical periodic function here.

Let's consider the following extended tropical meromorphic 1-periodic function.

Definition 4.2. Define

$$\pi_a(x) := \max\{(1-a)([x]-x), a([-x]-(-x))\}$$

for $0 \leq a < 1$. (Figure 4.1)

The number a control the skewness of $f(x)$. When $a = 0$, the graph is just the x-axis. It has the following properties.

Remark 4.1.

(a)

$$\pi_a(n) = 0 \quad for \quad each\, n \in \mathbb{Z}. \tag{4.1}$$

(b)

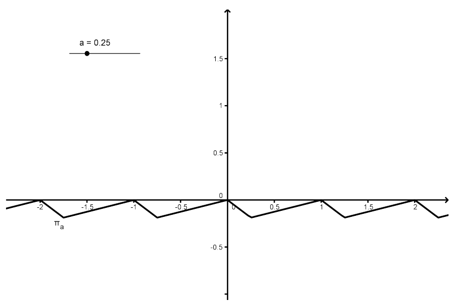
$$a - a^2 = \pi_a(a) \leq \pi_a(x) \leq 0 \quad for \quad 0 \leq x < 1. \tag{4.2}$$

(c)

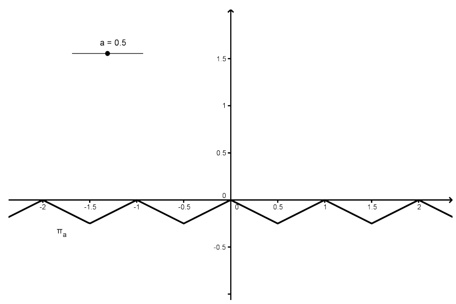
$$w_{\pi_a}(a) = \begin{cases} 1 & a \neq 0 \\ 0 & a = 0 \end{cases} \tag{4.3}$$

(d)

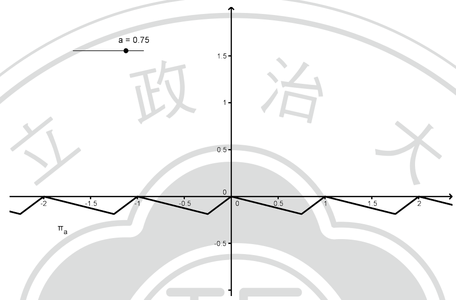
$$For \quad n \in \mathbb{Z}, \; w_{\pi_a}(n) = \begin{cases} -1 & a \neq 0 \\ 0 & a = 0 \end{cases} \tag{4.4}$$



(a) $a = 0.25$.



(b) $a = 0.5$.



(c) $a = 0.75$.

Figure 4.1: $\pi_a(x) = \max\{(1-a)([x]-x), a([-x]-(-x))\}$

(e)

$$w_{\pi_a}(x) = 0 \quad if \;\; x \;\; is \;\; not \;\; an \;\; integer \;\; and \;\; x \neq a. \tag{4.5}$$

Proof. We verify these properties by a straightforward computation.

(a)

$$\begin{aligned}\pi_a(n) &= \max\{(1-a)([n]-n), a([-n]-(-n))\} \\ &= \max\{(1-a)\times 0, a \times 0\} = \max\{0,0\} = 0\end{aligned}$$

(b)

$$\begin{aligned}\pi_a(a) &= \max\{(1-a)([a]-a), a([-a]-(-a))\} \\ &= \max\{(1-a)(-a), a(-1+a)\} \\ &= \max\{a^2-a, a^2-a\} = a^2-a\end{aligned}$$

If $0 \le x < 1$,

$$\pi_a(x) = \max\{(1-a)(0-x), a(-1+x)\} \tag{4.6}$$
$$= \max\{ax - x, ax - a\} \tag{4.7}$$
$$= \begin{cases} (a-1)x & x \le a \\ ax - a & x > a \end{cases} \tag{4.8}$$

$(a-1)x \le 0$ and $ax - a = a(x-1) \le 0$, hence, $\pi_a(x) \le 0$ for $0 \le x < 1$. If $x \le a$, $\pi_a(x) = (a-1)x \ge (a-1)a = a^2 - a$; if $x > a$, $\pi_a(x) = ax - a > a \times a - a = a^2 - a$, hence, $\pi_a(x) \ge a^2 - a$ for $0 \le x < 1$. That is, $a^2 - a$ is the minimum of $\pi_a(x)$.

(c) By (4.8), $w_{\pi_a}(a) = \lim_{\epsilon\to 0^+}(\pi'_a(a+\epsilon) - \pi'_a(a-\epsilon)) = \lim_{\epsilon\to 0^+}(a-(a-1)) = 1$ if $a \neq 0$. And since $\pi_0(x) = \max\{[x]-x, 0\} = 0$ for each x, then $w_{\pi_0}(0) = 0$.

(d) If $a \neq 0$,

$$\begin{aligned} w_{\pi_a}(n) &= \lim_{\epsilon\to 0^+}(\pi'_a(n+\epsilon) - \pi'_a(n-\epsilon)) \\ &= \lim_{\epsilon\to 0^+}(\pi'_a(0+\epsilon) - \pi'_a(1-\epsilon)) \\ &= \lim_{\epsilon\to 0^+}((a-1)-a) = -1. \end{aligned}$$

If $a = 0$,

$$\begin{aligned} \pi_0(x) &= \max\{[x]-x, 0\} = 0 \quad \forall x. \\ w_{\pi_0}(n) &= \lim_{\epsilon\to 0^+}(\pi'_0(n+\epsilon) - \pi'_0(n-\epsilon)) = 0 - 0 = 0. \end{aligned}$$

(e) It is clear that the support of w_f on $[0,1)$ only possible at $x = 0$ or $x = a$, so the conclusion holds. □

4.2 $\mathbb{R}$-linear Combination of Tropical 1-Periodic Functions

We first introduce some lemmas before proving the main theorem in this chapter.

Lemma 4.1. *Let* $f(x) = \pi_a(\frac{x}{T})$, $T > 0$, *then* $w_f(x) = \frac{1}{T} w_{\pi_a}(\frac{x}{T})$.

Proof. If $\frac{x}{T}$ is not a support of w_{π_a}, by Chain Rule, $f'(x) = \frac{1}{T}\pi'_a(\frac{x}{T})$.

$$\begin{aligned} w_f(x) &= \lim_{\epsilon \to 0^+} (f'(x+\epsilon) - f'(x-\epsilon)) \\ &= \lim_{\epsilon \to 0^+} (\frac{1}{T}\pi'_a(\frac{x+\epsilon}{T}) - \frac{1}{T}\pi'_a(\frac{x-\epsilon}{T})) \\ &= \frac{1}{T} \lim_{\epsilon \to 0^+} (\pi'_a(\frac{x}{T} + \frac{\epsilon}{T}) - \pi'_a(\frac{x}{T} - \frac{\epsilon}{T})) \\ &= \frac{1}{T} w_{\pi_a}(\frac{x}{T}) \end{aligned}$$

□

Lemma 4.2. *A non-constant tropical periodic function has as many roots and poles in a period interval, counting multiplicities. That is,*

$$\sum_{c \in (supp\, w_f) \bigcap [0,T)} w_f(c) = 0$$

for any tropical T*-periodic function* $f(x)$*. More generally,*

$$\sum_{c \in (supp\, w_f) \bigcap [k,T+k)} w_f(c) = \sum_{k \le c < T+k} w_f(x) = 0$$

for any real constant k*.*

Proof. Let $f(x)$ be a tropical T-periodic function, $\{c_i \mid i = 1, 2, \cdots, k\}$ is the support of $w_f(x)$ on $[0, T)$ with $0 \le c_1 < c_2 < \cdots < c_k < T$. Hence, $c_0 := c_k - T < c_1 < c_2 < \cdots < c_k < c_1 + T := c_{k+1}$, Since $f(x)$ is linear in each (c_i, c_{i+1}), then $f'(c_i^+) = f'(c_{i+1}^-), i = 0, 1, 2, \cdots, k$. In particular,

$$f'(c_k^+) = f'(c_{k+1}^-) = f'((c_1 + T)^-) = f'(c_1^-)$$

$$
\begin{aligned}
\sum_{c \in (supp\, w_f) \bigcap [0,T)} w_f(c) &= \sum_{i=1}^{k} w_f(c_i) \\
&= \sum_{i=1}^{k} \lim_{\epsilon \to 0^+} (f'(c_i + \epsilon) - f'(c_i - \epsilon)) \\
&= \sum_{i=1}^{k} (f'(c_i^+) - f'(c_i^-)) \\
&= (f'(c_1^+) - f'(c_1^-)) + (f'(c_2^+) - f'(c_2^-)) + \cdots + (f'(c_k^+) - f'(c_k^-)) \\
&= f'(c_k^+) - f'(c_1^-) = 0
\end{aligned}
$$

□

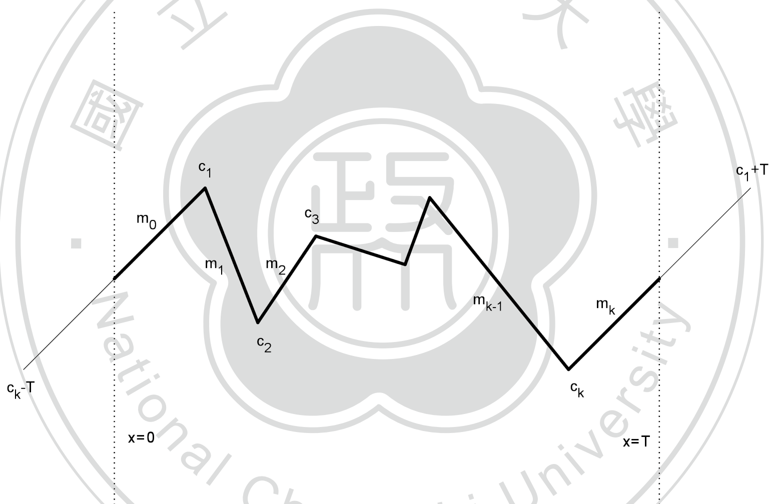


Figure 4.2:

Geometrically, it says that the sum of all changes of the slope is zero during a period.(Figure 4.2)

Theorem 4.3 (ℝ-linear Combination Of Tropical 1-Periodic Functions)**.** *Let $f(x)$ be an tropical 1-periodic function on $\mathbb{R}$, $\{c_i \mid i = 1, 2, \cdots, k\}$ is the support of $w_f(x)$ on $[0, 1)$ with $0 \leq c_1 < c_2 < \cdots < c_k < 1$. Then*

$$f(x) = (\sum_{i=1}^{k} w_f(c_i) \pi_{c_i}(x)) + f(0)$$

Proof. Define

$$\hat{f}(x) = (\sum_{i=1}^{k} w_f(c_i)\pi_{c_i}(x)) + f(0)$$

If $c_1 = 0$,

$$\begin{aligned}
w_{\hat{f}}(c_1) &= w_{\hat{f}}(0) && (4.9)\\
&= \sum_{i=1}^{k} w_f(c_i) w_{\pi_{c_i}}(0) && (4.10)\\
&= 0 \times w_f(c_1) + \sum_{i=2}^{k} w_f(c_i) \times (-1) && (4.11)\\
&= (-1) \times (\sum_{i=2}^{k} w_f(c_i)) && (4.12)\\
&= (-1) \times (-w_f(c_1)) && (4.13)\\
&= w_f(c_1) && (4.14)
\end{aligned}$$

(4.10) follows by (4.1), (4.11) follows by (4.4), and (4.13) follows by Lemma 4.2.
If $c_1 \neq 0$, then $c_j \neq 0, j = 1, 2, \cdots, k$.

$$\begin{aligned}
w_{\hat{f}}(c_j) &= \sum_{i=1}^{k} w_f(c_i) w_{\pi_{c_i}}(c_j) && (4.15)\\
&= w_f(c_j) w_{\pi_{c_j}}(c_j) && (4.16)\\
&= w_f(c_j) && (4.17)
\end{aligned}$$

(4.15) follows by linear property of $w_{\hat{f}}$, (4.16) and (4.17) follows by (4.3) and (4.5). And if $x \neq c_1, c_2, \cdots, c_k$, $w_{\pi_{c_j}}(x) = 0$ for each j, hence, $w_{\hat{f}}(x) = 0 = w_f(x)$. Therefore, for $x \in [0, 1)$,

$$w_{\hat{f}}(x) = 0 = w_f(x).$$

And since $\hat{f}(x)$ is a linear finite combination of π_a's which are of period one, so $\hat{f}(x)$ is of period one too. Therefore, for all $x \in \mathbb{R}$

$$w_{\hat{f}}(x) = 0 = w_f(x).$$

By Lemma 3.2, $\hat{f}(x) = f(x) + Ax + B$ for some constants A and B. Moreover,

$$\hat{f}(0) = (\sum_{i=1}^{k} w_f(c_i)\pi_{c_i}(0)) + f(0) = 0 + f(0) = f(0).$$

$\pi_{c_i}(0) = 0$ from (4.1), it follows $B = 0$. And,

$$\hat{f}(1) = (\sum_{i=1}^{k} w_f(c_i)\pi_{c_i}(1)) + f(0) = 0 + f(0) = f(0) = f(1).$$

It follows $A = 0$. Therefore,

$$f(x) = \hat{f}(x) = (\sum_{i=1}^{k} w_f(c_i)\pi_{c_i}(x)) + f(0)$$

□

Example 4.1. Let $f(x)$ be a tropical 1-periodic function, $\{c_i \mid i = 1, 2, \cdots, k\}$ is the support of $w_f(x)$ on $[0, 1)$ with $c_1 = 0.2, c_2 = 0.4, c_3 = 0.6, c_4 = 0.8$. And $w_f(c_1) = 1, w_f(c_2) = -3, w_f(c_3) = 3, w_f(c_4) = -1, f(0) = 0$ (Figure 4.3). Note that $w_f(c_1) + w_f(c_2) + w_f(c_3) + w_f(c_4) = 1 + (-3) + 3 + (-1) = 0$. Then by Theorem 4.3,

$$\begin{aligned} f(x) &= (\sum_{i=1}^{4} w_f(c_i)\pi_{c_i}(x)) + f(0) \\ &= 1 \times \pi_{0.2}(x) + (-3) \times \pi_{0.4}(x) + 3 \times \pi_{0.6}(x) + (-1) \times \pi_{0.8}(x) + 0 \\ &= \pi_{0.2}(x) - 3\pi_{0.4}(x) + 3\pi_{0.6}(x) - \pi_{0.8}(x) \end{aligned}$$

4.3 $\mathbb{R}$-linear Combination of Tropical T-Periodic Functions

Theorem 4.4 ($\mathbb{R}$-linear Combination Of Tropical T-Periodic Functions)**.** *Let $f(x)$ be an tropical T-periodic function on $\mathbb{R}$, $\{c_i \mid i = 1, 2, \cdots, k\}$ is the support of $w_f(x)$ on $[0, T)$ with $0 \leq c_1 < c_2 < \cdots < c_k < T$. Then*

$$f(x) = T \times (\sum_{i=1}^{k} w_f(c_i)\pi_{\frac{c_i}{T}}(\frac{x}{T})) + f(0)$$

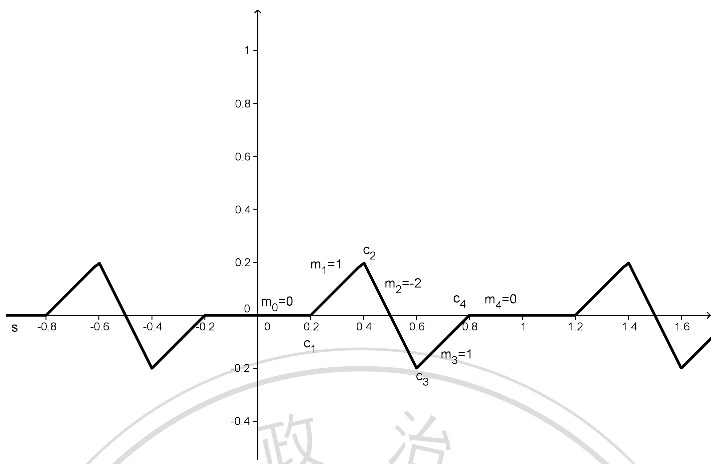


Figure 4.3:

Proof. The proof is very similar to Theorem 4.3. Let $f_{c_i}(x) = \pi_{\frac{c_i}{T}}(\frac{x}{T})$, $i = 1, 2, \cdots, k$. Define

$$\hat{f}(x) = T \times (\sum_{i=1}^{k} w_f(c_i)\pi_{\frac{c_i}{T}}(\frac{x}{T})) + f(0) = T \times (\sum_{i=1}^{k} w_f(c_i) f_{c_i}(x)) + f(0)$$

If $c_j \neq 0$

$$\begin{aligned}
w_{\hat{f}}(c_j) &= T \times (\sum_{i=1}^{k} w_f(c_i) w_{f_{c_i}}(c_j)) && (4.18)\\
&= T \times (\sum_{i=1}^{k} w_f(c_i) \times \frac{1}{T} w_{\pi_{\frac{c_i}{T}}}(\frac{c_j}{T})) && (4.19)\\
&= \sum_{i=1}^{k} w_f(c_i) w_{\pi_{\frac{c_i}{T}}}(\frac{c_j}{T}) && (4.20)\\
&= w_f(c_j) w_{\pi_{\frac{c_j}{T}}}(\frac{c_j}{T}) && (4.21)\\
&= w_f(c_j) \times 1 && (4.22)\\
&= w_f(c_j) && (4.23)
\end{aligned}$$

(4.19) follows by Lemma 4.1, (4.21) follows by (4.5), (4.22) follows by (4.3). If $c_1 = 0$,

$$\begin{aligned}
w_{\hat{f}}(c_1) &= w_{\hat{f}}(0) && (4.24)\\
&= T \times (\sum_{i=1}^{k} w_f(c_i) w_{f_{c_i}}(0)) && (4.25)\\
&= T \times (\sum_{i=1}^{k} w_f(c_i) \times \frac{1}{T} w_{\pi_{\frac{c_i}{T}}}(\frac{0}{T})) && (4.26)\\
&= \sum_{i=1}^{k} w_f(c_i) w_{\pi_{\frac{c_i}{t}}}(0) && (4.27)\\
&= 0 + \sum_{i=2}^{k} w_f(c_i) w_{\pi_{\frac{c_i}{T}}}(0) && (4.28)\\
&= \sum_{i=2}^{k} w_f(c_i) \times (-1) && (4.29)\\
&= w_f(c_1) && (4.30)
\end{aligned}$$

(4.26) follows by Lemma 4.1, (4.28) and (4.29) follows by (4.4), (4.30) follows by Lemma 4.2. And if $x \neq c_1, c_2, \cdots, c_k$, $w_{\pi_{\frac{c_i}{T}}}(\frac{x}{T}) = 0$ for each j, hence, $w_{\hat{f}}(x) = 0 = w_f(x)$. Therefore, for all x in $[0, T)$, $w_{\hat{f}}(x) = 0 = w_f(x)$. And since $\hat{f}(x)$ is a linear finite combination of $\pi_{\frac{c_i}{T}}(\frac{x}{T})$'s which are with period T, so is $\hat{f}(x)$. Therefore, for all $x \in \mathbb{R}$

$$w_{\hat{f}}(x) = 0 = w_f(x).$$

By Lemma 3.2, $\hat{f}(x) = f(x) + Ax + B$ for some constants A and B. Moreover,

$$\hat{f}(0) = (\sum_{i=1}^{k} w_f(c_i) \pi_{\frac{c_i}{T}}(\frac{0}{T})) + f(0) = 0 + f(0) = f(0).$$

$\pi_{\frac{c_i}{T}}(0) = 0$ from (4.1), it follows $B = 0$. And,

$$\hat{f}(T) = (\sum_{i=1}^{k} w_f(c_i)\pi_{\frac{c_i}{T}}(\frac{T}{T})) + f(0) = 0 + f(0) = f(0).$$

It follows $A = 0$. Therefore,

$$f(x) = \hat{f}(x) = T \times (\sum_{i=1}^{k} w_f(c_i)\pi_{\frac{c_i}{T}}(\frac{x}{T})) + f(0)$$

□

Example 4.2. Let's consider Figure 3.2(b) now, that is, $f(x) = |x|, -1 \leq x \leq 1; f(x+2) = f(x)$ for each $x \in \mathbb{R}$. We can find a finite $\mathbb{R}$-linear combination of π_a's for it. $f(x)$ is of period 2, and $\{0, 1\}$ are the support of w_f on $[0, 2)$, $w_f(0) = 1 - (-1) = 2$, $w_f(1) = -1 - 1 = -2$, $f(0) = 0$. Therefore,

$$\begin{aligned}
f(x) &= 2 \times (2 \times \pi_{\frac{0}{2}}(\frac{x}{2}) + (-2) \times \pi_{\frac{1}{2}}(\frac{x}{2})) + f(0) \\
&= -4\pi_{\frac{1}{2}}(\frac{x}{2}) = -4\pi_{0.5}(\frac{x}{2}) \\
&= -4 \times \max\{\frac{1}{2}([\frac{x}{2}] - \frac{x}{2}), \frac{1}{2}([\frac{-x}{2}] - (\frac{-x}{2}))\} \\
&= \min\{-2([\frac{x}{2}] - \frac{x}{2}), -2([\frac{-x}{2}] + \frac{x}{2})\} \\
&= \min\{-2[\frac{x}{2}] + x, -2[\frac{-x}{2}] - x\} \\
&= \min\{x, 2 - x\} \qquad for \quad 0 \leq x < 2 \\
&= \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}
\end{aligned}$$

We see that $-4\pi_{0.5}(\frac{x}{2})$ is really the formula we want to find.

Example 4.3. Given Figure 4.4, support of w_f are $0, 1, 2, 2.5, 3, 4$, and $w_f(0) = 2 - (-2) = 4$, $w_f(1) = 0 - 2 = -2$, $w_f(2) = 1 - 0 = 1$,

$w_f(2.5) = -1 - 1 = -2$, $w_f(3) = 0 - (-1) = 1$, $w_f(0) = -2 - 0) = -2$, $f(0) = 1$. Therefore,

$$
\begin{aligned}
f(x) =& 5 \times (4 \times \pi_{\frac{0}{5}}(\frac{x}{5}) + (-2) \times \pi_{\frac{1}{5}}(\frac{x}{5}) + 1 \times \pi_{\frac{2}{5}}(\frac{x}{5}) + (-2) \times \pi_{\frac{2.5}{5}}(\frac{x}{5}) \\
&+ 1 \times \pi_{\frac{3}{5}}(\frac{x}{5}) + (-2) \times \pi_{\frac{4}{5}}(\frac{x}{5})) + f(0) \\
=& 20\pi_0(\frac{x}{5}) - 2\pi_{0.2}(\frac{x}{5}) + \pi_{0.4}(\frac{x}{5}) - 2\pi_{0.5}(\frac{x}{5}) + \pi_{0.6}(\frac{x}{5}) - 2\pi_{0.8}(\frac{x}{5}) + 1
\end{aligned}
$$

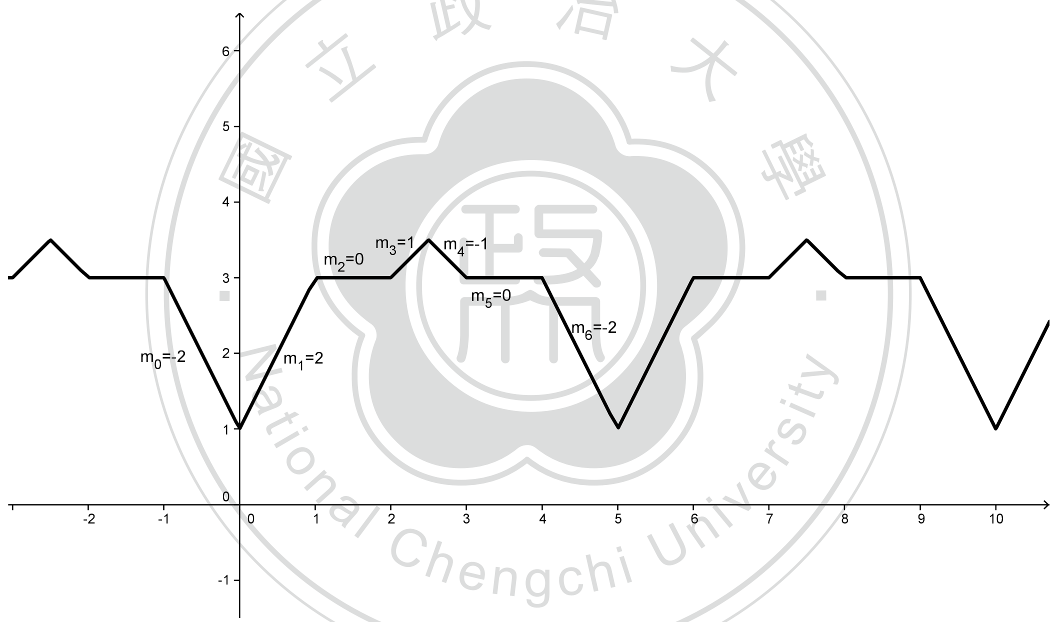


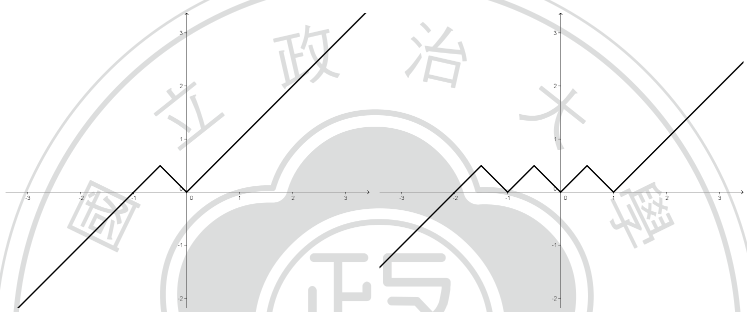
Figure 4.4:

Example 4.4. Let

$$f_N(x) = (-N) \odot (x \oplus (-\infty)) \odot \bigodot_{n=-N}^{N} ((x \oplus n)^{\odot 2} \oslash (x \oplus (n - \frac{1}{2}))^{\odot 2}),$$

See Figure 4.5 for $N = 1$ and $N = 2$. By Theorem 3.3 and Theorem 4.3,

$$\begin{aligned}\lim_{N\to\infty} f_N(x) &= 2\pi_0(x) + (-2)\pi_{0.5}(x) = -2\pi_{0.5}(x) \\ &= -2\max\{\frac{1}{2}([x]-x), \frac{1}{2}([-x]-(-x))\} \\ &= -\max\{[x]-x, [-x]-(-x)\} \\ &= 1_{\mathbb{T}} \oslash (([x]-x) \oplus ([-x]-(-x)))\end{aligned}$$



(a) $f_1(x)$. (b) $f_2(x)$.

Figure 4.5:

Chapter 5

Application on Difference Equations

In this chapter, we will consider difference equation of type

$$y(x+1) = y(x)^{\odot c} = cy(x) \qquad (c \in \mathbb{R})$$

It is called ultradiscrete equations in [3] and [4], and we want to find the solutions $y(x)$ which are tropical meromorphic functions. Before it, we consider certain special tropical meromorphic functions introduced in [4], Laine and Tohge define

$$e_\alpha(x) := \alpha^{[x]}(x-[x]) + \sum_{j=-\infty}^{[x]-1} \alpha^j = \alpha^{[x]}(x-[x]+\frac{1}{\alpha-1}),$$

where α is a real number with $|\alpha| > 1$. In a similar way, they also define

$$e_\beta(x) := \sum_{j=[x]}^{\infty} \beta^j - \beta^{[x]}(x-[x]) = \sum_{j=[x]+1}^{\infty} \beta^j + \beta^{[x]}(1-x+[x]) = \beta^{[x]}(\frac{1}{1-\beta}-x+[x]),$$

where β is real number with $|\beta| < 1$.
But now, we will multiply $e_\alpha(x)$ by α, and then rename it by $e_\alpha(x)$. It will see new $e_\alpha(x)$ still have the similar properties, and it will be better because of $w_{e_\alpha}(0) = \alpha - 1$, we will discover that at last.

5.1 Tropical Counterpart to The Exponential Function

Definition 5.1. Let α be a real number with $|\alpha| > 1$. Define a function $e_\alpha(x)$ on $\mathbb{R}$ by

$$e_\alpha(x) := \alpha^{1+[x]}(x - [x]) + \sum_{j=-\infty}^{[x]} \alpha^j = \alpha^{1+[x]}(x - [x] + \frac{1}{\alpha - 1}),$$

Then we will see

Remark 5.1.

(a)

$$e_\alpha(m) = \frac{\alpha^{1+m}}{\alpha - 1} \qquad \forall\, m \in \mathbb{Z}. \tag{5.1}$$

(b) For $x \in [m, m+1)$, $m \in \mathbb{Z}$,

$$e_\alpha(x) = \alpha^{1+m}x + \alpha^{1+m}(-m + \frac{1}{\alpha - 1}) \tag{5.2}$$

(c) $e_\alpha(x+1) = \alpha e_\alpha(x)\ \forall x \in \mathbb{R}$.

(d) $e_\alpha(x)$ is continuous on $\mathbb{R}$. And

$$w_{e_\alpha}(m) = \alpha^m(\alpha - 1), \quad m \in \mathbb{Z}. \tag{5.3}$$

$$w_{e_\alpha}(x) = 0, \quad x \notin \mathbb{Z}. \tag{5.4}$$

In particular, $w_{e_2}(0) = 1$.

Proof. These follows from Definition 5.1 by a straightforward computation.
(a) $e_\alpha(m) = \alpha^{1+m}(m - m + \frac{1}{\alpha} - 1) = \frac{\alpha^{1+m}}{\alpha - 1}$
(b) For $x \in [n, n+1)$, $n \in \mathbb{Z}$,
$e_\alpha(x) = \alpha^{1+n}(x - n + \frac{1}{\alpha - 1}) = \alpha^{1+n}x + \alpha^{1+n}(-n + \frac{1}{\alpha - 1})$

(c) For $x \in \mathbb{R}$,

$$\begin{aligned} e_\alpha(x+1) &= \alpha^{1+[x+1]}((x+1) - [x+1] + \frac{1}{\alpha-1}) \\ &= \alpha^{1+1+[x]}((x+1) - ([x]+1) + \frac{1}{\alpha-1}) \\ &= \alpha \times \alpha^{1+[x]}(x - [x] + \frac{1}{\alpha-1}) \\ &= \alpha e_\alpha(x) \end{aligned}$$

(d) From (b), linearity of $e_\alpha(x)$ on $[m, m+1)$ implies that it is continuous at non-integer points x, and it remains to verify that $e_\alpha(x)$ is continuous at integer points $x = m \in \mathbb{Z}$. Take $0 < \epsilon < 1$, then

$$e_\alpha(m+\epsilon) = \alpha^{1+m}(m + \epsilon - m + \frac{1}{\alpha-1}) = \alpha^{1+m}(\frac{1}{\alpha-1} + \epsilon)$$

$$e_\alpha(m-\epsilon) = \alpha^m((m-\epsilon) - (m-1) + \frac{1}{\alpha-1}) = \alpha^m(1 + \frac{1}{\alpha-1} - \epsilon) = \alpha^{1+m}(\frac{1}{\alpha-1} - \frac{\epsilon}{\alpha})$$

It follows

$$\lim_{\epsilon \to 0^+} e_\alpha(m+\epsilon) = \lim_{\epsilon \to 0^+} e_\alpha(m-\epsilon) = e_\alpha(m) = \frac{\alpha^{m+1}}{\alpha - 1}$$

Hence, $e_\alpha(x)$ is continuous on $\mathbb{R}$. And, by equation (5.2), for $m \in \mathbb{Z}$, we can get $e'_\alpha(m+\epsilon) = \alpha^{1+m}$, and $e'_\alpha(m-\epsilon) = \alpha^m$. Therefore,

$$\begin{aligned} w_{e_\alpha}(m) &= \lim_{\epsilon \to 0^+}(e'_\alpha(m+\epsilon) - e'_\alpha(m-\epsilon)) \\ &= \alpha^{1+m} - \alpha^m = \alpha^m(\alpha - 1) \end{aligned}$$

In particular, $w_{e_2}(0) = \alpha^0(2-1) = 1$. □

Example 5.1. The graph of $e_2(x)$ is an approximate function of 2^{1+x}. See Figure 5.1, let A_m be the point $(m, 2^{1+m})$, $m \in \mathbb{Z}$. $e_2(x)$ is the graph of all line segments by joining A_m and A_{m+1}, and $e_2(x) \geq 2^{1+x}$ for all $x \in \mathbb{R}$, the eqaul sign holds if and only if $x \in \mathbb{Z}$. The proof is given in the following. Suppose $n < x = n + \epsilon < n + 1$, for some $n \in \mathbb{Z}$, $0 < \epsilon < 1$.

$$e_2(x) = 2^{1+n}((n+\epsilon) - n + 1) = 2^{1+n}(1+\epsilon) > 2^{1+n} \times 2^\epsilon = 2^{1+n+\epsilon} = 2^{1+x}$$

The inequality holds because of $1 + \epsilon > 2^\epsilon$ for $0 < \epsilon < 1$. And if $x = m \in \mathbb{Z}$, $e_2(m) = 2^{1+m}$ follows from (5.1). For convenience, we omit 2 from $e_2(x)$, and denote $e(x) = e_2(x)$.

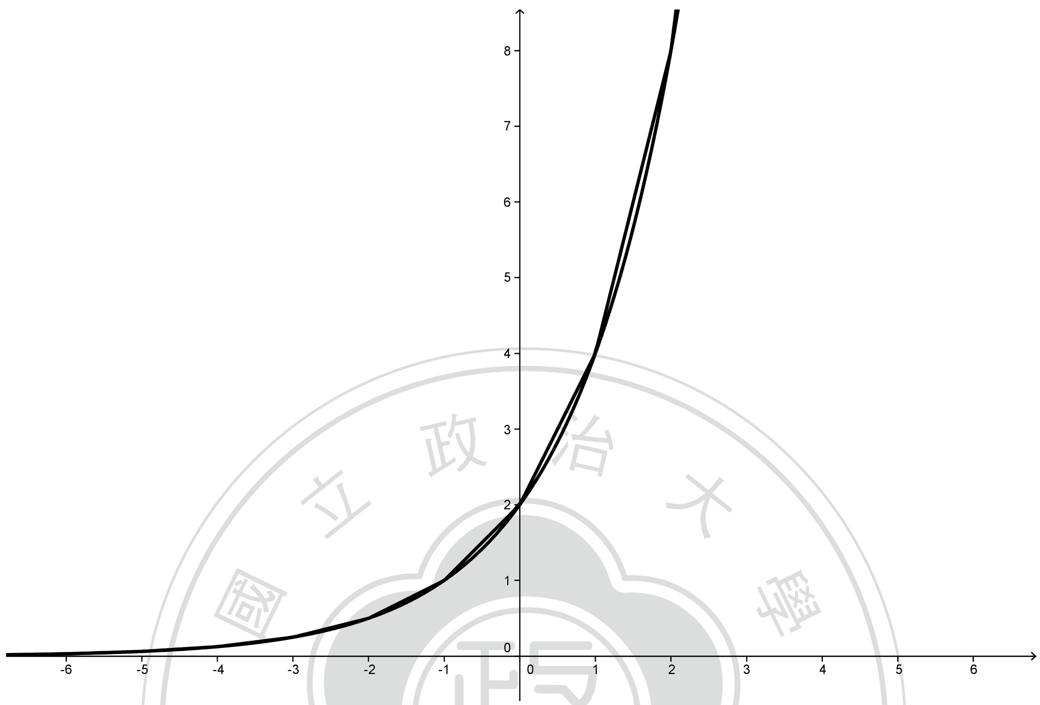


Figure 5.1: $e_2(x)$ is an aproximate function of 2^{1+x}.

5.2 Application on Difference Equations: First Order

Now, let's consider

$$y(x+1) = y(x)^{\odot c} = cy(x) \qquad (c \in \mathbb{R}). \tag{5.5}$$

In [3], lemma 4.1, Halburd and Southall have shown that equation (5.5) admits a nonconstant tropical meromorphic soluction on $\mathbb{R}$ if and only if $c = \pm 1$. For convenience and completeness, we carry the proof into here and make changes slightly.

Theorem 5.1. *The equation (5.5) admits a nonconstant tropical meromorphic soluction on* $\mathbb{R}$ *if and only if* $c = \pm 1$.

Proof. If $c = 0$ then $y \equiv 0$ is the only solution. If $c = 1$ then y is any tropical meromorphic 1-periodic function. If u is any 2-periodic tropical meromorphic function, then $y(x) := u(x+1) - u(x)$ is a tropical meromorphic solution of equation (5.5) with $c = -1$. Conversely, if $h(x)$ is a tropical meromorphic solution function of $y(x+1) = -y(x)$, then there exist a 2-periodic function $v(x)$ such that $h(x) = v(x+1) - v(x)$. We can define

$$v(x) = \begin{cases} 0 & 0 \le x < 1 \\ -h(x) & 1 \le x < 2 \end{cases}$$

and $v(x+2) = v(x)$ for each $x \in \mathbb{R}$. Then

$$v(x+1) = \begin{cases} -h(x+1) & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$$

$$v(x+1) - v(x) = \begin{cases} -h(x+1) = h(x) & 0 \le x < 1 \\ h(x) & 1 \le x < 2 \end{cases}$$

Hence, by periodicity of $h(x)$ and $v(x)$, $h(x) = v(x+1) - v(x)$ for all $x \in \mathbb{R}$.

If y is nonconstant then there is an $x_0 \in \mathbb{R}$ such that y' exists and is a nonzero integer m at x_0. It follows from equation (5.5) that for all $v \in \mathbb{Z}$, $y'(x_0 - v) = \dfrac{m}{c^v}$. Therefore if $c \neq \pm 1$ then for sufficiently large v, $0 < |y'(x_0 - v)| < 1$, and hence the slope is not an integer. $\square$

Note that if we allow the solution $y(x)$ with non-integer slope, then any extended tropical meromorphic 1-periodic function is a solution of equation $y(x+1) = y(x)$. And we can express the solutions with a linear combination of $\pi_a(x)$'s by Theorem 4.3.

The following theorem is the case of $c \neq 0, \pm 1$.

Theorem 5.2.

$$y(x+1) = cy(x), \quad c \neq 0, \pm 1. \tag{5.6}$$

Given an arbitrary extended tropical meromorphic solution f to equation (5.6) with discontinuities of slope at $x_1, x_2, \cdots, x_k$ in $[0,1)$, then f can be represented as a linear combination of finite shifts of the function $e_2(x)$, that is,

$$f(x) = \sum_{i=1}^{k} w_f(x_i) e_2(x - x_i) = \sum_{i=1}^{k} w_f(x_i) e(x - x_i)$$

Proof. Given a non-trivial tropical meromorphic solution f to equation (5.6), there are only finitely many points $x_1, x_2, \cdots, x_k$ in the interval on which $w_f(x) \neq 0$. And at least one point x such that $w_f(x) \neq 0$. If $w_f(x) = 0$ for all $x \in [0, 1)$, there exist $\epsilon > 0$ such that $f(x) = ax + b$ on $[-\epsilon, 1)$ for some constants a and b. For every δ with $0 < \delta < \epsilon$, equation (5.6) implies

$$y(1-\delta) = cy(-\delta)$$
$$a(1-\delta) + b = c(a(-\delta) + b)$$
$$c = \frac{a(1-\delta)+b}{a(-\delta)+b} = 1 + \frac{a}{b - a\delta}$$

but c is a constant, a contradiction.

Define $\hat{f}(x) = \sum_{i=1}^{k} w_f(x_i) e_2(x - x_i) = \sum_{i=1}^{k} w_f(x_i) e(x - x_i)$, then

$$\begin{aligned}
\hat{f}(x+1) &= \sum_{i=1}^{k} w_f(x_i) e(x + 1 - x_i) \\
&= \sum_{i=1}^{k} w_f(x_i) c e(x - x_i) \\
&= c \sum_{i=1}^{k} w_f(x_i) e(x - x_i) \\
&= c\hat{f}(x)
\end{aligned}$$

Hence, $\hat{f}$ is a solution of equation (5.6).

When $x \in [0, 1)$, we see that if $x \neq x_j$,

$$w_{\hat{f}}(x) = \sum_{i=1}^{k} w_f(x_i) w_e(x - x_i) = \sum_{i=1}^{k} w_f(x_i) \times 0 = 0$$

$e(x - x_i)$ is the shift of $e(x)$ to right for x_i unit, $w_e(x - x_i) \neq 0$ only for $x = x_i$ on $[0, 1)$, hence, if $x \neq x_1, x_2, \cdots, x_k$, $w_e(x - x_i) = 0$ for each $i = 1, 2, \cdots, k$.

And, for $j = 1, 2, \cdots, k$,

$$\begin{aligned}
w_{\hat{f}}(x_j) &= \sum_{i=1}^{k} w_f(x_i) w_e(x_j - x_i) \\
&= w_f(x_j) w_e(x_j - x_j) \\
&= w_f(x_j) w_e(0) = w_f(x_j) \times 1 = w_f(x_j)
\end{aligned}$$

$w_e(0) = 1$ follows by equality (5.3).

Therefore, applying Lemma 3.2 to conclude that $f(x) = \hat{f}(x) + Ax + B$ for some constants $A, B \in \mathbb{R}$. And for each $x \in \mathbb{R}$,

$$\begin{aligned} f(x+1) - cf(x) &= \hat{f}(x+1) + A(x+1) + B - c(\hat{f}(x) + Ax + B) \\ &= (\hat{f}(x+1) - c\hat{f}(x)) + (1-c)Ax + A + (1-c)B \\ &= (1-c)Ax + A + (1-c)B = 0 \end{aligned}$$

We conclude $A = 0$ and $B = 0$ since $c \neq 1$.

Hence, $f = \hat{f}$ on $[0, 1)$. If x is a real number such that $n \leq x < n+1$ for some $n \in \mathbb{Z}$, let $x = n + x_0$, then $0 \leq x_0 < 1$, $f(x) = f(n + x_0) = c^n f(x_0) = c^n \hat{f}(x_0) = \hat{f}(n + x_0) = \hat{f}(x)$. Therefore, $f(x) = \hat{f}(x)$ for all $x \in \mathbb{R}$. □

Remark 5.2. When applying Theorem 5.2, we prefer to write the representation for f in the form

$$f(x) = \sum_{i=1}^{k} a_i e(x - x_i)$$

where $a_i = w_f(x_i)$.

Now, let's consider the equation

$$y(x+1) = y(x) + b \qquad (b \in \mathbb{R}) \tag{5.7}$$

It is clear that $y(x) = bx$ is a solution function of equation (5.7), and if $h(x)$ is an extended tropical meromorphic function such that $h(x+1) = h(x)$, then $h(x) + bx$ is an extended tropical meromorphic solution function of equation (5.7). Moreover, all solution functions of equation (5.7) is the form $h(x) + bx$, where $h(x)$ is a tropical 1-periodic function.

And now the equation

$$y(x+1) = ay(x) + b \qquad (a, b \in \mathbb{R}, a \neq 0, \pm 1) \tag{5.8}$$

can be turned into

$$y(x+1) - \frac{b}{1-a} = a(y(x) - \frac{b}{1-a}) \tag{5.9}$$

Let $z(x) = y(x) - \dfrac{b}{1-a}$, equation (5.9) turn into

$$z(x+1) = az(x) \tag{5.10}$$

By Theorem 5.2, the solution function of equation (5.10) can be represented as $z(x) = \sum_{i=1}^{k} a_i e(x - x_i)$, where $0 \le x_i < 1$, $a_i \in \mathbb{R} \setminus \{0\}$. Hence, the solution function of equation (5.8) can be represented as

$$y(x) = (\sum_{i=1}^{k} a_i e(x - x_i)) + \frac{b}{1-a}.$$

Last case, consider the equation

$$y(x+1) = -y(x) + b \qquad (b \in \mathbb{R}). \tag{5.11}$$

Subtract both sides by $-\frac{b}{2}$, it turns into

$$y(x+1) - \frac{b}{2} = -(y(x) - \frac{b}{2})$$

Hence, by Theorem 5.1, the solution function of equation (5.11) can be represented as

$$y(x) = u(x+1) - u(x) + \frac{b}{2}$$

where $u(x)$ is any 2-periodic tropical meromorphic function. So far we have discussed all the circumstances of equation $y(x+1) = ay(x) + b$.

For the example of second order difference equations, one can see [4], Laine and Tohge consider equation

$$y(x+1) + y(x-1) = cy(x)$$

for $c \in \mathbb{R}$.

5.3 Tropical Approximated Function

We found that each continuous function defined on $\mathbb{R}$ can be approximated by extended tropical meromorphic functions. Given a continuous function $f(x)$ on $\mathbb{R}$, connect the line between the point $(m, f(m))$ and $(m+1, f(m+1))$ for each integer m. This will be a function of an extended tropical approximated of original function. The function is

$$(f([x+1]) - f([x]))(x - [x]) + f([x]). \tag{5.12}$$

where $[x]$ represent the greatest integer not exceeding x.

If every $\frac{1}{n}$ as a line linking, then this function is

$$Tf_n(x) := \frac{f(\frac{1}{n}[n(x+\frac{1}{n})]) - f(\frac{1}{n}[nx])}{\frac{1}{n}}(x - \frac{1}{n}[nx]) + f(\frac{1}{n}[nx])$$

It is continuous piecewise linear, and $\lim_{n\to\infty} Tf_n(x) = f(x)$ for each $x \in \mathbb{R}$.

Theorem 5.3. *Given a continuous function $f(x)$ on $\mathbb{R}$, define*

$$Tf_n(x) := \frac{f(\frac{1}{n}[n(x+\frac{1}{n})]) - f(\frac{1}{n}[nx])}{\frac{1}{n}}(x - \frac{1}{n}[nx]) + f(\frac{1}{n}[nx])$$

then $Tf_n(x)$ is continuous piecewise linear for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} Tf_n(x) = f(x)$ for each $x \in \mathbb{R}$.

Proof. If $n \in \mathbb{N}$, $x = \frac{m}{n}$ for some $m \in \mathbb{Z}$, then

$$Tf_n(x) = Tf_n(\frac{m}{n}) = f(\frac{m}{n}) = f(x).$$

If $x \in (\frac{m}{n}, \frac{m+1}{n})$,

$$Tf_n(x) = \frac{f(\frac{m+1}{n}) - f(\frac{m}{n})}{\frac{1}{n}}(x - \frac{m}{n}) + f(\frac{m}{n}).$$

If $x \in (\frac{m-1}{n}, \frac{m}{n})$,

$$Tf_n(x) = \frac{f(\frac{m}{n}) - f(\frac{m-1}{n})}{\frac{1}{n}}(x - \frac{m-1}{n}) + f(\frac{m-1}{n}).$$

Hence,

$$\lim_{x\to(\frac{m}{n})^+} Tf_n(x) = f(\frac{m}{n}),$$

and

$$\lim_{x\to(\frac{m}{n})^-} Tf_n(x) = f(\frac{m}{n}) - f(\frac{m-1}{n}) + f(\frac{m-1}{n}) = f(\frac{m}{n}).$$

Therefore, $Tf_n(x)$ is continuous piecewise linear for each $n \in \mathbb{N}$.

Next, if $x \in \mathbb{Z}$, then $Tf_n(x) = f(x)$ for each $n \in \mathbb{N}$, and clearly $\lim_{n\to\infty} Tf_n(x) = f(x)$. If $x \in (t, t+1)$ for some $t \in \mathbb{Z}$, continuity of $f(x)$ implies that $f(x)$ is uniformly continuous on $[t, t+1]$, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|x - y| < \delta_1$ and $x, y \in [t, t+1]$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $\delta_2 = \min\{x-t, t+1-x\}$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta := \min\{\delta_1, \delta_2\}$, for each $n \geq N$, there exists $m_n \in \mathbb{Z}$ such that $t < \frac{m_n}{n} \leq x < \frac{m_n+1}{n} < t+1$, and

$$\begin{aligned}
|Tf_n(x) - f(x)| &= |Tf_n(x) - Tf_n(\frac{m_n}{n}) + Tf_n(\frac{m_n}{n}) - f(x)| \\
&\leq |Tf_n(x) - Tf_n(\frac{m_n}{n})| + |Tf_n(\frac{m_n}{n}) - f(x)| \\
&= |\frac{f(\frac{m_n+1}{n}) - f(\frac{m_n}{n})}{\frac{1}{n}}(x - \frac{m_n}{n})| + |f(\frac{m_n}{n}) - f(x)| \\
&\leq |f(\frac{m_n+1}{n}) - f(\frac{m_n}{n})| + |f(\frac{m_n}{n}) - f(x)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} Tf_n(x) = f(x)$ for each $x \in \mathbb{R}$. □

In fact, $\{Tf_n\}$ converges to f uniformly on any closed bounded interval.

Example 5.2. If $f(x) = 2^{x+1}$, then

$$\begin{aligned}
Tf_1(x) &= f([x+1] - f([x]))(x - [x]) + f([x]) \\
&= (2^{[x+1]+1} - 2^{[x+1]})(x - [x]) + 2^{[x]+1} \\
&= 2^{[x]+1}(x - [x]) + 2^{[x]+1} \\
&= 2^{[x]+1}(x - [x] + 1)
\end{aligned}$$

It is precisely the function $e_2(x)$ of section 5.1.

Given first-order linear difference equations with general solutions, we can use Formula (5.12) to find their tropical approximated solutions. We can then use these tropical approximated solutions to generate all the extended tropical meromorphic solutions of the given difference equations. For the future research, we hope these results can be extended to difference equations in general.

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