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Optimal Two-Level Fractional Factorial Designs for Location Main Effects with Dispersion Factors

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Abstract

In two-level fractional factorial designs, homogeneous variance is a commonly made assumption in analysis of variance. When the variance of the response variable changes when a factor changes from one level to another, we call that factor the dispersion factor. The problem of finding optimal designs when dispersion factors present is relatively unexplored, however. In this article, we focus on finding optimal designs for the estimation of all location main effects when there are one or two dispersion factors, in the class of regular unreplicated two-level fractional factorial designs of resolution III and higher. We show that by an appropriate choice of the defining contrasts, *A*-optimal and *D*-optimal designs can be identified. Efficiencies of an arbitrary design are also investigated.

Key Words: Generator; Defining relation; Dispersion effect; Location effect.

1. Introduction

Two-level fractional factorial design is one of the most commonly and widely used designs to identify important location factors in industrial, agricultural and business experiments. The assumption of constant variance is a standard one when performing the analysis. In practice, however, situations when variance of the response variable differs from one treatment combination to another do happen. Factors that are responsible for such differences are called dispersion factors.

Identification of the dispersion factors has been extensively studied recently. Box and Meyer (1986) studied the logarithm of the ratio of the residual variance and proposed an informal method to identify dispersion factors. Montgomery (1990) achieved the same goal by plotting these statistics on a normal probability plot. Wang (1989) developed a large sample test statistic to identify dispersion factors. More recently, Bergman and Hynên (1997), Liao (2000), and McGrath and Lin (2001a) developed test procedures to identify dispersion factors in unreplicated regular 2^{n-p} fractional factorial designs. Pan (1999) and McGrath and Lin (2001b) stressed the importance of identifying the location effects before studying dispersion effects.

All of the aforementioned papers focused on identifying dispersion effects, not until Liao and Iyer (2000), the optimality property for the estimation of dispersion effects has been studied. Even though there is a growing interest in studying the optimality property for dispersion effects, the optimality property for location effects when dispersion factors present is relatively unexplored. Lin (2005) formed D-optimal designs for estimating a specific set of location effects with one dispersion factor. In this article, our interest is focused on finding *A*-optimal and *D*-optimal designs for estimating all location main effects when one or two dispersion factors present in the class of unreplicated regular 2^{n-p} fractional factorial designs of resolution III or higher.

In the next section, the notation used throughout this article and the information matrix for the estimation of all location main effects are stated. Section 3 gives the *A*-optimal and *D*-optimal designs for the estimation of all location main effects with one dispersion factors. *A*-efficiency and *D*-efficiency for an arbitrary design are also given. In section 4, *A*-optimal and *D*-optimal designs for estimating all location main effects with two dispersion factors are given. Catalogues of 16-run and 32-run *D*-efficient 2_{III}^{n-p} designs are provided.

2. Preliminaries

Let F_1, F_2, … , F_n denote the n two-level factors and the main effects of the corresponding factors as well. Let $F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n}$ denote the general effect with e_i = 1 if F_i appears in the effect, and e_i = 0, otherwise. Without loss of generality, we use F_1, F_2, …, and F_a to denote the a factors that are responsible for the dispersion effects.

Running a full factorial design may not be desirable, especially when n is large. Instead, running a fraction of the full factorial design, which is called a fractional factorial design, is sufficed when our interest is to estimate main effects and few low-order interactions of the design. A 2^{n-p} fractional factorial design with $N = 2^{n-p}$ runs can be determined by appropriately selecting p independent generators, and the design can uniquely be determined by its defining relation. For example, the treatment combinations of a 2^{6-3} design are determined when the following three generators F_4 = F_1F_2, F_5 = F_1F_3, and F_6 = F_2F_3 are selected. It's corresponding defining relation is

$$I = F_1F_2F_4 = F_1F_3F_5 = F_2F_3F_6 = F_4F_5F_6 = F_2F_3F_4F_5 = F_1F_3F_4F_6 = F_1F_2F_5F_6,$$

where I denotes the general mean and is the identity column. In this design, all of the main effects, and some of the low-order interactions can be estimated.

The resolution of a design depends on the alias structure. In the defining relation, an effect that is aliased with the general mean, is called a word, and the number of letters in a word is called the word length. The minimum length of all the words in the defining relation is called the resolution of the design for two-level factional factorial designs. The example above is a design of resolution III, and is denoted as 2_{III}^{6-3}.

In a regular 2^{n-p} fractional factorial design setting, let $\vec{Y}$ denote the response vector, and the model we employ here is

$$\vec{Y} = X\vec{\beta} + \varepsilon ,$$

where $\vec{\beta}$ is the $(n+1)\times 1$ vector of the overall mean and all location main effects; $X = [\vec{x}_0, \vec{x}_1, \cdots, \vec{x}_n]$ is the $N \times (n+1)$ model matrix, $\vec{x}_0 = (1, 1, \cdots, 1)'$, and $\vec{x}_j = (x_{1j}, x_{2j}, \cdots, x_{Nj})'$ with $x_{ij} = +1$ or -1 depends on whether factor j appears at its high level or low level in the i^{th} response; and $\vec{\varepsilon}$ is the $N \times 1$ vector of uncorrelated random error with $E(\vec{\varepsilon}) = \vec{0}$ and

$$Var(\vec{\varepsilon}) = \gamma_0 I + \gamma_1 D_1 + \gamma_2 D_2 + \cdots + \gamma_a D_a ,$$

where γ_0 is the dispersion mean, γ_j is the dispersion main effect of factor F_j by Liao and Iyer (2000), and D_j is the $N \times N$ diagonal matrix whose diagonal elements are $x_{1j}, x_{2j}, \cdots,$ and x_{Nj}.

Under the assumption of constant variance, that is, $\gamma_0 = \sigma^2$ and $\gamma_1 = \ldots = \gamma_a = 0$ then $Var(\vec{Y}) = \sigma^2 I$. The best linear unbiased estimator $\hat{\vec{\beta}}$ of $\vec{\beta}$ and the corresponding covariance matrix are

$$\hat{\vec{\beta}} = (X'X)^{-1} X'\vec{Y}, \text{ and}$$

$$Var(\hat{\vec{\beta}}) = (X'X)^{-1}\sigma^2, \text{ respectively.}$$

In here $X'X = NI_{n+1}$ since the designs we consider here are of resolution III or higher, and I_n is the $n \times n$ identity matrix.

Example 2.1. A 2_{III}^{5-2} fractional factorial design with generators $F_4 = F_1F_2$, and $F_5 = F_1F_3$, then the defining relation is

$$I = F_1F_2F_4 = F_1F_3F_5 = F_2F_3F_4F_5 .$$

The design matrix X is thus determined, and is

$$X = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

One can see that $X'X = 8I_6$, $\hat{\vec{\beta}} = (1/8)X'\vec{Y}$, and $Var(\hat{\vec{\beta}}) = (\sigma^2/8)I_6$.

For more general cases, that is, $Var(\vec{Y}) = \gamma_0 I + \gamma_1 D_1 + \cdots + \gamma_a D_a = V$, say, the best linear unbiased estimator $\hat{\vec{\beta}}$ of $\vec{\beta}$, and the corresponding covariance matrix are

$$\hat{\vec{\beta}} = (X'V^{-1}X)^{-1}X'V^{-1}\vec{Y}, \text{ and}$$

$$Var(\hat{\vec{\beta}}) = (X'V^{-1}X)^{-1}, \text{ respectively.}$$

Let M be the information matrix for the estimation of $\vec{\beta}$, then $M = X'V^{-1}X$.

3. Regular 2^{n-p} fractional factorial design with one dispersion factor

In this section, we focus on regular unreplicated 2^{n-p} fractional factorial designs with one dispersion factor. Without loss of generality, we assume F_1 is responsible for the dispersion effect, that is, $\gamma_0 > \gamma_1 \neq 0$, $\gamma_2 = \ldots = \gamma_a = 0$, and $Var(\vec{Y}) = V = \gamma_0 I_N + \gamma_1 D_1$. Then

$$V^{-1} = \frac{1}{\gamma_0^2 - \gamma_1^2}(\gamma_0 I_N - \gamma_1 D_1) \text{, and}$$

$M = (m_{ij}),\ i, j = 0, \cdots, n,$ can be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{'} & M_{22} \end{bmatrix}, \text{ where } M_{11} = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_0 \end{bmatrix}$$

is a 2 × 2 matrix with $m_0 = \gamma_0 N / (\gamma_0^2 - \gamma_1^2)$ and $m_1 = -\gamma_1 N / (\gamma_0^2 - \gamma_1^2)$; M_{12} is a 2 × $(n+1)$ matrix of zeroes; M_{22} is a $(n-1)\times(n-1)$ matrix, with $m_{ii} = m_0$, $i = 2, \ldots, n$, and for all $i \neq j = 2, \ldots, n$,

$$m_{ij} = \begin{cases} m_1, \text{ if } F_1 F_i F_j \text{ is a word in the defining relation,} \\ 0, \text{ otherwise.} \end{cases}$$

For the derivation of M, see Appendix.

Example 3.1. The same 2_{III}^{5-2} fractional factorial design as in Example 2.1. Then D_1 and M, respectively, are

$$D_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} m_0 & m_1 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_1 \\ 0 & 0 & m_1 & 0 & m_0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & m_0 \end{pmatrix}.$$

Since $F_1F_2F_4$ and $F_1F_3F_5$ are words in the defining relation, $m_{24} = m_{24} = m_{35} = m_{53} = m_1$, and all the other off-diagonal entries in M_{22} are zeroes.

3.1. Optimal 2^{n-p} fractional factorial design with one dispersion factor

Let θ be the number of length three words in the defining relation involving F_1. That is, θ is the number of words in the defining relation of form $F_1F_iF_j$, for $2 \le i < j \le n$. Then through some row and column operations, M can be transformed into M_T, where

$$M_T = \begin{bmatrix} I_{\theta+1} \otimes Q & 0 \\ 0 & m_0 I_{n-2\theta-1} \end{bmatrix}, \quad Q = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_0 \end{bmatrix},$$

and "⊗" is the Kronecker product. The eigenvalues of M can thus easily be obtained, and they are $m_0 - m_1$ with multiplicities θ+1, $m_0 + m_1$ with multiplicities θ+1, and m_0 with multiplicities $n - 2\theta - 1$.

Example 3.1. (Continued) For this 2^{5-2}_{III} design, one can see that n = 5, θ = 2, and M_T is given below. The eigenvalues of M are $m_0 - m_1$ with multiplicities 3, and $m_0 + m_1$ with multiplicities 3.

$$M_T = \begin{pmatrix} m_0 & m_1 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_0 & m_1 & 0 & 0 \\ 0 & 0 & m_1 & m_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_0 & m_1 \\ 0 & 0 & 0 & 0 & m_1 & m_0 \end{pmatrix}.$$

D-optimal design is the design that minimizes the determinant of M^{-1}, or equivalently maximizes the determinant of M. A-optimal design is the design that minimizes the trace of M^{-1}. For most of the eigenvalues based optimality criteria, for example, D-optimality and A-optimality, one can see that the smaller the value of θ is, the "better" the corresponding design is. The smallest possible value of θ is 0, and the D-optimal and A-optimal designs are those having the following information matrix

M^*,

$$M^* = \begin{bmatrix} Q & 0 \\ 0 & m_0 I_{n-1} \end{bmatrix}.$$

It is thus appropriate to define the D-efficiency, D_e, and A-efficiency, A_e, for an arbitrary design as

$$D_e = \frac{\det(M)}{\det(M^*)}, \quad A_e = \frac{tr(M^*)^{-1}}{tr(M)^{-1}}.$$

Through some calculation, one has

$$D_e = (1-(\gamma_1/\gamma_0)^2)^\theta, \text{ and } A_e = 1 - \frac{2\theta}{(n+1)(\gamma_0/\gamma_1)^2 + 2\theta + 1 - n}.$$

One can observe that both D_e and A_e are decreasing in θ, that is, when there are more length three words involving F_1 in the defining relation, the less efficient the corresponding design is. Also, both D_e and A_e are decreasing in γ_1, that is, the larger the dispersion effect is, the less efficient the corresponding design is.

Example 3.2. A 2_{III}^{6-2} factional factorial design with generators $F_5 = F_2F_3$, and $F_6 = F_1F_2F_4$. The defining relation is

$$I = F_2F_3F_5 = F_1F_2F_4F_6 = F_1F_3F_4F_5F_6 .$$

Since there is no length three word of form $F_1F_iF_j$ in the defining relation, hence $\theta = 0$, and the corresponding information matrix M is of the optimal form, that is

$$M = M^* = \begin{bmatrix} Q & 0 \\ 0 & m_0 I_5 \end{bmatrix}.$$

The 2_{III}^{6-2} fractional factorial design above is thus both D-optimal and A-optimal in estimating all location main effects when F_1 is responsible for the dispersion effect in the model.

Example 3.3. A 2_{III}^{6-2} factional factorial design with generators $F_5 = F_1F_2$, and $F_6 =$

F_3F_4. The defining relation and the corresponding information matrix M are

$$I = F_1F_2F_5 = F_3F_4F_6 = F_1F_2F_3F_4F_5F_6,$$

$$M = \begin{pmatrix} m_0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_0 \end{pmatrix}, \text{ respectively.}$$

One can see that there is one length three word of form $F_1F_iF_j$ in the defining relation, hence $\theta = 1$, and

$$D_e = 1 - (\gamma_1/\gamma_0)^2, \text{ and } A_e = 1 - \frac{2}{7(\gamma_0/\gamma_1)^2 - 3}.$$

Example 3.4. A 2_{III}^{6-2} factional factorial design with generators $F_5 = F_1F_2$, and $F_6 = F_1F_3$. The defining relation and the corresponding information matrix M are

$$I = F_1F_2F_5 = F_1F_3F_6 = F_2F_3F_5F_6,$$

$$M = \begin{pmatrix} m_0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & m_0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 & m_0 \end{pmatrix}, \text{ respectively.}$$

Since there are two length three words of form $F_1F_iF_j$ in the defining relation, hence $\theta = 2$, and

$$D_e = (1 - (\gamma_1/\gamma_0)^2)^2, \text{ and } A_e = 1 - \frac{4}{7(\gamma_0/\gamma_1)^2 - 1}.$$

One can observed that for fixed values of γ_0 and γ_1, $\gamma_0 > \gamma_1$, the design in Example 3.3 is more efficient than the design in Example 3.4 since both of its A_e and

D_e are larger. We say that the design in Example 3.3 is *D*-better and *A*-better than the design in Example 3.4

Remark：For designs of resolution IV or higher, the shortest word in the defining relation is of length at least four, hence all their information matrices are of form M^*. Thus, resolution IV or higher designs are "robust" against single dispersion factor when our interest is to estimate all location main effects.

4. Regular 2^{n-p} fractional factorial design with two dispersion factors

In this section, we focus on regular unreplicated 2^{n-p} fractional factorial designs with two dispersion factors. Without loss of generality, we assume that F_1 and F_2 are responsible for the dispersion effects, that is, $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $\gamma_3 = \ldots = \gamma_a = 0$, and $Var(\vec{Y}) = V = \gamma_0 I + \gamma_1 D_1 + \gamma_2 D_2$. Then $\gamma_0 > \gamma_1 + \gamma_2$, and

$$NV^{-1} = m_0 I + m_1 D_1 + m_2 D_2 + m_3 D_1 D_2,$$

where

$$m_0 = \frac{\gamma_0(\gamma_0^2 - \gamma_1^2 - \gamma_2^2)N}{\gamma_0^2(\gamma_0^2 - \gamma_1^2 - \gamma_2^2) + \gamma_1^2(\gamma_1^2 - \gamma_2^2 - \gamma_0^2) + \gamma_2^2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2)},$$

$$m_1 = \frac{\gamma_1(\gamma_1^2 - \gamma_2^2 - \gamma_0^2)N}{\gamma_0^2(\gamma_0^2 - \gamma_1^2 - \gamma_2^2) + \gamma_1^2(\gamma_1^2 - \gamma_2^2 - \gamma_0^2) + \gamma_2^2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2)},$$

$$m_2 = \frac{\gamma_2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2)N}{\gamma_0^2(\gamma_0^2 - \gamma_1^2 - \gamma_2^2) + \gamma_1^2(\gamma_1^2 - \gamma_2^2 - \gamma_0^2) + \gamma_2^2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2)},$$

$$m_3 = \frac{2\gamma_0\gamma_1\gamma_2 N}{\gamma_0^2(\gamma_0^2 - \gamma_1^2 - \gamma_2^2) + \gamma_1^2(\gamma_1^2 - \gamma_2^2 - \gamma_0^2) + \gamma_2^2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2)}.$$

The information matrix $M = (m_{ij}),\ i, j = 0, \cdots, n,$ for the estimation of $\vec{\beta}$ again can be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{bmatrix}, \text{ where } M_{11} = \begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_0 & m_3 \\ m_2 & m_3 & m_0 \end{bmatrix};$$

M_{12} is a 3 × (n–2) matrix and for i = 0, 1, 2, j = 3, … , n, $m_{0j} = m_3$, $m_{1j} = m_2$, and $m_{2j} = m_1$ if $F_1F_2F_j$ is a word in the defining relation, otherwise $m_{ij} = 0$; M_{22} is a $(n-2)\times(n-2)$ matrix whose diagonal elements are m_0 and off-diagonal elements m_{ij}, $i \neq j$ = 3, … , n, is

$$m_{ij} = \begin{cases} m_1, & \text{if } F_1F_iF_j \text{ is a word in the defining relation,} \\ m_2, & \text{if } F_2F_iF_j \text{ is a word in the defining relation,} \\ m_3, & \text{if } F_1F_2F_iF_j \text{ is a word in the defining relation,} \\ 0, & \text{otherwise.} \end{cases}$$

Some characteristics concerning M are listed below.

1. There is at most one j in M_{12} such that $m_{ij} \neq 0$. That is, M_{12} is either a matrix of zeroes, or a matrix with exactly one column of form $[m_3, m_2, m_1]'$ and all the other entries are zeroes.
2. In M_{22}, the number of appearances of m_i, i = 1, 2, 3, is at most one in each row and each column. That is, it is not possible to have two m_1's, two m_2's, or two m_3's in any row or column.
3. In M_{22}, if $m_{ij} = m_1$ (or m_2), $m_{ik} = m_2$ (or m_1), then $m_{jk} = m_3$, $3 \leq i < j < k \leq n$.

The derivation of M and its characteristics are given in the Appendix .

Example 4.1. A 2_{III}^{6-2} design with generators $F_5 = F_1F_3$ and $F_6 = F_2F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_3F_5 = F_2F_4F_6 = F_1F_2F_3F_4F_5F_6,$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m_2 & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & m_2 \\ 0 & 0 & 0 & m_1 & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & m_0 \end{bmatrix}, \text{ respectively.}$$

There is no length three word of form $F_1F_2F_j$, hence, M_{12} is a zero matrix. As for M_{22}, since $F_1F_3F_5$ is in the defining relation, $m_{35} = m_{53} = m_1$, and since $F_2F_4F_6$ is in the defining relation, $m_{46} = m_{64} = m_2$.

Example 4.2. A 2_{III}^{6-2} design with generators $F_5 = F_1F_3$ and $F_6 = F_2F_3$, the defining relation and the corresponding information matrix M are

$$I = F_1F_3F_5 = F_2F_3F_6 = F_1F_2F_5F_6, \text{ and}$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m_2 & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_1 & m_2 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & m_0 & m_3 \\ 0 & 0 & 0 & m_2 & 0 & m_3 & m_0 \end{bmatrix}, \text{ respectively.}$$

There is no length three word of form $F_1F_2F_j$, hence, M_{12} is a zero matrix. As for M_{22}, since $F_1F_3F_5$ is in the defining relation, $m_{35} = m_{53} = m_1$, and since $F_2F_3F_6$ is in the defining relation, $m_{36} = m_{63} = m_2$. Now since both $F_1F_3F_5$ and $F_2F_3F_6$ are in the defining relation, $F_1F_2F_5F_6$ is automatically in the defining relation, and hence $m_{56} = m_{65} = m_3$.

Example 4.3. A 2_{III}^{6-3} design with generators $F_4 = F_1F_2$, $F_5 = F_1F_3$, and $F_6 = F_2F_3$, the defining relation and the corresponding information matrix M are

$$I = F_1F_2F_4 = F_1F_3F_5 = F_2F_3F_6 = F_4F_5F_6 = F_2F_3F_4F_5 = F_1F_3F_4F_6 = F_1F_2F_5F_6, \text{ and}$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & 0 & m_3 & 0 & 0 \\ m_1 & m_0 & m_3 & 0 & m_2 & 0 & 0 \\ m_2 & m_3 & m_0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_1 & m_2 \\ m_3 & m_2 & m_1 & 0 & m_0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & m_0 & m_3 \\ 0 & 0 & 0 & m_2 & 0 & m_3 & m_0 \end{bmatrix}, \text{ respectively.}$$

There is one length three word $F_1F_2F_4$ of form $F_1F_2F_j$ in the defining relation, hence there is one column of $[m_3, m_2, m_1]'$ in M_{12}. The structure of M_{22} in here is the same as in the previous example.

4.1. Optimal 2^{n-p} fractional factorial designs with two dispersion factors

Same as in section 3, for most of the eigenvalues based optimality criteria, the "optimal" information matrix, M^*, if exists, is of the following form

$$M^* = \begin{bmatrix} M_{11} & 0 \\ 0 & m_0 I_{n-2} \end{bmatrix},$$

that is, the defining relation of the corresponding optimal design does not contain any length three word involving F_1 and F_2, and length four word involving both F_1 and F_2.

Let λ_1, λ_2, and λ_3 be the eigenvalues of M_{11}. Then

$$\begin{aligned} \det(M^*) &= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot m_0^{\,n-2} \\ &= m_0^{\,n+1} - m_0^{\,n-1}(m_1^2 + m_2^2 + m_3^2) + 2m_0^{\,n-2} m_1 m_2 m_3, \text{ and} \end{aligned}$$

$$\begin{aligned} tr(M^*)^{-1} &= \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + (n-2)m_0^{-1} \\ &= \frac{3m_0^2 - m_1^2 - m_2^2 - m_3^2}{m_0^3 - (m_1^2 + m_2^2 + m_3^2)m_0 + 2m_1 m_2 m_3} + \frac{n-2}{m_0}. \\ &= \varphi + (n-2)m_0^{-1}, \text{ say.} \end{aligned}$$

4.2. Efficient resolution IV designs

For resolution IV designs, there is no length three word in the defining relation, and the transformed information matrix M_T is thus of the following simpler form

$$M_T = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22(IV)} \end{bmatrix}, \text{ where } M_{22(IV)} = \begin{bmatrix} I_\delta \otimes T & 0 \\ 0 & m_0 I_{n-2\delta-2} \end{bmatrix}, \ T = \begin{bmatrix} m_0 & m_3 \\ m_3 & m_0 \end{bmatrix}, \text{ and}$$

δ is the number of length four words in the defining relation involving both F_1 and F_2. The eigenvalues of M are λ_1, λ_2, λ_3, and $m_0 - m_3$ with multiplicities δ, $m_0 + m_3$ with multiplicities δ, and m_0 with multiplicities $n - 2\delta - 2$. Then D_e and A_e for an arbitrary design are thus determined,

$$D_e = \frac{\det(M)}{\det(M^*)} = (1 - (m_3 / m_0)^2)^\delta = \left(1 - \left(\frac{2\gamma_1\gamma_2}{\gamma_0^2 - \gamma_1^2 - \gamma_2^2}\right)^2\right)^\delta, \text{ and}$$

$$A_e = \frac{tr(M^*)^{-1}}{tr(M)^{-1}} = 1 - \frac{2\delta}{(\varphi m_0 + n - 2)((m_0 / m_3)^2 - 1) + 2\delta}.$$

Both D_e and A_e are decreasing in δ, that is, when there are more length four words involving both F_1 and F_2 in the defining relation, the less efficient the corresponding design is. Also, D_e is decreasing in γ_1 and γ_2, that is, the larger the dispersion effects are, the less efficient the corresponding design is.

Example 4.4. A 2_{IV}^{6-2} design with generators $F_5 = F_1F_2F_3$, and $F_6 = F_1F_3F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_2F_3F_5 = F_1F_3F_4F_6 = F_2F_4F_5F_6, \text{ and}$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m_2 & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & 0 & m_3 & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_0 \end{bmatrix}, \text{ respectively.}$$

There is one word $F_1F_2F_3F_5$ of form $F_1F_2F_iF_j$ in the defining relation, hence, m_{35} = m_{53} = m_3, and δ = 1, then

$$D_e = 1 - \left(\frac{2\gamma_1\gamma_2}{\gamma_0^2 - \gamma_1^2 - \gamma_2^2} \right)^2, \text{ and}$$

$$A_e = 1 - \frac{2}{(\varphi m_0 + 4)((m_0 / m_3)^2 - 1) + 2}.$$

Example 4.5. A 2_{IV}^{6-2} design with generators F_5 = $F_1F_2F_3$, and F_6 = $F_1F_2F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_2F_3F_5 = F_1F_2F_4F_6 = F_3F_4F_5F_6, \text{ and}$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & 0 & 0 & 0 & 0 \\ m_1 & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m_2 & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & m_3 \\ 0 & 0 & 0 & m_3 & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & m_3 & 0 & m_0 \end{bmatrix}, \text{ respectively.}$$

There are two length four words, $F_1F_2F_3F_5$ and $F_1F_2F_4F_6$, involving both F_1 and F_2, hence m_{35} = m_{53} = m_{46} = m_{64} = m_3, and δ = 2, then

$$D_e = \left(1 - \left(\frac{2\gamma_1\gamma_2}{\gamma_0^2 - \gamma_1^2 - \gamma_2^2} \right)^2 \right)^2, \text{ and}$$

$$A_e = 1 - \frac{4}{(\varphi m_0 + 4)((m_0/m_3)^2 - 1) + 4}.$$

This design has more length four words of form $F_1F_2F_iF_j$, it is thus less efficient than the design in Example 4.3. As one can also see from the values of D_e and A_e of the design in Example 4.4, both of them are smaller than those in Example 4.3.

Remark：Resolution V or higher designs are robust against two dispersion effects, if our interest is focused on estimating location main effects.

4.3. Efficient resolution III designs

Efficiencies of resolution III designs depend not only on the values of γ_0, γ_1 and γ_2, but also on the number of length three words of forms $F_1F_2F_j$, $F_1F_iF_j$, $F_2F_iF_j$, and the number of length four words of form $F_1F_2F_iF_j$ in the defining relation. Due to the complexity in calculating the efficiencies of an arbitrary design, we focus on investigating the D-efficiency of designs when $\gamma_1 = \gamma_2 = \gamma$, say. Then $m_1 = m_2 = m$, say, and

$$T = \begin{bmatrix} m_0 & m_3 \\ m_3 & m_0 \end{bmatrix},\quad Q = \begin{bmatrix} m_0 & m \\ m & m_0 \end{bmatrix},\quad M_{11} = \begin{bmatrix} m_0 & m & m \\ m & m_0 & m_3 \\ m & m_3 & m_0 \end{bmatrix}. \text{ Now, let}$$

$$U = \begin{bmatrix} m_0 & m & m & m_3 \\ m & m_0 & m_3 & m \\ m & m_3 & m_0 & m \\ m_3 & m & m & m_0 \end{bmatrix},$$

apply some row and column operations on M, M can be transformed into M_T, where

$$M_T = \begin{bmatrix} I_{\delta_1} \otimes U & 0 & 0 & 0 & 0 \\ 0 & I_{\delta_2 \otimes M_{11}} & 0 & 0 & 0 \\ 0 & 0 & I_{\delta_3} \otimes T & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_4} \otimes Q & 0 \\ 0 & 0 & 0 & 0 & m_0 I_{\delta_5} \end{bmatrix},$$

δ_1 is the number of U matrices in M_T, δ_2 is the number of M_{11} matrices in M_T, and so on. δ_1, δ_2, δ_3, and δ_4 are functions of the number of words of forms $F_1F_2F_j$, $F_1F_iF_j$, $F_2F_iF_j$, and $F_1F_2F_iF_j$ in the defining relation, where $\delta_1 + \delta_2 \geq 1$ and $4\delta_1 + 3\delta_2 + 2\delta_3 + 2\delta_4 + \delta_5 = n+1$. Then

$$\begin{aligned} \det(M) &= (\det(U))^{\delta_1} (\det(M_{11}))^{\delta_2} (\det(T))^{\delta_3} (\det(Q))^{\delta_4} m_0^{\delta_5} \\ &= (\det(M_{11}))^{\delta_1+\delta_2} (m_0 - m_3)^{\delta_1+\delta_3} (m_0 + m_3)^{\delta_3} (m_0^2 - m^2)^{\delta_4} m_0^{\delta_5}, \end{aligned}$$

where $\det(T) = m_0^2 - m_3^2$, $\det(Q) = m_0^2 - m^2$,

$$\det(M_{11}) = m_0(m_0^2 - 2m^2 - m_3^2) + 2m^2 m_3,$$

$$\det(U) = (m_0 - m_3)\det(M_{11}), \text{ and}$$

D_e can thus be determined

$$D_e = \frac{\det M}{\det M^*} = \frac{(\det(M_{11}))^{\delta_1+\delta_2-1}(m_0 - m_3)^{\delta_1+\delta_3}(m_0 + m_3)^{\delta_3}(m_0^2 - m^2)^{\delta_4}}{m_0^{n-\delta_5-2}}.$$

Example 4.1. (Continued) For this 2_{III}^{6-2} design, δ_1 = 0, δ_2 = 1, δ_3 = 0, δ_4 = 2, and δ_5 = 0. Then $\det(M) = (\det(M_{11}))(m_0^2 - m^2)^2$, and $D_e = (1-(m/m_0)^2)^2$.

Example 4.2. (Continued) For this 2_{III}^{6-2} design, δ_1 = 0, δ_2 = 2, δ_3 = δ_4 = 0, and δ_5 = 1. Then

$$\det(M) = (\det(M_{11}))^2 m_0, \text{ and } D_e = \frac{m_0(m_0^2 - 2m^2 - m_3^2) + 2m^2 m_3}{m_0^3}.$$

Example 4.6. A 2_{III}^{6-2} design with generators F_5 = F_1F_3, and F_6 = $F_2F_3F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_3F_5 = F_2F_3F_4F_6 = F_1F_2F_4F_5F_6 \text{, and}$$

$$M = \begin{bmatrix} m_0 & m & m & 0 & 0 & 0 & 0 \\ m & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_0 \end{bmatrix} \text{, respectively.}$$

For this design $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = 0$, $\delta_4 = 1$, and $\delta_5 = 2$. Then

$$\det(M) = (\det(M_{11}))(m_0^2 - m^2)m_0^2 \text{, and } D_e = 1 - (m/m_0)^2 .$$

Example 4.7. A 2_{III}^{6-2} design with generators $F_5 = F_1F_2$, and $F_6 = F_2F_3F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_2F_5 = F_2F_3F_4F_6 = F_1F_3F_4F_5F_6 \text{, and}$$

$$M = \begin{bmatrix} m_0 & m & m & 0 & 0 & m_3 & 0 \\ m & m_0 & m_3 & 0 & 0 & m & 0 \\ m & m_3 & m_0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & m_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & 0 \\ m_3 & m & m & 0 & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_0 \end{bmatrix} \text{, respectively.}$$

For this design $\delta_1 = 1$, $\delta_2 = \delta_3 = \delta_4 = 0$, and $\delta_5 = 3$. Then

$$\det(M) = (\det(M_{11}))(m_0 - m_3)m_0^3 \text{, and}$$

$$D_e = \frac{m_0^2(m_0^2 - 2m^2 - m_3^2) + m_3^2(m_3^2 + 2m^2 - m_0^2) + 4m^2 m_0 m_3 - 2m^2(m_0 - m_3)}{m_0(m_0(m_0^2 - 2m^2 - m_3^2) + 2m^2 m_3)} .$$

Through some straightforward calculations, one can see that the D_e of the design in Example 4.6 is the highest, the design in Example 4.1 is the second highest, and then the design in Example 4.2. The D_e of the design in Example 4.7 is the lowest among the four designs.

The rankings, according to the values of D_e, of the above four 2_{III}^{6-2} designs with different generators are not hard to obtain. In the beginning, we think that values of $\delta_1, \ldots, \delta_5$ determine the "structure" of the defining relation, hence the design. If we can find an ordering of all possible structures, then the corresponding *D*-better designs can be determined regardless of the values of γ_0 and γ. However, after having been extensively investigating many designs, we realize that it is generally not true. Example 4.7 above and Example 4.8 below show how values of γ_0 and γ affect the values of D_e of two designs with fixed defining relations.

Example 4.8. A 2_{III}^{6-2} design with generators $F_5 = F_1F_3$, and $F_6 = F_1F_2F_4$. The defining relation and the corresponding information matrix M are

$$I = F_1F_3F_5 = F_2F_4F_6 = F_1F_2F_3F_4F_5F_6\text{, and}$$

$$M = \begin{bmatrix} m_0 & m & m & 0 & 0 & 0 & 0 \\ m & m_0 & m_3 & 0 & 0 & 0 & 0 \\ m & m_3 & m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m_0 & 0 & m_3 \\ 0 & 0 & 0 & m & 0 & m_0 & 0 \\ 0 & 0 & 0 & 0 & m_3 & 0 & m_0 \end{bmatrix}\text{, respectively.}$$

For this design $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = \delta_4 = 1$, and $\delta_5 = 0$. Then

$$\det(M) = (\det(M_{11}))(m_0^2 - m^2)(m_0^2 - m_3^2)\text{, and}$$

$$D_e = 1 - (m/m_0)^2 - (m_3/m_0)^2 + (mm_3/m_0^2)^2\,.$$

When $\gamma_0/\gamma > 2.590901$, the D_e of this design is larger than the D_e of the design in Example 4.7, that is, Example 4.8 is *D*-better than Example 4.7. Whereas, when $\gamma_0/\gamma \leq 2.590901$, the D_e of this design is smaller, that is, Example 4.7 is *D*-better.

Although the ordering of the D_e of 2_{III}^{n-p} fractional factorial designs varies when $\delta_1, \ldots, \delta_5$, γ_0, and γ vary, it seems to the authors that a design tends to have a larger value of D_e if δ_1 is small and δ_5 is large.

Based on the catalogues of the 16-run and 32-run 2^{n-p} fractional factorial designs in Chen, Sun, and Wu (1993), we provide tables of the factors that we suggest to be named as the factors that are responsible for the dispersion effects, such that the D-efficiency of the resulting designs are higher. For example, design 6-2.2 in Table 4.1 with generators $F_5 = F_1F_2$, and $F_6 = F_1F_3F_4$, if we assign F_3 and F_4, or F_3 and F_6, or F_4 and F_6 as the factors that are responsible for the dispersion effects, the D_e of the resulting designs are higher or even the highest depending on the values of γ_0 and γ. The designs with bold faced dispersion factors are D-optimal designs, regardless of the values of γ_0 and γ.

There are cases when D_e depends on the size of the dispersion effects. For example, design 9-4.7 in Table 4.2, when $\gamma \geq \gamma_0/\sqrt{6}$, that is, the dispersion effects are relatively large, the D_e of the designs when dispersion factors are named as (F_2,F_3), (F_2,F_4), or (F_6,F_9) are higher; and when $\gamma < \gamma_0/\sqrt{6}$, that is, the dispersion effects are relatively moderate, the D_e of the designs when dispersion factors are named as (F_3,F_4), (F_3,F_5), or (F_8,F_9) are higher. These factors with moderate dispersion effects are shadowed in the table.

Table 4.1. Suggested dispersion factors for 16-run 2_{III}^{n-p} fractional factorial designs.

Design	Generators	Dispersion Factors
5-1.3	$F_5=F_1F_2$	**(F_3,F_4)**
6-2.2	$F_5=F_1F_2$, $F_6=F_1F_3F_4$	(F_3,F_4), (F_3,F_6), (F_4,F_6)
6-2.3	$F_5=F_1F_2$, $F_6=F_3F_4$	(F_1,F_2), (F_1,F_5), (F_2,F_5), (F_3,F_4), (F_3,F_6), (F_4,F_6)
6-2.4	$F_5=F_1F_2$, $F_6=F_1F_3$	(F_2,F_4), (F_3,F_4), (F_4,F_5), (F_4,F_6)
7-3.2	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3F_4$	(F_4,F_7)
7-3.3	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_4$	(F_3,F_4), (F_3,F_7), (F_4,F_6), (F_6,F_7)
7-3.4	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_1F_4$	(F_2,F_3), (F_2,F_4), (F_2,F_6), (F_2,F_7), (F_3,F_4), (F_3,F_5), (F_3,F_7), (F_4,F_5), (F_4,F_6), (F_5,F_6), (F_5,F_7), (F_6,F_7)
7-3.5	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$	(F_1,F_4), (F_2,F_4), (F_3,F_4), (F_4,F_5), (F_4,F_6), (F_4,F_7)
8-4.2	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_1F_4$, $F_8=F_2F_3F_4$	(F_2,F_8), (F_3,F_8), (F_4,F_8), (F_5,F_8), (F_6,F_8), (F_7,F_8)
8-4.3	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_4$, $F_8=F_3F_4$	(F_5,F_8), (F_6,F_7)
8-4.4	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_2F_3F_4$	(F_4,F_8)
8-4.5	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$	(F_4,F_7), (F_4,F_8), (F_7,F_8)
8-4.6	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_2F_3$	(F_1,F_4), (F_2,F_4), (F_3,F_4), (F_4,F_5), (F_4,F_6), (F_4,F_7), (F_4,F_8)
9-5.1	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_1F_4$, $F_8=F_2F_3F_4$, $F_9=F_1F_2F_3F_4$	(F_2,F_3), (F_2,F_4), (F_2,F_5), (F_2,F_6), (F_2,F_7), (F_2,F_8), (F_2,F_9), (F_3,F_4), (F_3,F_5), (F_3,F_6), (F_3,F_7), (F_3,F_8), (F_3,F_9), (F_4,F_5), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_9), (F_5,F_6), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_6,F_7), (F_6,F_8), (F_6,F_9), (F_7,F_8), (F_7,F_9), (F_8,F_9)
9-5.2	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_4$, $F_8=F_3F_4$, $F_9=F_1F_2F_3F_4$	(F_1,F_2), (F_1,F_3), (F_1,F_5), (F_1,F_6), (F_2,F_4), (F_2,F_5), (F_2,F_7), (F_3,F_4), (F_3,F_6), (F_3,F_8), (F_4,F_7), (F_4,F_8), (F_5,F_8), (F_5,F_9), (F_6,F_7), (F_6,F_9), (F_8,F_9)
9-5.3	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$	(F_8,F_9)
9-5.4	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_4$	(F_3,F_8), (F_3,F_9), (F_4,F_6), (F_4,F_7), (F_6,F_9), (F_7,F_8)
9-5.5	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_2F_3$, $F_9=F_1F_4$	(F_4,F_9)
10-6.1	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$	(F_7,F_9), (F_7,F_{10}), (F_9,F_{10})
10-6.2	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_4$, $F_{10}=F_1F_3F_4$	(F_3,F_4), (F_3,F_{10}), (F_4,F_{10})
10-6.3	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_4$, $F_{10}=F_3F_4$	(F_1,F_7), (F_1,F_9), (F_1,F_{10}), (F_2,F_6), (F_2,F_8), (F_2,F_{10}), (F_3,F_5), (F_3,F_8), (F_3,F_9), (F_4,F_5), (F_4,F_6), (F_4,F_7), (F_5,F_{10}), (F_6,F_9), (F_7,F_8)
10-6.4	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_4$	(F_4,F_9), (F_4,F_{10}), (F_9,F_{10})
11-7.1	$F_5=F_1F_2$, $F_6=F_1F_3$, $F_7=F_2F_3$, $F_8=F_1F_4$, $F_9=F_2F_4$, $F_{10}=F_1F_3F_4$, $F_{11}= F_2F_3F_4$	(F_1,F_2), (F_1,F_6), (F_1,F_8), (F_1,F_{11}), (F_2,F_7), (F_2,F_9), (F_2,F_{10}), (F_6,F_7), (F_6,F_9), (F_6,F_{10}), (F_7,F_8), (F_7,F_{11}), (F_8,F_9), (F_8,F_{10}), (F_9,F_{11}), (F_{10},F_{11})

Table 4.1. (Continued)

Design	Generators	Dispersion Factors
11-7.2	F_5=F_1F_2, F_6=F_1F_3, F_7=F_2F_3, F_8=$F_1F_2F_3$, F_9=F_1F_4, F_{10}=F_2F_4, F_{11}= F_3F_4	(F_4,F_8), (F_4,F_9), (F_4,F_{10}), (F_4,F_{11}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_9,F_{10}), (F_9,F_{11}), (F_{10},F_{11})
11-7.3	F_5=F_1F_2, F_6=F_1F_3, F_7=F_2F_3, F_8=$F_1F_2F_3$, F_9=F_1F_4, F_{10}=F_2F_4, F_{11}= $F_1F_2F_4$	(F_3,F_4), (F_3,F_6), (F_3,F_7), (F_3,F_8), (F_3,F_9), (F_3,F_{10}), (F_3,F_{11}), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_9), (F_4,F_{10}), (F_4,F_{11}), (F_6,F_7), (F_6,F_8), (F_6,F_9), (F_6,F_{10}), (F_6,F_{11}), (F_7,F_8), (F_7,F_9), (F_7,F_{10}), (F_7,F_{11}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_9,F_{10}), (F_9,F_{11}), (F_{10},F_{11})
12-8.1	F_5=F_1F_2, F_6=F_1F_3, F_7=F_2F_3, F_8=F_1F_4, F_9=F_2F_4, F_{10}=$F_1F_3F_4$, F_{11}= $F_2F_3F_4$, F_{12}=$F_1F_2F_3F_4$	(F_1,F_2), (F_1,F_3), (F_1,F_4), (F_1,F_5), (F_1,F_6), (F_1,F_7), (F_1,F_8), (F_1,F_9), (F_1,F_{10}), (F_1,F_{11}), (F_1,F_{12}), (F_2,F_3), (F_2,F_4), (F_2,F_5), (F_2,F_6), (F_2,F_7), (F_2,F_9), (F_2,F_{10}), (F_2,F_{11}), (F_2,F_{12}), (F_3,F_4), (F_3,F_5), (F_3,F_6), (F_3,F_7), (F_3,F_8), (F_3,F_9), (F_3,F_{10}), (F_3,F_{11}), (F_3,F_{12}), (F_4,F_5), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_9), (F_4,F_{10}), (F_4,F_{11}), (F_4,F_{12}), (F_5,F_6), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_5,F_{10}), (F_5,F_{11}), (F_5,F_{12}), (F_6,F_7), (F_6,F_9), (F_6,F_{10}), (F_6,F_{11}), (F_6,F_{12}), (F_7,F_8), (F_7,F_9), (F_7,F_{10}), (F_7,F_{11}), (F_7,F_{12}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_8,F_{12}), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{11},F_{12})
12-8.2	F_5=F_1F_2, F_6=F_1F_3, F_7=F_2F_3, F_8=$F_1F_2F_3$, F_9=F_1F_4, F_{10}=F_2F_4, F_{11}= $F_1F_2F_4$, F_{12}=F_3F_4	(F_3,F_4), (F_3,F_9), (F_3,F_{10}), (F_3,F_{11}), (F_3,F_{12}), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_{12}), (F_6,F_9), (F_6,F_{10}), (F_6,F_{11}), (F_6,F_{12}), (F_7,F_9), (F_7,F_{10}), (F_7,F_{11}), (F_7,F_{12}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_8,F_{12}), (F_9,F_{12}), (F_{10},F_{12}), (F_{11},F_{12})

Note: Designs with bold face dispersion factors are D-optimal designs.

Table 4.2. Suggested dispersion factors for 32-run 2^{n-p}_{III} fractional factorial designs

Design	Generators	Dispersion Factors
7-2.4	$F_6=F_1F_2$, $F_7=F_1F_3F_4F_5$	**(F_3,F_4), (F_3,F_5), (F_3,F_7), (F_4,F_5), (F_4,F_7), (F_5,F_7)**
7-2.5	$F_6=F_1F_2$, $F_7=F_3F_4F_5$	(F_3,F_4), (F_3,F_5), (F_3,F_7), (F_4,F_5), (F_4,F_7), (F_5,F_7)
7-2.6	$F_6=F_1F_2$, $F_7=F_1F_3F_4$	**(F_3,F_5), (F_4,F_5), (F_5,F_7)**
7-2.7	$F_6=F_1F_2$, $F_7=F_3F_4$	(F_1,F_5), (F_2,F_5), (F_3,F_5), (F_4,F_5), (F_5,F_6), (F_5,F_7)
7-2.8	$F_6=F_1F_2$, $F_7=F_1F_3$	**(F_4,F_5)**
8-3.5	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_2F_3F_5$	**(F_4,F_5), (F_4,F_8), (F_5,F_7), (F_7,F_8)**
8-3.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3F_4F_5$	**(F_4,F_5), (F_4,F_8), (F_5,F_8)**
8-3.7	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_3F_5$	(F_3,F_4), (F_3,F_5), (F_3,F_7), (F_3,F_8), (F_4,F_5), (F_4,F_8), (F_5,F_7), (F_7,F_8)
8-3.8	$F_6=F_1F_2$, $F_7=F_3F_4$, $F_8=F_1F_3F_5$	(F_5,F_8)
8-3.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4F_5$	(F_4,F_5), (F_4,F_8), (F_5,F_8)
8-3.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4F_5$	(F_4,F_5), (F_4,F_8), (F_5,F_8)
9-4.6	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_2F_5$, $F_9=F_2F_4F_5$	**(F_3,F_9)**
9-4.7	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_3F_5$, $F_9=F_1F_4F_5$	(F_2,F_3), (F_2,F_4), (F_2,F_5), (F_2,F_7), (F_2,F_8), (F_2,F_9), (F_3,F_6), (F_4,F_6), (F_5,F_6), (F_6,F_7), (F_6,F_8), (F_6,F_9) (F_3,F_4), (F_3,F_5), (F_3,F_7), (F_3,F_8), (F_3,F_9), (F_4,F_5), (F_4,F_7), (F_4,F_8), (F_4,F_9), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_7,F_8), (F_7,F_9), (F_8,F_9)
9-4.8	$F_6=F_1F_2$, $F_7=F_3F_4$, $F_8=F_1F_3F_5$, $F_9=F_2F_4F_5$	(F_5,F_8), (F_5,F_9), (F_8,F_9)
9-4.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4F_5$	**(F_5,F_9)**
9-4.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4$, $F_9=F_3F_4F_5$	(F_5,F_9)
10-5.5	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_3F_5$, $F_9=F_1F_4F_5$, $F_{10}=F_3F_4F_5$	(F_1,F_2), (F_1,F_6), (F_2,F_3), (F_2,F_4), (F_2,F_5), (F_2,F_6), (F_2,F_7), (F_2,F_8), (F_2,F_9), (F_2,F_{10}), (F_3,F_4), (F_3,F_5), (F_3,F_6), (F_3,F_7), (F_3,F_8), (F_3,F_9), (F_3,F_{10}), (F_4,F_5), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_9), (F_4,F_{10}), (F_5,F_6), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_5,F_{10}), (F_6,F_7), (F_6,F_8), (F_6,F_9), (F_6,F_{10}), (F_7,F_8), (F_7,F_9), (F_7,F_{10}), (F_8,F_9), (F_8,F_{10}), (F_9,F_{10})
10-5.6	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_3F_5$, $F_9=F_1F_4F_5$, $F_{10}=F_2F_3F_4F_5$	(F_3,F_{10}), (F_4,F_{10}), (F_5,F_{10}), (F_7,F_{10}), (F_8,F_{10}), (F_9,F_{10})
10-5.7	$F_6=F_1F_2$, $F_7=F_1F_3F_4$, $F_8=F_1F_3F_5$, $F_9=F_2F_4F_5$, $F_{10}=F_1F_2F_3F_4F_5$	(F_1,F_2), (F_1,F_6), (F_2,F_6), (F_3,F_4), (F_3,F_5), (F_3,F_7), (F_4,F_7) (F_5,F_8), (F_5,F_9), (F_5,F_{10}), (F_8,F_9), (F_8,F_{10}), (F_9,F_{10})
10-5.8	$F_6=F_1F_2$, $F_7=F_3F_4$, $F_8=F_2F_3F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_1F_4F_5$	(F_4,F_9), (F_4,F_{10}), (F_5,F_8), (F_5,F_{10}), (F_8,F_{10}), (F_9,F_{10})
10-5.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3F_4$, $F_9=F_2F_3F_4F_5$, $F_{10}=F_2F_4F_5$	(F_4,F_5), (F_4,F_9), (F_5,F_8), (F_8,F_9)
10-5.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4F_5$	(F_5,F_9), (F_5,F_{10}), (F_9,F_{10})
11-6.3	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_1F_4F_5$, $F_{11}=F_1F_2F_3F_4F_5$	(F_4,F_5), (F_4,F_9), (F_4,F_{10}), (F_4,F_{11}), (F_5,F_8), (F_5,F_{10}), (F_5,F_{11}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_9,F_{10}), (F_9,F_{11})

Table 4.2. (Continued)

Design	Generators	Dispersion Factors
11-6.4	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_2F_4F_5$, $F_{11}=F_1F_3F_4F_5$	(F_4,F_{10}), (F_4,F_{11}), (F_5,F_{10}), (F_5,F_{11}), (F_8,F_{10}), (F_8,F_{11}), (F_9,F_{10}), (F_9,F_{11}), (F_{10},F_{11})
11-6.5	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_2F_4F_5$, $F_{11}=F_3F_4F_5$	(F_4,F_6), (F_4,F_7), (F_5,F_6), (F_5,F_7), (F_6,F_8), (F_6,F_9), (F_6,F_{10}), (F_6,F_{11}), (F_7,F_8), (F_7,F_9), (F_7,F_{10}), (F_7,F_{11}) (F_4,F_5), (F_4,F_9), (F_4,F_{10}), (F_4,F_{11}), (F_5,F_8), (F_5,F_{10}), (F_5,F_{11}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_9,F_{10}), (F_9,F_{11})
11-6.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_1F_2F_4F_5$, $F_{11}=F_3F_4F_5$	(F_5,F_{11}), (F_9,F_{11})
11-6.7	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_2F_4F_5$, $F_{11}=F_1F_3F_4F_5$	(F_5,F_9), (F_5,F_{10}), (F_5,F_{11}), (F_9,F_{10}), (F_9,F_{11}), (F_{10},F_{11})
11-6.8	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_5$, $F_{10}=F_2F_4F_5$, $F_{11}=F_3F_4F_5$	(F_2,F_7), (F_2,F_8), (F_3,F_6), (F_3,F_8), (F_4,F_6), (F_4,F_7) (F_5,F_9), (F_5,F_{10}), (F_5,F_{11}), (F_9,F_{10}), (F_9,F_{11}), (F_{10},F_{11})
11-6.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_3F_5$, $F_{11}=F_2F_4F_5$	(F_5,F_9), (F_9,F_{10}), (F_9,F_{11})
11-6.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_1F_3F_4F_5$	(F_9,F_{11})
12-7.3	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_3F_5$, $F_{11}=F_2F_4F_5$, $F_{12}=F_1F_3F_4F_5$	(F_5,F_9), (F_5,F_{10}), (F_5,F_{11}), (F_5,F_{12}), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{11},F_{12})
12-7.4	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_3F_5$, $F_{11}=F_2F_4F_5$, $F_{12}=F_3F_4F_5$	(F_5,F_9), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12})
12-7.5	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_3F_5$, $F_{11}=F_1F_2F_4F_5$, $F_{12}=F_1F_3F_4F_5$	(F_5,F_{11}), (F_5,F_{12}), (F_{10},F_{11}), (F_{10},F_{12})
12-7.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_3F_5$, $F_{11}=F_1F_4F_5$, $F_{12}=F_1F_2F_3F_4F_5$	(F_5,F_{12}), (F_{10},F_{11})
12-7.7	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_2F_3F_4$, $F_{10}=F_1F_2F_3F_5$, $F_{11}=F_1F_4F_5$, $F_{12}=F_2F_3F_4F_5$	(F_4,F_5), (F_4,F_{10}), (F_4,F_{11}), (F_4,F_{12}), (F_5,F_9), (F_5,F_{11}), (F_5,F_{12}), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12}), (F_{10},F_{11}), (F_{10},F_{12})
12-7.8	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_5$, $F_{11}=F_2F_3F_5$, $F_{12}=F_2F_4F_5$	(F_9,F_{11}), (F_9,F_{12}), (F_{11},F_{12})
12-7.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_3F_5$, $F_{12}=F_2F_4F_5$	(F_2,F_5), (F_2,F_{11}), (F_2,F_{12}), (F_3,F_5), (F_3,F_{11}), (F_3,F_{12}), (F_4,F_5), (F_4,F_{11}), (F_4,F_{12}), (F_5,F_6), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_5,F_{10}), (F_6,F_{11}), (F_6,F_{12}), (F_7,F_{11}), (F_7,F_{12}), (F_9,F_{11}), (F_9,F_{12}), (F_{10},F_{11}), (F_{10},F_{12}) (F_5,F_{11}), (F_5,F_{12}), (F_{11},F_{12})
12-7.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_1F_2F_3F_4F_5$	(F_9,F_{12})
13-8.2	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_5$, $F_{11}=F_2F_3F_5$, $F_{12}=F_2F_4F_5$, $F_{13}=F_3F_4F_5$	(F_9,F_{11}), (F_9,F_{12}) (F_9,F_{13}), (F_{11},F_{12}), (F_{11},F_{13}), (F_{12},F_{13})

Table 4.2. (Continued)

Design	Generators	Dispersion Factors
13-8.3	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_3F_5$, $F_{12}=F_2F_4F_5$, $F_{13}=F_1F_3F_4F_5$	(F_5,F_{11}), (F_5,F_{12}), (F_5,F_{13}), (F_{11},F_{12}), (F_{11},F_{13}), (F_{12},F_{13})
13-8.4	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_3F_5$, $F_{12}=F_2F_4F_5$, $F_{13}=F_3F_4F_5$	(F_2,F_5), (F_2,F_{11}), (F_2,F_{12}), (F_2,F_{13}), (F_3,F_5), (F_3,F_{11}), (F_3,F_{12}), (F_3,F_{13}), (F_4,F_5), (F_4,F_{11}), (F_4,F_{12}), (F_4,F_{13}), (F_5,F_6), (F_5,F_7), (F_5,F_8), (F_5,F_9), (F_5,F_{10}), (F_6,F_{11}), (F_6,F_{12}), (F_6,F_{13}), (F_7,F_{11}), (F_7,F_{12}), (F_7,F_{13}), (F_8,F_{11}), (F_8,F_{12}), (F_8,F_{13}), (F_9,F_{11}), (F_9,F_{12}), (F_9,F_{13}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{10},F_{13}) (F_5,F_{11}), (F_5,F_{12}), (F_5,F_{13}), (F_{11},F_{12}), (F_{11},F_{13}), (F_{12},F_{13})
13-8.5	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_4F_5$, $F_{13}=F_3F_4F_5$	(F_{12},F_{13})
13-8.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_4$, $F_9=F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_3F_5$, $F_{12}=F_1F_2F_4F_5$, $F_{13}=F_1F_3F_4F_5$	(F_5,F_{11}), (F_5,F_{12}), (F_5,F_{13}), (F_{11},F_{12}), (F_{11},F_{13}), (F_{12},F_{13})
13-8.7	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_4F_5$, $F_{13}=F_1F_2F_3F_4F_5$	(F_9,F_{13})
13-8.8	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_1F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_4F_5$, $F_{13}=F_2F_3F_4F_5$	(F_6,F_{13}), (F_8,F_{13}) (F_9,F_{13})
13-8.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_1F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_4F_5$, $F_{13}=F_1F_2F_3F_4F_5$	(F_8,F_{13})
13-8.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_4F_5$, $F_{13}=F_1F_3F_4F_5$	(F_{12},F_{13})
14-9.2	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_3F_4F_5$	(F_{12},F_{13}), (F_{12},F_{14}), (F_{13},F_{14})
14-9.3	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_1F_2F_3F_5$, $F_{14}=F_2F_4F_5$	(F_2,F_{14}), (F_3,F_{14}), (F_4,F_{14}), (F_5,F_{14}), (F_6,F_{14}), (F_7,F_{14}), (F_8,F_{14}), (F_9,F_{14}), (F_{10},F_{14}), (F_{11},F_{14}), (F_{12},F_{14}), (F_{13},F_{14})
14-9.4	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_1F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_4F_5$, $F_{13}=F_2F_3F_4F_5$, $F_{14}=F_1F_2F_3F_4F_5$	(F_6,F_7), (F_6,F_8), (F_6,F_9), (F_6,F_{10}), (F_6,F_{11}), (F_6,F_{12}), (F_6,F_{13}), (F_7,F_8), (F_7,F_9), (F_7,F_{10}), (F_7,F_{11}), (F_7,F_{12}), (F_7,F_{13}), (F_8,F_9), (F_8,F_{10}), (F_8,F_{11}), (F_8,F_{12}), (F_8,F_{13}), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12}), (F_9,F_{13}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{10},F_{13}), (F_{11},F_{12}), (F_{11},F_{13}), (F_{12},F_{13})

Table 4.2. (Continued)

Design	Generators	Dispersion Factors
14-9.5	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_5$, $F_{12}=F_3F_5$, $F_{13}=F_4F_5$, $F_{14}=F_1F_2F_3F_4F_5$	(F_6,F_{12}), (F_6,F_{13}), (F_6,F_{14}), (F_7,F_{11}), (F_7,F_{13}), (F_7,F_{14}), (F_8,F_{11}), (F_8,F_{12}), (F_8,F_{14}), (F_9,F_{11}), (F_9,F_{12}), (F_9,F_{13})
14-9.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_1F_3F_4F_5$	(F_4,F_9), (F_4,F_{10}), (F_5,F_{11}), (F_5,F_{12}), (F_{13},F_{14})
14-9.7	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_5$, $F_{12}=F_3F_5$, $F_{13}=F_4F_5$, $F_{14}=F_2F_3F_4F_5$	(F_6,F_{12}), (F_6,F_{13}), (F_6,F_{14}), (F_7,F_{11}), (F_7,F_{13}), (F_7,F_{14}), (F_8,F_{11}), (F_8,F_{12}), (F_8,F_{14}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{10},F_{13})
14-9.8	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_2F_3F_4$, $F_{12}=F_1F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_1F_3F_4F_5$	(F_2,F_{13}), (F_2,F_{14}), (F_7,F_{13}), (F_7,F_{14}) (F_5,F_{13}), (F_5,F_{14}), (F_{12},F_{13}), (F_{12},F_{14}), (F_{13},F_{14})
14-9.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_1F_2F_4F_5$	(F_9,F_{12}), (F_{10},F_{11}), (F_{10},F_{13}), (F_{10},F_{14}), (F_{12},F_{13}), (F_{12},F_{14})
14-9.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_2F_3F_4$, $F_{12}=F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_1F_3F_4F_5$	(F_{13},F_{14})
15-10.2	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_1F_2F_3F_5$, $F_{14}=F_2F_4F_5$, $F_{15}=F_3F_4F_5$	(F_{14},F_{15})
15-10.3	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_1F_2F_3F_5$, $F_{14}=F_2F_4F_5$, $F_{15}=F_1F_2F_4F_5$	(F_2,F_7), (F_2,F_8), (F_2,F_9), (F_7,F_8), (F_7,F_9), (F_8,F_9)
15-10.4	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_5$, $F_{12}=F_2F_3F_5$, $F_{13}=F_2F_4F_5$, $F_{14}=F_1F_2F_4F_5$, $F_{15}=F_3F_4F_5$	(F_9,F_{12}), (F_9,F_{15}), (F_{10},F_{11}), (F_{10},F_{14}), (F_{11},F_{15}), (F_{12},F_{14})
15-10.5	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_2F_3F_4$, $F_{12}=F_1F_5$, $F_{13}=F_2F_3F_5$, $F_{14}=F_2F_4F_5$, $F_{15}=F_1F_3F_4F_5$	(F_{14},F_{15})
15-10.6	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_2F_3F_4$, $F_{12}=F_1F_5$, $F_{13}=F_2F_3F_5$, $F_{14}=F_2F_4F_5$, $F_{15}=F_1F_2F_4F_5$	(F_{13},F_{14}), (F_{13},F_{15})
15-10.7	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_2F_5$, $F_{11}=F_3F_5$, $F_{12}=F_1F_2F_3F_5$, $F_{13}=F_4F_5$, $F_{14}=F_1F_2F_4F_5$, $F_{15}=F_1F_3F_4F_5$	(F_2,F_9), (F_3,F_9), (F_4,F_9), (F_9,F_{12}), (F_9,F_{14}), (F_9,F_{15})

Table 4.2. (Continued)

Design	Generators	Dispersion Factors
15-10.8	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_2F_5$, $F_{12}=F_1F_2F_3F_5$, $F_{13}=F_4F_5$, $F_{14}=F_1F_2F_4F_5$, $F_{15}=F_1F_2F_3F_4F_5$	(F_6,F_{10}), (F_9,F_{11}), (F_9,F_{14}), (F_{10},F_{11}), (F_{10},F_{14}) (F_9,F_{10})
15-10.9	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_1F_4$, $F_9=F_2F_3F_4$, $F_{10}=F_1F_2F_3F_4$, $F_{11}=F_2F_5$, $F_{12}=F_3F_5$, $F_{13}=F_1F_2F_3F_5$, $F_{14}=F_4F_5$, $F_{15}=F_2F_3F_4F_5$	(F_2,F_6), (F_2,F_8), (F_2,F_{10}), (F_2,F_{11}), (F_2,F_{14}), (F_2,F_{15}), (F_3,F_7), (F_3,F_{10}), (F_3,F_{12}), (F_3,F_{14}), (F_3,F_{15}), (F_4,F_6), (F_4,F_7), (F_4,F_8), (F_4,F_{11}), (F_4,F_{12}), (F_4,F_{14}), (F_6,F_9), (F_6,F_{12}), (F_6,F_{14}), (F_6,F_{15}), (F_7,F_9), (F_7,F_{11}), (F_7,F_{14}), (F_7,F_{15}), (F_8,F_{11}), (F_8,F_{12}), (F_8,F_{15}), (F_9,F_{10}), (F_9,F_{11}), (F_9,F_{12}), (F_9,F_{15}), (F_{10},F_{11}), (F_{10},F_{12}), (F_{10},F_{14})
15-10.10	$F_6=F_1F_2$, $F_7=F_1F_3$, $F_8=F_2F_3$, $F_9=F_1F_4$, $F_{10}=F_2F_3F_4$, $F_{11}=F_1F_2F_3F_4$, $F_{12}=F_1F_5$, $F_{13}=F_2F_3F_5$, $F_{14}=F_1F_2F_4F_5$, $F_{15}=F_2F_4F_5$	(F_4,F_{15}), (F_{10},F_{15})

Note: 1. Designs with bold face dispersion factors are D-optimal designs.
2. Shadowed dispersion factors are assigned when moderate dispersion occurs.

Appendix

Derivation of the information matrix with one dispersion factor

Now $M = X'V^{-1}X = N^{-1}X'(m_0 I_N + m_1 D_1)X = (m_{ij}),\ i, j = 0, \cdots, n$, where $m_{ij} = N^{-1}(m_0(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0) + m_1(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1))$, and "$\circ$" denote the general inner product of vectors, that is, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = \sum_{k=1}^{N} x_{ik} x_{jk} x_{1k}$.

For $i = j$, $m_{ii} = N^{-1}(m_0(\vec{x}_i \circ \vec{x}_i \circ \vec{x}_0) + m_1(\vec{x}_i \circ \vec{x}_i \circ \vec{x}_1)) = N^{-1}(Nm_0) = m_0$. For $i \neq j$, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0 = 0$, and $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = N$, if $F_1F_iF_j$ is a word, otherwise $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = 0$. Hence

$$m_{ij} = \begin{cases} m_1, & \text{if } F_1F_iF_j \text{ is a word,} \\ 0, & \text{otherwise.} \end{cases}$$

Also note that since the designs we consider here are of resolution III or higher, there is at most one nonzero off-diagonal entry in each row and column of M, that is, it is not possible to have two words of forms $F_1F_iF_j$, $F_1F_{i'}F_j$, or of forms $F_1F_iF_j$, $F_1F_iF_{j'}$, respectively, in the defining relation.

Derivation of the information matrix with two dispersion factors

Now $M = (m_{ij}),\ i, j = 0, \cdots, n,$

$$= N^{-1}X'(m_0 I_N + m_1 D_1 + m_2 D_2 + m_3 D_1 D_2)X$$

$$= \begin{bmatrix} M_{11} & M_{12} \\ M'_{12} & M_{22} \end{bmatrix}$$

where

$$m_{ij} = N^{-1}(m_0(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0) + m_1(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1) + m_2(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_2) + m_3(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2)).$$

For $i = j$, $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_0 = N$, $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_1 = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_2 = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_1 \circ \vec{x}_2 = 0$, hence $m_{ii} = m_0$.

For $i \neq j$, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0 = 0$. In M_{11}, that is, $0 \le i < j \le 2$,

$$\vec{x}_0 \circ \vec{x}_i \circ \vec{x}_i = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_j \circ \vec{x}_j = N\ ,$$

$$\vec{x}_0 \circ \vec{x}_i \circ \vec{x}_0 = \vec{x}_0 \circ \vec{x}_i \circ \vec{x}_j = \vec{x}_i \circ \vec{x}_j \circ \vec{x}_j = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_j = 0\ ,$$

$$\vec{x}_0 \circ \vec{x}_i \circ \vec{x}_j \circ \vec{x}_j = \vec{x}_0 \circ \vec{x}_i \circ \vec{x}_i \circ \vec{x}_j = 0\ .$$

Hence $m_{01} = m_{10} = m_1$, $m_{02} = m_{20} = m_2$, $m_{12} = m_{21} = m_3$, and

$$M_{11} = \begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_0 & m_3 \\ m_2 & m_3 & m_0 \end{bmatrix}.$$

In M_{12}, that is, $0 \le i \le 2 < j \le n$, if $F_1F_2F_j$ is a word, then

$$m_{0j} = N^{-1}(m_1(\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_1) + m_2(\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_2) + m_3(\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2)) = m_3\ ,$$

$$m_{1j} = N^{-1}(m_1(\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_1) + m_2(\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_2) + m_3(\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2)) = m_2\ ,$$

$$m_{2j} = N^{-1}(m_1(\vec{x}_2 \circ \vec{x}_j \circ \vec{x}_1) + m_2(\vec{x}_2 \circ \vec{x}_j \circ \vec{x}_2) + m_3(\vec{x}_2 \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2)) = m_1\ .$$

If $F_1F_2F_j$ is not a word, $m_{ij} = 0$. Since the designs we consider here are of resolution III or higher, it is not possible to have two length three words of form $F_1F_2F_j$ in the defining relation. Hence there is at most one column of form $[m_3, m_2, m_1]'$ in M_{12}.

In M_{22}, that is, $3 \le i < j \le n$, values of $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1$, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_2$, and $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2$ depends on whether the respective corresponding effects $F_1F_iF_j$, $F_2F_iF_j$, and $F_1F_2F_iF_j$ are words in the defining relation, have four possibilities. We summarize them into the following table with "*I*" indicates that the corresponding effect is a word, and "×" indicates that it is not.

$F_1F_iF_j$	$F_2F_iF_j$	$F_1F_2F_iF_j$	$\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1$	$\vec{x}_i \circ \vec{x}_j \circ \vec{x}_2$	$\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2$	m_{ij}
I	$\times$	$\times$	N	0	0	m_1
$\times$	I	$\times$	0	N	0	m_2
$\times$	$\times$	I	0	0	N	m_3
$\times$	$\times$	$\times$	0	0	0	0

Again, since we only consider designs of resolution III or higher, it is not possible to have two length three words $F_1F_iF_j$, $F_1F_iF_{j'}$, or $F_1F_iF_j$, $F_1F_{i'}F_j$ ($F_2F_iF_j$, $F_2F_iF_{j'}$, or $F_2F_iF_j$, $F_2F_{i'}F_j$) in the defining relation, also it is not possible to have two length four words $F_1F_2F_iF_j$, $F_1F_2F_iF_{j'}$, or $F_1F_2F_iF_j$, $F_1F_2F_{i'}F_j$ in the defining relation. Hence, each of the m_0, m_1, m_2, and m_3 appears at most once in each row and each column.

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