國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

Some Value Distribution of Meromorphic Functions of Class $\mathcal{A}$

$\mathcal{A}$ 類半純函數之某些值分佈

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Abstract

In this thesis, we study the basic theory of value distribution of meromorphic function of class $\mathcal{A}$. We prove that every meromorphic function of class $\mathcal{A}$ has at most two multiple values and the result is sharp. Also, we prove that if a meromorphic function f of class $\mathcal{A}$ and its derivative $f^{(k)}$ share a non-zero complex value, then $f \equiv f^{(k)}$.

中文摘要

在這篇論文裡，我們探討 $\mathcal{A}$ 類半純函數的值分佈基本理論。我們證明了每一個 $\mathcal{A}$ 類半純函數最多有兩個重值，而這個結果是最佳的情形。進而，我們證明若一個 $\mathcal{A}$ 類半純函數 f 與其導數 $f^{(k)}$ 共非零的複數值，則 $f \equiv f^{(k)}$。

Contents

1 Introduction

In this thesis, a meromorphic function will mean meromorphic in the whole complex plane $\mathbb{C}$. Given two non-constant meromorphic functions f and g and $a \in \mathbb{C}_\infty$, the extended complex plane. We say that f and g share a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros ignoring multiplicities. We say that f and g share a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities.

In 1929, R. Nevanlinna [8] proved the following remarkable results which play an important role in the area of value distribution of meromorphic functions. Thus the theory of value distribution became a fascinating topic. Thereafter, more and more people participated in the research of this theory.

Theorem 1.1 [8] *Let f and g be two non-constant meromorphic functions. If f and g share five distinct values in $\mathbb{C}_\infty$, then $f \equiv g$.*

Theorem 1.2 [8] *Let f and g be two non-constant meromorphic functions. If f and g share four distinct values a_1, a_2, a_3, and a_4 CM, then f is a Möbius transformation of g, two of the values, say a_1 and a_2, must be Picard exceptional values of f and g, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

After having these results, one question may be arised. What happen if two non-constant meromorphic functions share four values but not all CM ? Does the conclusion in Theorem 1.2 still hold for the other cases?

In 1979 and 1983, G. G. Gundersen [4, 5] proved the following results. The results say that two non-constant meromorphic functions sharing either three values CM and one value IM or two values CM and two values IM are not different from sharing all four values CM.

Theorem 1.3 [4] *Let f and g be two non-constant meromorphic functions. If f and g share three values CM and share a forth value IM, then they share all four values CM, and hence the conclusion of Thereom 1.2 holds.*

Theorem 1.4 [5] *Let f and g be two non-constant meromorphic functions sharing four values a_1, a_2, a_3, and a_4. If f and g share a_1, a_2 CM, and a_3, a_4 IM, then f and g share all four values CM, and hence the conclusion of Theorem 1.2 holds.*

The remaining case, f and g share one value CM and the other three values IM, is still open and is an interesting research problem.

In view of Nevanlinna and Gundersen's results, it is natural to ask what happen if two meromorphic functions share the number of values less than four.

In this thesis, we will study some value distribution of meromorphic functions of class $\mathcal{A}$. Some well-known properties will be discussed and some new results will be obtained.

The thesis will be divided into five sections. In section 1, we give some introductions of the sharing value problems. In section 2, we review some basic theory of value distribution. In section 3, we discuss the basic properties of meromorphic functions of class $\mathcal{A}$. In section 4, we study the multiple values of meromorphic functions of class $\mathcal{A}$. In section 5, we consider the unicity of meromorphic functions of class $\mathcal{A}$. In the end of section 4 and section 5, we get our main results in this thesis.

2 Basic Theory of Value Distribution

In this section, we introduce and review some basic facts and notations in complex analysis and value distribution which will be used throughout the rest of the thesis. For the sake of brevity, proofs are omitted because they are standard and can be found in [1, 3, 6, 9, 10].

In Nevanlinna's value distribution theory, the following Poisson-Jensen's formula plays a very important role.

Theorem 2.1 (Poisson-Jensen's formula) *Let $0 < R < \infty$ and f be meromorphic in $|z| < R$ and a_μ and b_ν be the zeros and poles of f in $|z| < R$, $1 \le \mu \le M$, $1 \le \nu \le N$, respectively. If $z = re^{i\theta}$, $0 \le r < R$, and $f(z) \neq 0, \infty$, then we have*

$$\begin{aligned}\log|f(z)| = \ & \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi \\ & + \sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \overline{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 - \overline{b}_\nu z}\right|.\end{aligned}$$

By taking $z = 0$ in Theorem 2.1, we get the Jensen's formula.

Theorem 2.2 (Jensen's formula) *Under the assumptions of Theorem 2.1, if $f(0) \neq 0, \infty$, then we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|a_\mu|} + \sum_{\nu=1}^{N} \log\frac{R}{|b_\nu|}.$$

The assumption $f(0) \neq 0, \infty$ in Theorem 2.1 can be eliminated. In fact, for $0 \le r < \infty$, let $n(r, f)$ denote the number of poles of f in $|z| \le r$ counting multiplicities. Consider the Laurent expansion of f at the origin

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \dots$$

Note that $\lambda = n(0, \frac{1}{f}) - n(0, f)$. Consider the function

$$g(z) = \begin{cases} f(z)\left(\frac{R}{z}\right)^{\lambda} & \text{if } z \neq 0 \\ c_\lambda R^\lambda & \text{if } z = 0, \end{cases}$$

then we have the generalized Jensen's formula.

Theorem 2.3 (Generalized Jensen's formula) *Under the assumptions of Theorem 2.1 without the condition $f(0) \neq 0, \infty$, then we have*

$$\begin{aligned} \log|c_\lambda| = \ & \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|a_\mu|} - n(0, \frac{1}{f})\log R \\ & + \sum_{\nu=1}^{N} \log\frac{R}{|b_\nu|} + n(0, f)\log R, \end{aligned}$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

From now on, meromorphic function means meromorphic in the whole complex plane. First of all, we introduce the positive logarithmic function.

Definition 2.4 *For $x \geq 0$,*

$$\log^+ x = \max\{\log x, 0\} = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

Obviously, $\log^+ x$ is a continuous non-negative increasing function on $[0, \infty)$ satisfying $\log x = \log^+ x - \log^+ \frac{1}{x}$ and $|\log x| = \log^+ x + \log^+ \frac{1}{x}$.

Let f be a meromorphic function, Nevanlinna [8] introduced the following notations.

Definition 2.5 *For $0 < r < \infty$,*

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta.$$

Definition 2.6 *For* $0 < r < \infty$,

$$N(r,f) = \int_0^r \frac{n(t,f) - n(o,f)}{t} dt + n(0,f)\log r,$$

where $n(t,f)$ *denotes the number of poles of* f *in the disc* $|z| \leq t$ *counting multiplicities.* $N(r,f)$ *is called the counting function of* f.

For $0 \leq r < \infty$, $n(r,f)$ *denotes the number of poles of* $f(z)$ *in* $|z| \leq r$ *counting multiplicities;* $\overline{n}(r,f)$ *denotes the number of poles of* $f(z)$ *in* $|z| \leq r$ *ignoring multiplicities;* $n_{k)}(r,1/f)$ *(resp.* $n_{(k}(r,1/f)$*) denotes the number of zeros of* $f(z)$ *in* $|z| \leq r$ *with order* $\leq k$ *(resp.* $\geq k$*) counting multiplicities;* $\overline{n}_{k)}(r,1/f)$ *(resp.* $\overline{n}_{(k}(r,1/f)$*) denotes the number of zeros of* $f(z)$ *in* $|z| \leq r$ *with order* $\leq k$ *(resp.* $\geq k$*) ignoring multiplicities.*

Definition 2.7 *For* $0 < r < \infty$*, the function* $T(r,f)$ *defined by*

$$T(r,f) = m(r,f) + N(r,f)$$

is called the (Nevanlinna) characteristic function of f.

It is clear that $T(r,f)$ is a non-negative increasing function and a convex function of $\log r$. Let f be given in Theorem 2.1. It follows from the integration by parts in Riemann-Stieltjes integral, we have

$$\sum_{\mu=1}^{M} \log \frac{R}{|a_\mu|} = \int_0^R \frac{n(t,\frac{1}{f}) - n(0,\frac{1}{f})}{t} dt$$

and

$$\sum_{\nu=1}^{M} \log \frac{R}{|b_\nu|} = \int_0^R \frac{n(t,f) - n(0,f)}{t} dt.$$

On the other hand, the generalized Jensen's formula can be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} log^+ |\, f(Re^{i\varphi}) \,|\, d\varphi + \sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} + n(0,f)\log R$$

$$= \frac{1}{2\pi}\int_0^{2\pi} log^+ \left| \frac{1}{f(Re^{i\varphi})} \right| d\varphi + \sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} + n(0,\frac{1}{f})\log R + \log |\, c_\lambda \,|.$$

Therefore, we obtain

$$m(R, f) + N(R, f) = m(R, \frac{1}{f}) + N(R, \frac{1}{f}) + \log |c_\lambda|,$$

that is,

$$T(R, f) = T(R, \frac{1}{f}) + \log |c_\lambda|,$$

which is another form of the generalized Jensen's formula and is also known as the Nevanlinna-Jensen's formula.

Theorem 2.8 (Nevanlinna-Jensen's formula) *Let f be a meromorphic function, then for $r > 0$,*

$$T(r, f) = T(r, \frac{1}{f}) + \log |c_\lambda|,$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

By the Nevanlinna-Jensen's formula, we can get the Nevalinna's first fundamental theorem.

Theorem 2.9 (Nevanlinna's First Fundamental Theorem) *Let f be a meromorphic function and a be a finite complex number. Then, for $r > 0$, we have*

$$T(r, \frac{1}{f-a}) = T(r, f) + \log |c_\lambda| + \varepsilon(a, r),$$

where c_λ is the first non-zero coefficient of the Laurent expansion of $\frac{1}{f-a}$ at 0, and

$$|\varepsilon(a.r)| \leq \log^+ |a| + \log 2.$$

Usually, Nevanlinna's first fundamental theorem is written as

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1).$$

Now, we come to the most important theorem in the theory of value distribution, namely, Nevanlinna's second fundamental theorem.

Theorem 2.10 (Nevanlinna's Second Fundamental Theorem) *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}$, $1 \le j \le q$, be q distinct finite values $(q \ge 2)$. Then*

$$m(r,f) + \sum_{j=1}^{q} m(r, \frac{1}{f-a_j}) \le 2T(r,f) - N_1(r) + S(r,f),$$

where $N_1(r) = 2N(r,f) - N(r,f') + N(r, \frac{1}{f'})$ *and*

$$S(r,f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}) + O(1).$$

Given $a \in \mathbb{C}$, by Nevanlinna's first fundamental theorem,

$$m(r, \frac{1}{f-a}) = T(r,f) - N(r, \frac{1}{f-a}) + O(1).$$

Hence, Nevanlinna's second fundamental theorem can be rewritten as follows.

Theorem 2.11 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values $(q \ge 3)$. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) + S(r,f),$$

where $N_1(r)$ and $S(r,f)$ are given as in Theorem 2.10.

Note that, in Theorem 2.11, if some $a_j = \infty$, then $N(r, \frac{1}{f-a_j})$ should be read as $N(r,f)$.

Let $n_1(t) = 2n(t,f) - n(t,f') + n(t, \frac{1}{f'})$ and let $\overline{n}(t,f)$ denote the number of distinct poles of f in $|z| \le t$. Define

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t}\, dt + \overline{n}(0,f) \log r,$$

which is called the reduced counting function of f.

Note that, if z_0 is a pole of f of order k in $|z| \leq t$, then z_0 is counted $k-1$ times by $n_1(r)$. Similarly, for a finite value a, if z_0 is a zero of $f-a$ of order k in $|z| \leq t$, then z_0 is also counted $k-1$ times by $n_1(r)$. Hence,

$$\sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) \leq \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}).$$

Therefore, we have the third form of Nevanlinna's second fundamental theorem.

Theorem 2.12 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty, 1 \leq j \leq q$, be q distinct values $(q \geq 3)$. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

where $S(r,f)$ is given as in Theorem 2.10.

In Nevanlinna's second fundamental theorem, the remainder term $S(r,f)$ is a complicated object which can be estimated by using the method of logarithmic derivative. It turns out that $S(r,f)$ is small comparing to $T(r,f)$. In order to make it clear, we need the concept of the growth of meromorphic function.

Classically, we use the maximum modulus to measure the growth of an entire function.

Definition 2.13 *Let f be a meromorphic function. The order λ of f is defined to be*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and the lower order μ of f is defined to be

$$\mu = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Definition 2.14 *Let $f(z)$ and $a(z)$ be meromorphic functions. If $T(r,a) = S(r,f)$, then $a(z)$ is called a small function of $f(z)$.*

Let f be an entire function. Define, for $r \geq 0$,

$$M(r, f) = \max_{|z| \leq r} |f(z)|.$$

Then the relation between $M(r, f)$ and $T(r, f)$ is given as follows.

Theorem 2.15 *Let* $0 \leq r < R < \infty$ *and* f *be an entire function, we have*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f).$$

In particular,

$$T(r, f) \leq \log^+ M(r, f) \leq 3T(2r, f).$$

By Theorem 2.15, the order and lower order of an entire function are unambiguous. Now, we can state the properties of $S(r, f)$.

Lemma 2.16 *Let* f *be a non-constant meromorphic function. If* f *is of finite order, then*

$$m(r, \frac{f'}{f}) = O(\log r), \quad (r \to \infty).$$

If f *is of infinite order, then*

$$m(r, \frac{f'}{f}) = O(\log(rT(r, f))), \quad (r \to \infty, r \notin E),$$

where E *is a set of finite measure.*

Theorem 2.17 *Let* f *be a non-constant meromorphic function and* $S(r, f)$ *be defined in Theorem 2.10. If* f *is of finite order, then*

$$S(r, f) = O(\log r), \quad (r \to \infty).$$

If f *is of infinite order, then*

$$S(r, f) = O(\log (rT(r, f))), \quad (r \to \infty, r \notin E),$$

where E *is a set of finite measure.*

In the thesis, we will denote by $S(r,f)$ any quantity satisfy $S(r,f) = o(T(r,f))$ as $r \to \infty$ if f is of finite order, and $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$ if f is of infinite order, where E is a set of finite measure.

By Lemma 2.16, $m(r, \frac{f'}{f}) = S(r,f)$. Moreover, Milloux [7] proved the following.

Theorem 2.18 *Let f be a non-constant meromorphic function and k be a positive integer and let*

$$\Psi(z) = \sum_{i=1}^{k} a_i(z) f^{(i)}(z),$$

where $a_1(z), a_2(z), \ldots, a_k(z)$ are small functions of f. Then

$$m(r, \frac{\Psi}{f}) = S(r,f).$$

For three small functions, we still have the generalization of second fundamental theorem.

Theorem 2.19 [9] *Let f be a non-constant meromorphic function and $a_1(z), a_2(z)$ and $a_3(z)$ are three distinct small functions. Then*

$$T(r,f) < \sum_{j=1}^{3} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f).$$

In 1929, Nevanlinna [9] introduced the quantity $\delta(a,f)$ to measure the degree of a meromorphic function misses a value a.

Definition 2.20 *Let f be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. The deficiency of a with respect to f is defined by*

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}.$$

Definition 2.21 *Let f be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. We define*

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)},$$

and

$$\theta(a, f) = \liminf_{r\to\infty} \frac{N(r, \frac{1}{f-a}) - \overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

Clearly, $0 \le \delta(a, f) \le 1, 0 \le \Theta(a, f) \le 1$ and $0 \le \theta(a, f) \le 1$. Also, $0 \le \delta(a, f) + \theta(a, f) \le \Theta(a, f)$. By Theorem 2.12, we have

Theorem 2.22 *Let f be a non-constant meromorphic function. Then*

$$\sum_a \delta(a, f) + \theta(a, f) \le \sum_a \Theta(a, f) \le 2.$$

In order to study uniqueness theorems of meromorphic functions, we state a Nevanlinna theorem which plays an important role.

Theorem 2.23 [9] *Suppose $f_1, \ldots, f_n$ are linearly independent meromorphic functions satisfying the following identity*

$$\sum_{j=1}^{n} f_j \equiv 1.$$

Then for $1 \le j \le n$, we have

$$\begin{aligned} T(r, f_j) \le& \sum_{k=1}^{n} N(r, \frac{1}{f_k}) + N(r, f_j) + N(r, D) \\ &- \sum_{k=1}^{n} N(r, f_k) - N(r, \frac{1}{D}) + o(T(r)), \end{aligned}$$

where D is the Wronskian of $f_1, \ldots, f_n$, and

$$T(r) = \max_{1\le k\le n} \{T(r, f_k)\},$$

E is a set with finite linear measure.

Moreover, we can get a useful result in the uniqueness theorem of meromorphic functions.

Theorem 2.24 [9] *Let $f_j (j = 1, 2, 3)$ be meromorphic functions where f_1 be not a constant function. If*

$$\sum_{j=1}^{3} f_j(z) \equiv 1,$$

and

$$\sum_{j=1}^{3} N(r, \frac{1}{f_j}) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))(T(r)) \quad (r \in I),$$

where $\lambda < 1$,

$$T(r) = \max_{1\le j\le 3}\{T(r, f_j)\},$$

and I is a set of $r \in (0, \infty)$ with infinite measure, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Finally, we review some theorems which will be needed in the following sections.

Theorem 2.25 [9] *Suppose that f is a meromorphic function in $|z| < R$ and $g(z) = \dfrac{af(z) + b}{cf(z) + d}$, where $a, b, c,$ and d are constant satisfying $ad - bc \neq 0$. Then for $0 < r < R$, we have*

$$T(r, g) = T(r, f) + O(1).$$

Theorem 2.26 [9] *If f is a transcendental meromorphic function in the complex plane, then*

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty.$$

3 Meromorphic Functions of Class $\mathcal{A}$

Let $\mathcal{A}$ denote the collection of all non-constant meromorphic functions f satisfying

$$\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f).$$

Such functions are called meromorphic functions of class $\mathcal{A}$. Clearly, $e^z, ze^z, \ldots$ are functions of class $\mathcal{A}$.

Proposition 3.1 *If f is a non-constant rational function, write*

$$f(z)=\frac{a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0}{b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0},$$

where $a_p(\neq 0), a_{p-1}, \ldots, a_0$ and $b_q(\neq 0), b_{q-1}, \ldots, b_0$ are complex numbers, p, q are non-negative integers satisfying $p+q\geq 1$, and $a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0$ and $b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0$ have no common factors. Then

$$m(r,f)=\begin{cases}(p-q)\log r+O(1) & \text{if } p>q,\\ O(1) & \text{if } p\leq q\end{cases}$$

and

$$N(r,f)=q\log r+O(1)$$

holds for a sufficiently large r. Thus,

$$T(r,f)=\max\{p,q\}\log r+O(1).$$

Proof. First, we prove that $m(r,f)=\begin{cases}(p-q)\log r+O(1) & \text{if } p>q,\\ O(1) & \text{if } p\leq q,\end{cases}$

$P(z)=a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0$ and $Q(z)=b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0$.

Let $A(r)=\frac{|a_{p-1}|}{|a_p|}\frac{1}{r}+\cdots+\frac{|a_0|}{|a_p|}\frac{1}{r^p}$ and $B(r)=\frac{|b_{q-1}|}{|b_q|}\frac{1}{r}+\cdots+\frac{|b_0|}{|b_q|}\frac{1}{r^q}$.

Given $\varepsilon > 0$, there exists $r_0 > 0$, such that $|A(r)| < \varepsilon$ and $|B(r)| < \varepsilon$ for $r \geq r_0$.
So for all $r \geq r_0$ and $|z| = r$,

$$(1-\varepsilon)|a_p|r^p \leq |P(z)| \leq (1+\varepsilon)|a_p|r^p$$

and

$$(1-\varepsilon)|b_q|r^q \leq |Q(z)| \leq (1+\varepsilon)|b_q|r^q.$$

Let $\alpha = \dfrac{(1-\varepsilon)|a_p|}{(1+\varepsilon)|b_q|}$ and $\beta = \dfrac{(1+\varepsilon)|a_p|}{(1-\varepsilon)|b_q|}$, then for all $r \geq r_0$,

$$\alpha r^{p-q} \leq |f(z)| \leq \beta r^{p-q}.$$

If $p > q$ and $r \geq 0$, then there exists $M > 0$, such that

$$|m(r,f) - (p-q)\log r| \leq M,$$

which means that $m(r,f) = (p-q)\log r + O(1)$.
If $p = q$, then for all $r \geq r_0$ and $|z| = r$,

$$\alpha \leq |f(z)| \leq \beta.$$

So we get $\log\alpha^+ \leq m(r,f) \leq \log\beta^+$, which means that $m(r,f) = O(1)$.
If $p < q$, then by choosing $r > 0$, such that $0 < \alpha r^{p-q} < 1$ and $r^{p-q} < 1$, we get $m(r,f) = O(1)$.

Now, we prove $N(r,f) = q\log r + O(1)$ for a sufficiently large r.
Choose $r_0 > 0$ such that $Q(z)$ has q zeros in $|z| < r_0$.
We may assume that Q has a zero at $z = 0$ of multiple $m \geq 0$. Then for all $r \geq r_0$,

$$\begin{aligned} N(r,f) &= \int_0^r \frac{n(t,f)-n(o,f)}{t}dt + n(0,f)\log r \\ &= \int_\delta^r \frac{n(t,f)-n(o,f)}{t}dt + n(0,f)\log r \\ &= (q-m)(\log r - \log\delta) + m\log r \\ &= q\log r - (q-m)\log\delta \\ &= q\log r + O(1), \end{aligned}$$

where $\delta > 0$ is small.

Therefore, we have $T(r, f) = \max\{p, q\} \log r + O(1)$ for a sufficiently large r.

❑

In general, the following class of meromorphic function is of class $\mathcal{A}$.

Proposition 3.2 *Let α be a non-constant entire function and h be a non-zero rational function. Then*

$$f(z) = h(z)e^{\alpha(z)}$$

is a meromorphic function of class $\mathcal{A}$.

Proof. Let $h(z) = \dfrac{P(z)}{Q(z)}$ be a nonzero rational function, $\deg P(z) = p$, $\deg Q(z) = q$, $(P(z), Q(z)) = 1$. Choose $r_0 > 0$, so that all zeros of $P(z)$ and $Q(z)$ lie in $|z| < r_0$. Then by Theorem 3.1, for all $r \geq r_0$, we have

$$N(r, h) = q \log r$$

and

$$N(r, \frac{1}{h}) = p \log r.$$

By assumption, $e^{\alpha(z)}$ is an entire function without zeros, so we have

$$\overline{N}(r, f) = \overline{N}(r, h) \leq N(r, h) = q \log r$$

and

$$\overline{N}(r, \frac{1}{f}) = \overline{N}(r, \frac{1}{h}) \leq N(r, \frac{1}{h}) = p \log r,$$

which imples that

$$\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) \leq q \log r + p \log r.$$

Therefore, we get

$$\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f)$$

by Theorem 2.26. So we conclude that $f \in \mathcal{A}$. ❑

Proposition 3.3 *If f is a meromorphic function of class $\mathcal{A}$, then so is $\frac{1}{f}$.*

Proof. Since $f \in \mathcal{A}$, we have $\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f)$.

By the Theorem 2.25 , $T(r,f) = T(r,\frac{1}{f}) + O(1)$, we have

$$\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) \;\; = \;\; S(r,\frac{1}{f}),$$

and $1/f$ is a meromorphic function of class $\mathcal{A}$. ❑

However, if f and g are of functions of class $\mathcal{A}$, $f+g$ and fg may not be of class $\mathcal{A}$; for example, $f(z) = e^z$, $g(z) = -e^z$ and $h(z) = e^{-z}$ are of meromorphic functions of class $\mathcal{A}$, but $f+g$ and fh are not meromorphic functions of class $\mathcal{A}$.

Proposition 3.4 *All functions in $\mathcal{A}$ are transcendental meromorphic functions.*

Proof. Let $f(z) = \frac{P(z)}{Q(z)}$ be a non-constant rational function, $\deg P(z) = p$, $\deg Q(z) = q$, $(P(z), Q(z)) = 1$ and $p + q \geq 1$. Choose $r_0 > 0$, so that all zeros of $P(z)$ and $Q(z)$ lie in $|z| \leq r_0$. Then by Theorem 3.1, for all $r \geq r_0$,

$$N(r,f) = q \log r,$$

$$N(r,\frac{1}{f}) = p \log r,$$

and

$$T(r,f) = \max\{p,q\} \log r + O(1).$$

Now, assume that $P(z)$ has s distinct zeros and $Q(z)$ has t distinct zeros, then, $s \leq p$, $t \leq q$, and for all $r \geq r_0$, we have

$$\overline{N}(r,f) = t \log r$$

and

$$\overline{N}(r,\frac{1}{f}) = s \log r.$$

Therefore,

$$\lim_{r\to\infty} \frac{\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})}{T(r,f)} = \frac{r+s}{\max\{p,q\}}$$

which is not zero. So f is not a meromorphic function of class $\mathcal{A}$. ❑

Proposition 3.5 *Let f be a meromorphic function with $\Theta(0,f)=\Theta(\infty,f)=1$, then $f\in\mathcal{A}$.*

Proof. Since $\Theta(0,f)=\Theta(\infty,f)=1$, we have

$$\Theta(0,f) = 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})}{T(r,f)} = 1$$

and

$$\Theta(\infty,f) = 1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)} = 1,$$

which imply

$$\begin{aligned}\lim_{r\to\infty}\frac{\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})}{T(r,f)} &\le \limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}+\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})}{T(r,f)}\\ &= 1-\Theta(0,f)+1-\Theta(\infty,f)\\ &= 0.\end{aligned}$$

Therefore, $\overline{N}(r,f)+\overline{N}(r,\dfrac{1}{f}) = o(T(r,f)) = S(r,f)$ and $f\in\mathcal{A}$. ❑

Remark. In the literature, a non-constant meromorphic function f satisfying $\Theta(0,f)=\Theta(\infty,f)=1$ is called meromorphic functions of class $\mathcal{K}$.

Proposition 3.6 *Let f be a meromorphic function with $\delta(0,f)=\delta(\infty,f)=1$, then $f\in\mathcal{A}$.*

Proof. Since $\delta(0,f)=\delta(\infty,f)=1$, we have

$$\Theta(0,f)=\Theta(\infty,f)=1.$$

By Proposition 3.5, we get $f\in\mathcal{A}$. ❑

Remark. In the literature, a non-constant meromorphic function f satisfying $\delta(0,f)=\delta(\infty,f)=1$ is called meromorphic functions of class $\mathcal{F}$.

For further properties of meromorphic functions of class $\mathcal{A}$, we recall the following proposition.

Proposition 3.7 [9] *If $f\in\mathcal{A}$ and k is a positive integer, then*

(i) $T(r,\frac{f^{(k)}}{f})=S(r,f)$;

(ii) $T(r,f^{(k)})=T(r,f)+S(r,f)$;

(iii) $f^{(k)}(z)\in\mathcal{A}$.

Proof. Since $f\in\mathcal{A}$, we have $\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f)$. In particular, $\overline{N}(r,f)=S(r,f)$ and $\overline{N}(r,\frac{1}{f})=S(r,f)$.

By Lemma 2.16 , $m(r,\frac{f^{(k)}}{f})=S(r,f)$. Therefore,

$$\begin{aligned}T(r,\frac{f^{(k)}}{f}) &= N(r,\frac{f^{(k)}}{f})+m(r,\frac{f^{(k)}}{f})\\ &\le k\{\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})\}+S(r,f)\\ &= S(r,f),\end{aligned}$$

which implies (i).

By the basic property of characteristic function & (i), we have

$$T(r,f^{(k)})\le T(r,\frac{f^{(k)}}{f})+T(r,f)\le T(r,f)+S(r,f).$$

Similarly, we have

$$\begin{aligned}T(r,f) &\le T(r,f^{(k)})+T(r,\frac{f}{f^{(k)}})\\ &= T(r,f^{(k)})+T(r,\frac{f^{(k)}}{f})+O(1)\\ &= T(r,f^{(k)})+S(r,f).\end{aligned}$$

We obtain $T(r,f^{(k)}) = T(r,f) + S(r,f)$. So, (ii) holds.

Finally, by (i) & (ii), we have

$$\overline{N}(r,f^{(k)}) = \overline{N}(r,f) = S(r,f) = S(r,f^{(k)}),$$

and

$$\begin{aligned}\overline{N}(r,\frac{1}{f^{(k)}}) &\le \overline{N}(r,\frac{f}{f^{(k)}}) + \overline{N}(r,\frac{1}{f})\\ &\le T(r,\frac{f}{f^{(k)}}) + S(r,f)\\ &\le T(r,\frac{f^{(k)}}{f}) + S(r,f)\\ &= S(r,f).\end{aligned}$$

Therefore, $\overline{N}(r,f^{(k)}) + \overline{N}(r,\frac{1}{f^{(k)}}) = S(r,f^{(k)})$ and (iii) holds. ❑

Now, we can prove the main result in this section.

Theorem 3.8 *Let a and b be distinct complex numbers and f be a non-constant meromorphic function satisfies $\overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,\frac{1}{f-b}) = S(r,f)$. Then f is a Möbius transformation of a function in class $\mathcal{A}$*

Proof. Consider the meromorphic function g defined by

$$g(z) = \frac{f(z)-b}{f(z)-a}.$$

Then, by Theorem 2.25, $T(r,f) = T(r,g) + O(1)$. Obviously, we have

$$\overline{N}(r,g) = \overline{N}(r,\frac{1}{f-a})$$

and

$$\overline{N}(r,\frac{1}{g}) = \overline{N}(r,\frac{1}{f-b}).$$

Therefore, by assumption, we have

$$\begin{aligned}\overline{N}(r,g)+\overline{N}(r,\frac{1}{g}) &= \overline{N}(r,\frac{1}{f-a})+\overline{N}(r,\frac{1}{f-b})\\ &= S(r,f)\\ &= S(r,g),\end{aligned}$$

which says that g is a function of class $\mathcal{A}$. By a simple calculation, we get

$$f=\frac{ag-b}{g-1}$$

which says that f is a Möbius transformation of g. ❑

4 Multiple Values of Meromorphic Functions of Class $\mathcal{A}$

Definition 4.1 *Let f be a non-constant meromorphic function, and $a \in \mathbb{C}_\infty$. We say that a is a multiple value of f if all the zeros of $f(z) - a$ are multiple.*

Example 4.2 0 *is a multiple value of* $f(z) = (z-1)^2(z+1)^4$.

Example 4.3 $0, \infty$ *are multiple values of* e^z

For general meromorphic functions, we have the following well-known result about multiple values.

Theorem 4.4 *Let f be a non-constant meromorphic function, then f has at most four distinct multiple values.*

Proof. Suppose that f has five distinct multiple values, say $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C}_\infty$.

By Theorem 2.12,

$$\begin{aligned}(5-2)T(r,f) &< \sum_{j=1}^{5} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f) \\ &\leq \frac{1}{2}\sum_{j=1}^{5} N(r, \frac{1}{f-a_j}) + S(r,f) \\ &\leq \frac{1}{2}\sum_{j=1}^{5} T(r,f) + S(r,f) \\ &= \frac{5}{2} T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction. So f has at most four multiple values. ❑

In fact, there exists a meromorphic function which has exact four multiple values, namely, the well-known Weierstrass $\wp$-function $\wp(z)$ which satisfies the differential equation

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),$$

where e_1, e_2, e_3 are distinct constants. It is obvious that e_1, e_2, e_3 and ∞ are multiple values of $\wp(z)$. Therefore, Theorem 4.4 is sharp.

Now, we consider the case of meromorphic function of class $\mathcal{A}$ and prove our main result in this section as follows.

Theorem 4.5 *Let f be a meromorphic function of class $\mathcal{A}$, then f has at most two multiple values.*

Proof. Suppose that f has three distinct multiple values, say $a_1, a_2, a_3 \in \mathbb{C}_\infty$. Since $f \in \mathcal{A}$, we have

$$\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f).$$

Case 1. a_1, a_2, a_3 are different from 0 and ∞. Then, by Theorem 2.12,

$$\begin{aligned}(5-2)T(r,f) &\leq \sum_{j=1}^{3} \overline{N}(r,\frac{1}{f-a_j}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f) \\ &\leq \frac{1}{2}\sum_{j=1}^{3} N(r,\frac{1}{f-a_j}) + S(r,f) \\ &\leq \frac{3}{2}T(r,f) + S(r,f),\end{aligned}$$

which is impossible.

Case 2. One of a_1, a_2, a_3 is 0 or ∞.

$$\begin{aligned}(4-2)T(r,f) &\leq \sum_{j=1}^{2} \overline{N}(r,\frac{1}{f-a_j}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f) \\ &\leq \frac{1}{2}\sum_{j=1}^{2} N(r,\frac{1}{f-a_j}) + S(r,f) \\ &\leq T(r,f) + S(r,f),\end{aligned}$$

which is also impossible.

Case 3. Two of a_1, a_2, a_3 are 0 and ∞.

$$\begin{aligned}(3-2)T(r,f) &\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-a})+S(r,f)\\ &= \overline{N}(r,\frac{1}{f-a})+S(r,f)\\ &\leq \frac{1}{2}N(r,\frac{1}{f-a})+S(r,f)\\ &\leq \frac{1}{2}T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction.

Therefore, f has at most two multiple values. ❑

The function $f(z) = z^2e^z$ is of class $\mathcal{A}$ by Proposition 3.2, and it has exact two multiple values, namely, 0 and ∞. Therefore, Theorem 4.5 is sharp.

5 The Unicity of Meromorphic Functions of Class $\mathcal{A}$

In this section, we will discuss the sharing value problem of meromorphic function of class $\mathcal{A}$ and obtain some results.

In order to state and prove the theorems, we need some preliminaries.

Lemma 5.1 [9] *If $f \in \mathcal{A}$ and a is a finite non-zero number, then*

$$\overline{N}_{1)}(r, \frac{1}{f-a}) = T(r,f) + S(r,f),$$

where $\overline{N}_{1)}(r, \frac{1}{f-a})$ denotes the counting function of simple zeros of $f-a$.

The following result is stated without proof in [9]. For completeness, we give a proof.

Theorem 5.2 [9] *Let f and g be meromorphic functions of class $\mathcal{A}$ and a be a non-zero complex number. If f and g share a IM, then either $f \equiv g$ or $fg \equiv a^2$.*

Proof. By considering $\dfrac{f}{a}$ and $\dfrac{g}{a}$ if necessary, we may assume that $a = 1$.

By Lemma 5.1, we have

$$\overline{N}_{1)}(r, \frac{1}{f-1}) = T(r,f) + S(r,f)$$

and

$$\overline{N}_{1)}(r, \frac{1}{g-1}) = T(r,g) + S(r,g).$$

Hence,

$$N_{(2}(r, \frac{1}{f-1}) = S(r,f)$$

and

$$N_{(2}(r, \frac{1}{g-1}) = S(r,g),$$

where $N_{(2}(r, \frac{1}{f-1})$ is the counting function of $f-1$ with multiplicities greater or equal to 2, similarly for $N_{(2}(r, \frac{1}{g-1})$.

Since f and g are meromorphic functions of class $\mathcal{A}$ and they share 1 IM, by Theorem 2.12, we have

$$T(r,g) = T(r,f) + S(r,f).$$

Set

$$h(z) = \frac{f(z)-1}{g(z)-1}.$$

Obviously, we have

$$\overline{N}(r,h) \leq \overline{N}(r,f) + \overline{N}_{(2}(r, \frac{1}{g-1}) = S(r,f),$$

$$\overline{N}(r,\frac{1}{h}) \leq \overline{N}(r,g) + \overline{N}_{(2}(r, \frac{1}{f-1}) = S(r,f),$$

and

$$T(r,h) \leq T(r,f) + T(r,g) + O(1) \leq 2T(r,f) + S(r,f).$$

Let $f_1 = f$, $f_2 = h$, $f_3 = -hg$. Then $\sum_{j=1}^{3} f_j \equiv 1$ and

$$\sum_{j=1}^{3} N(r, \frac{1}{f_j}) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) = S(r,f).$$

By Theorem 2.24, we conclude that either $f_2 \equiv 1$ or $f_3 \equiv 1$ which imply that either $f \equiv g$ or $fg \equiv 1$ and the proof is completed. ❑

Finally, we consider the sharing value problem of a meromorphic function with its derivative.

The following well-known result has been proved by Frank-Weissenborn [2] in 1986.

Theorem 5.3 *Let f be a non-constant meromorphic function and $k \geq 1$. If f and $f^{(k)}$ share distinct finite value a and b CM, then $f \equiv f^{(k)}$.*

For meromorphic functions of class $\mathcal{A}$, we can use Theorem 5.2 to obtain the following result.

Theorem 5.4 *Let f be a non-constant meromorphic function of class $\mathcal{A}$, a be a non-zero complex number and $k \geq 1$. If f and $f^{(k)}$ share a IM, then $f \equiv f^{(k)}$.*

Proof. Since $f \in \mathcal{A}$, by Proposition 3.7, $f^{(k)} \in \mathcal{A}$.

Thus, we can apply Theorem 5.2 to conclude that either $f \equiv f^{(k)}$ or $ff^{(k)} \equiv a^2$. If $ff^{(k)} \equiv a^2$ holds, then

$$\begin{aligned} T(r, \frac{f^{(k)}}{f}) &= T(r, \frac{a^2}{f^2}) \\ &= 2T(r, f) + O(1). \end{aligned}$$

Which contradicts to Proposition 3.7, namely, $T(r, \frac{f^{(k)}}{f}) = S(r, f)$. Therefore, we must have $f \equiv f^{(k)}$. ❑

References

[1] C.-T. Chuang and C.-C. Yang, *Fix-points and factorization of meromorphic functions*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1990. Translated from the Chinese.

[2] G. Frank and G. Weissenborn, *Meromorphe Funktionen, die mit einer ihrer Ableitungen Werte teilen*, Complex Variables Theory Appl., 7 (1986), pp. 33–43.

[3] F. Gross, *Factorization of meromorphic functions*, Mathematics Research Center, Naval Research Laboratory, Washington, D. C., 1972.

[4] G. G. Gundersen, *Meromorphic functions that share three or four values*, J. London Math. Soc. (2), 20 (1979), pp. 457–466.

[5] ——, *Meromorphic functions that share four values*, Trans. Amer. Math. Soc., 277 (1983), pp. 545–567.

[6] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[7] H. Milloux, *Les fonctions méromorphes et leurs dérivées. Extensions d'un théorème de M. R. Nevanlinna. Applications*, Actualités Sci. Ind., no. 888, Hermann et Cie., Paris, 1940.

[8] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Chelsea Publishing Co., New York, 1974. Reprinting of the 1929 original.

[9] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, vol. 557 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 2003.

[10] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993. Translated and revised from the 1982 Chinese original.